VOLUMES AND AREAS OF LIPSCHITZ METRICS

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ABSTRACT. We develop and generalize methods of estimating (Riemannian and Finsler) filling volumes using nonexpanding maps to Banach spaces of L^∞ type. For every Finsler volume functional (such as the Busemann volume or the Holmes–Thompson volume) we construct a natural extension from the class of Finsler metrics to all Lipschitz metrics and define the notion of area for Lipschitz surfaces in a Banach space. We establish a correspondence between minimal fillings and minimal surfaces in L^∞ type spaces. We introduce a Finsler volume functional for which Riemannian and Finsler filling volumes are equal and prove that this functional is semi-elliptic.

INTRODUCTION

0.1. **Motivations.** This paper is motivated by filling minimality and boundary rigidity problems for Riemannian manifolds. Let (M, g) be a compact Riemannian manifold with boundary, $S = \partial M$. Denote by d_g the associated distance function, $d_g : M \times M \to \mathbf{R}$.

Let $d : S \times S \to \mathbf{R}$ be an arbitrary metric on S. We say that a compact Riemannian manifold (M, g) is a *filling* of a metric space (S, d) if $d_g(x, y) \geq d(x, y)$ for all $x, y \in S$. The *filling volume* $\mathrm{FillVol}(S, d)$ of (S, d) is defined by

$$\mathrm{FillVol}(S, d) = \inf\{\mathrm{vol}(M, g) : (M, g) \text{ is a filling of } (S, d)\},$$

cf. [17]. (This definition makes sense only for null-cobordant manifolds S, in general one should let M range over all pseudo-manifolds or complete non-compact manifolds.) We say that (M, g) is a *minimal filling* if it realizes the above infimum, that is, $\mathrm{vol}(M, g) = \mathrm{FillVol}(\partial M, d_g|_{S\times S})$.

Many classic inequalities can be formulated in terms of minimal fillings. For instance, Besikovitch' inequality [5] means that a bounded region in $\mathbf{R}^n$ with the Euclidean metric is a minimal filling of its boundary (equipped with either Euclidean or ℓ_∞ metric), Pu's inequality [22] is equivalent to the fact that the standard hemisphere is a minimal filling of an intrinsic metric of the circle (within the class of fillings homeomorphic to a disc).

Definition 0.1. Let (M, g) be a Riemannian manifold (possibly with boundary). We say that (M, g) has the *geodesic minimality property* if every geodesic segment is a shortest curve among all curves with the same endpoints.

We say that (M, g) has the *strong geodesic minimality property* if the geodesic minimality property is satisfied for some manifold containing M in the interior (and equipped with an extension of the metric g).

Recent results (cf. [19, 10]) indicate that the following conjecture is plausible.

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Conjecture 0.2. *If a manifold* (M, g) *has the geodesic minimality property then it is a minimal filling. Moreover if* (M, g) *has the strong geodesic minimality property then it is a unique* (*up to an isometry*) *minimal filling of its boundary.*

Substituting the definition of a minimal filling yields the following formulation of the conjecture. Suppose that (M, g) has the strong geodesic minimality property and let a Riemannian manifold (M', g') be such that $\partial M' = \partial M = S$ and $d_{g'}|_{S\times S} \ge d_g|_{S\times S}$. Then $\mathrm{vol}(M', g') \ge \mathrm{vol}(M, g)$ and in the case of equality the manifolds (M, g) and (M', g') are isometric.

It is easy to check that, if Conjecture 0.2 is true for (M, g) then g is uniquely (up to an isometry) is determined by the boundary distance function $d_g|_{S\times S}$. Thus Conjecture 0.2 would imply the following well-known Michel's boundary rigidity conjecture [21].

Conjecture 0.3. *If* (M, g) *has the strong geodesic minimality property and a manifold* (M', g') *is such that* $\partial M' = \partial M = S$ *and* $d_{g'}|_{S\times S} = d_g|_{S\times S}$, *then* (M, g) *and* (M', g') *are isometric.*

0.2. **Auxiliary embeddings.** M. Gromov [17] introduced a technique where filling volumes are estimated using Kuratowski's construction which allows one to isometrically embed any metric space X into a Banach space $C^0(X) \subset \ell_\infty(X)$. (This construction applied to the problems we consider is described below in §1.2.) Variants of this construction were used in [19] and [10] to prove partial cases of the conjectures 0.2 and 0.3. In [17] Gromov showed that the filling volume of a Riemannian manifold S equals, up to a multiplicative constant (depending on the dimension), to the infimum of the areas of surfaces spanning an isometric image of S in a suitable Banach space. One of the purposes of this paper is to sharpen this result (in particular, to get rid of the constant), namely to prove the following theorem.

Theorem 0.4. 1. *Let* d *be a Lipschitz metric on a manifold* S (*cf. Definition* 2.1), f *an isometric embedding of* (S, d) *to a Banach space* $\mathcal{L} = L^\infty(\mu)$ *where* μ *is an arbitrary finite measure. Then, for a suitable definition of the notion of area in* $\mathcal{L}$, *the filling volume* $\mathrm{FillVol}(S, d)$ *equals the infimum of the areas of Lipschitz surfaces spanning* $f(S)$ *in* $\mathcal{L}$.

2. *Let* (M, g) *be a Riemannian manifold with boundary,* $f : (M, g) \to \mathcal{L}$ *be an isometric embedding* (*where* $\mathcal{L}$ *is the same as above*). *Then* (M, g) *is a minimal filling if and only if the surface* $f(M)$ *minimizes the area among all Lipschitz surfaces in* $\mathcal{L}$ *with the same boundary.*

Remark 0.5. For a manifold M having the strong geodesic minimality property, there is a natural smooth map to $L^\infty(\partial M)$, namely the *boundary distance representation*, cf. Example 1.9. One can show (similarly to the arguments in [10]), that the surface defined by this map is minimal in a variational sense.

Remark 0.6. It is easy to formulate the equality case of Conjecture 0.2 in terms of auxiliary embeddings. Manifolds (M, g) and (M', g') with a common boundary are isometric if and only if their images under boundary distance representation coincide.

The formulation of Theorem 0.4 is preliminary, the complete formulation is given in §5 (Theorem 5.6 and Corollary 5.7). The definition of the area of a Lipschitz surface $f : M \to L^\infty(\mu)$ is a nontrivial issue which is the subject of a major part of this paper. For such a definition to be useful, it should agree with both the extrinsic geometry of the surface (that is, be computable in terms of derivatives of f) and with the intrinsic geometry (i.e. with the metric on M induced by f).

One of the difficulties that one meets is the lack of Rademacher's theorem (about differentiability almost everywhere) for L^∞-valued Lipschitz maps. Another one is that the metrics induced on M are not sufficiently regular, essentially they are arbitrary Lipschitz metrics. In §2 and §3 we develop a technique for handling these difficulties. In §2 we consider the *tangent Finsler structure* of an arbitrary Lipschitz metric on M. The construction goes along the same lines, with minor modifications, that the one in [15]; the goal of §2 is to prove the technical results for further use. In §3 we consider the notion of weak differentiability of a Lipschitz map $f : M \to L^\infty$ (this notion is obtained from the usual differentiability by switching to a weak topology) and prove a theorem about weak differentiability almost everywhere (Theorem 3.3). The main result of §3 is that the weak differential of a map and the tangent Finsler structure of the induced metric on M agree almost everywhere (Theorem 3.7).

0.3. **Finsler volumes.** Even for a smooth map of manifold to a Banach space, the metric induced on the manifold is not Riemannian since the arising norms in tangent spaces are not Euclidean. Therefore it is natural to consider filling volumes for Finsler metrics.

Recall that a (symmetric) *Finsler structure* on a smooth manifold M is a continuous function $\Phi : TM \to \mathbf{R}_+$ such that for every $p \in M$ the restriction $\Phi|_{T_pM}$ is a norm. Finsler structures are also called *Finsler metrics.* A manifold equipped with a Finsler structure is called a *Finsler manifold.*

Riemannian metrics are a partial case of Finsler ones. Namely, for a Riemannian manifold M, one sets $\Phi(v)$ to be equal to the Riemannian length of a tangent vector v. A Finsler metric Φ is Riemannian if and only if its restrictions on all fibers of the tangent bundle are Euclidean norms.

Unlike in the Riemannian case, there are different (non-equivalent) definitions of the volume for Finsler manifolds, for instance, the Busemann volume [12], the Holmes–Thompson volume [18], Gromov's mass and comass [17], etc. Different applications need different definitions of volume. At the same time, many properties do not depend on the choice of a specific definition and hold for all "natural" notions of volume. Being "natural" includes a set of requirements given in Definition 4.1, the most important one is the monotonicity of volume with respect to the metric.

In §4 we give the necessary definitions and construct an extension of a Finsler volume functional from the class of Finsler metrics to all Lipschitz metrics. This allows one to define the area of a Lipschitz surface in a Banach space. In §5 we prove that the filling volume within the class of all Lipschitz metrics and coincides with that within the class of smooth strictly convex Finsler metrics (Theorem 5.2). Also there we prove an analogue of Theorem 0.4 for Finsler manifolds (Theorem 5.6). The Riemannian version (Corollary 5.7) is obtained from the Finsler one by using a special definition of volume, namely the inscribed Riemannian volume (cf. Example 4.4).

In §6 we give a brief survey of semi-ellipticity problems for Finsler volumes and prove Theorem 6.2 which asserts that the inscribed Riemannian volume has the compression property: for every Banach space X and every n-dimensional linear subspace $V \subset X$ there exists an area non-increasing linear projection $P : X \to V$.

0.4. **Notation and conventions.** Throughout the paper we use the following notation:

ω_n denotes the Lebesgue measure of a unit ball in $\mathbf{R}^n$;

$\mathbf{R}^n_\infty$ denotes the normed vector space $(\mathbf{R}^n, \|\cdot\|_\infty)$ where the norm $\|\cdot\|_\infty$ is defined by $\|(x_1, \dots, x_n)\|_\infty = \max_{1 \le i \le n} |x_i|$. The distance defined by this norm is denoted by d_∞;

$\mathcal{N}(V)$ and $\mathcal{N}_0(V)$, where V is a finite-dimensional vector space, denote the set of all norms and semi-norms on V, respectively. These sets are regarded with the topology of point-wise convergence (which is the same as uniform convergence on compact sets).

All measures on manifolds are meant to be Borel ones, the term "measurable" always means measurable with respect to the Borel σ-algebra.

1. Metric spaces

This section contains preliminaries from metric geometry. A detailed exposition of most subjects considered here can be found in the book [11]. All matters presented in this section are well-known, however the most general formulations of some facts are hard to find in the literature and we supply them with proofs.

We use the terms "metric space" and "metric" in an extended sense, namely we allow zero distances between different points.

Definition 1.1. A *metric* on a set X is a function $d : X \times X \to [0, +\infty)$ satisfying the following conditions:

1. $d(x, x) = 0$ for all $x \in X$.
2. Symmetry: $d(x, y) = d(y, x)$ for all $x, y \in X$.
3. Triangle inequality: $d(x, y) + d(y, z) \ge d(x, z)$ for all $x, y, z \in X$.

A *metric space* is a set equipped with a metric on it.

If the metric d is clear from the context, we write $|xy|$ or $|x, y|$ instead of $d(x, y)$.

The standard definitions and theorems about metric spaces are easy to generalize to the case of metrics with zero distances. Metrics satisfying the condition $d(x, y) > 0$ for $x \ne y$ (i.e., metrics in the usual sense), are called *positive* metrics.

We often consider metrics or sequences of metrics defined on a set X with a prescribed topology (for example, on a smooth manifold). In such a context, we always assume that a metric agrees with the topology in the following sense.

Definition 1.2. We say that a metric d on a topological space X *agrees with the topology* if the topology defined by d is (non-strictly) weaker that the topology of X.

It is easy to see that a metric d agrees with the topology of X if and only if the function $d : X \times X \to \mathbf{R}$ is continuous (with respect to the product topology).

1.1. **Isometric maps.**

Definition 1.3. Let X, Y be metric spaces. A map $f : X \to Y$ is said to be *nonexpanding* if it does not increase distances, that is, $|f(x)f(y)| \le |xy|$ for all $x, y \in X$.

A map $f : X \to Y$ is said to be *isometric* if it preserves the distances, that is, $|f(x)f(y)| = |xy|$ for all $x, y \in X$.

If (Y, d) is a metric space and X is an arbitrary set, than every map $f : X \to Y$ defines a metric d' on X for which f is isometric, namely $d'(x, y) = d(f(x), f(y))$ for all $x, y \in X$. We refer to d' as the metric *induced by f from d* and denote it by f^*d.

Note that the above notion of an isometric map differs from terms "isometric embedding" and "isometric immersion" used in differential geometry where one usually means preserving the lengths of curves and not of the distances.

A well-knows Kuratowski's construction allows one to map any metric space (X, d) isometrically to a Banach space. Namely consider the space $C(X)$ of bounded continuous functions on X with the standard norm $\|f\| = \sup_X |f|$ and fix a point $x_0 \in X$. An isometric map $F : X \to C(X)$ is defined by $F(x)(y) = d(x, y) - d(x_0, y)$. In the case of a bounded X one can use a simpler formula $F(x)(y) = d(x, y)$. We need finite-dimensional approximations of this construction.

Proposition 1.4. *Let (X, d) be a separable metric space. Then there exists a non-decreasing sequence $\{d_n\}$ of metrics on X converging to d uniformly on compact sets and such that for every n the space (X, d_n) admits an isometric map to $\mathbf{R}^n_\infty$.*

Proof. Let $P = \{p_n\}_{n=1}^\infty$ be a countable dense set in X. For each n, consider a function $f_n : X \to \mathbf{R}$ given by $f_n(x) = d(x, p_n)$ and define a map $F_n : X \to \mathbf{R}^n_\infty$ by

$$F_n(x) = (f_1(x), f_2(x), \dots, f_n(x)).$$

Define $d_n = F_n^* d_\infty$. Then F_n is an isometric map of (X, d_n) to $\mathbf{R}^n_\infty$. The construction yields that $d_{n+1}(x, y) \geq d_n(x, y)$, that is, $\{d_n\}$ is a non-decreasing sequence.

It remains to prove that d_n converges to d uniformly on compact sets. The triangle inequality implies that the function f_n is nonexpanding. Therefore

$$d_n(x, y) = \sup_{i \leq n} |f_i(x) - f_i(y)| \leq d(x, y).$$

Let K be a compact subset of X, and let $\varepsilon > 0$. Since K is compact and P is dense, there is an integer $n_0 > 0$ such that the set $\{p_1, p_2, \dots, p_{n_0}\}$ is an ε-net for K. Let $x, y \in K$, then there exists an $i \leq n_0$ such that $d(x, p_i) \leq \varepsilon$. By the triangle inequality, $d(y, p_i) \geq d(x, y) - d(x, p_i) \geq d(x, y) - \varepsilon$. Hence

$$f_i(y) - f_i(x) = d(y, p_i) - d(x, p_i) > d(x, y) - 2\varepsilon,$$

then $d_n(x, y) = \|F_n(x) - F_n(y)\| \geq d(x, y) - 2\varepsilon$ for all $n \geq n_0$. Thus $d - 2\varepsilon \leq d_n \leq d$ on K for all $n \geq n_0$. Since ε is arbitrary, this means d_n converges to d uniformly on $K \times K$. $\square$

Remark 1.5. If the space (X, d) is bounded, then the above maps $F_n : X \to \mathbf{R}^n_\infty$ converge in a natural sense to an isometric map $F : X \to \ell_\infty$. In the general case, one can make the maps converging by subtracting a constant $d(x_0, p_n)$ from f_n where $x_0 \in X$ is a fixed point. Identifying ℓ_∞ with $\ell_\infty(P)$ and observing that the space $C(X)$ is isometrically mapped to $\ell_\infty(P)$ by the restriction operator, one easily sees that the limit map F coincide with Kuratowski's map.

1.2. **Extending Lipschitz maps.**

Proposition 1.6. *Let μ be a measure on an arbitrary set S, X a separable metric space, $Y \subset X$, $f : Y \to L^\infty(\mu)$ a nonexpanding map. Then there exists a nonexpanding map $F : X \to L^\infty(\mu)$ such that $F|_Y = f$.*

Proof. Choose a countable dense subset $Y' \subset Y$. For $x \in X$ and $s \in S$ define

$$F(x)(s) = \inf\{f(y)(s) + |xy| : y \in Y'\}.$$

For every $x \in X$, this formula defines a function $F(x) : S \to [-\infty, +\infty]$, furthermore this function is μ-measurable as an infimum of a countable collection of μ-measurable functions. Observe that

$$F(x)(s) = f(x)(s) \quad \text{for all } x \in Y' \text{ and } \mu\text{-almost all } s \in S. \tag{1.7}$$

Indeed, for all $x, y \in Y'$ and μ-almost all $s \in S$ one has

$$|f(x)(s) - f(y)(s)| \le |xy|$$

since f is nonexpanding, whence $f(y)(s) + |xy| \ge f(x)(s)$. Taking the infimum over y yields that $F(x)(s) \ge f(x)(s)$. On the other hand,

$$F(x)(s) = \inf\{f(y)(s) + |xy| : y \in Y'\} \le f(x)(s) + |xx| = f(x)(s)$$

for every $x \in Y'$, and (1.7) follows.

Now let us prove that

$$|F(x)(s) - F(x')(s)| \le |xx'| \tag{1.8}$$

for all $x, x' \in X$ and $s \in S$ (in the case $F(x)(s) = F(x')(s) = \pm\infty$ we assume the difference to be zero). Indeed, for every $y \in Y'$ one has

$$\big|(f(y)(s) + |xy|) - (f(y)(s) + |x'y|)\big| = \big||xy| - |x'y|\big| \le |xx'|$$

by the triangle inequality. Therefore the Hausdorff distance in $\mathbf{R}$ between the sets $T = \{f(y)(s) + |xy| : y \in Y'\}$ and $T' = \{f(y)(s) + |x'y| : y \in Y'\}$ is no greater than $|xy|$. Since $F(x)(s) = \inf T$ and $F(x')(s) = \inf T'$, this implies (1.8).

Substituting an arbitrary point of Y' for x' in (1.8) yields that the value $F(x)(s)$ is finite for almost every $s \in S$. Thus F is a map from X to $L^\infty(\mu)$, and (1.8) means that this map is nonexpanding. It remains to observe that f and F agree on Y since they agree on Y' by (1.7) and continuous. $\square$

Example 1.9. Let S be a closed smooth manifold, d a metric on S, (M, g) a Riemannian manifold filling (S, d), that is, $\partial M = S$ and $d_g|_{S\times S} \ge d$ (cf. Introduction). Consider a Kuratowski embedding $f : (S, d) \to L^\infty(S)$ given by $f(x)(y) = d(x, y)$. The inequality $d_g|_{S\times S} \ge d$ implies that this map is nonexpanding with respect to the metric d_g. Therefore there exists a nonexpanding map $F : (M, g) \to L^\infty(S)$ extending f.

Now suppose that (M, g) has the geodesic minimality property. Then such a map $F : (M, g) \to L^\infty(S)$ can be defined by the formula $F(x)(s) = d_g(x, s)$, $x \in M$, $s \in S$. It is easy to verify that this map is isometric. Indeed, for all $x, y \in M$ and $s \in S$ one has

$$|F(x)(s) - F(y)(s)| = |d_g(x, s) - d_g(y, s)| \le d_g(x, y)$$

by the triangle inequality, whence $\|F(x) - F(y)\|_\infty \le d_g(x, y)$. On the other hand, let s_0 be a point where the geodesic passing through x and y hits the boundary of the manifold. Then, by the geodesic minimality property, $d_g(x, y) = |d_g(x, s_0) - d_g(y, s_0)|$, i.e. $|F(x)(s_0) - F(y)(s_0)| = d_g(x, y)$, hence $\|F(x) - F(y)\|_\infty = d_g(x, y)$. The so constructed isometric map $F : (M, g) \to L^\infty(S)$ is referred to as the *boundary distance representation* of (M, g).

Now let (M, g) and (M', g') be as in Conjecture 0.2, that is, (M, g) has the geodesic minimality property, $\partial M = \partial M' = S$ and $d_{g'}|_{S\times S} \ge d_g|_{S\times S}$. Then, as shown above, there exists an isometric map $F : (M, g) \to L^\infty(S)$ and a non-expanding map $F' : (M', g') \to L^\infty(S)$ whose restrictions on S coincide with Kuratowski's embedding of $(S, d_g|_{S\times S})$. For any natural definition of the area $\mathrm{area}(F)$ of a surface $F : M \to L^\infty(S)$, isometric maps should preserve the area and nonexpanding maps should not increase it, hence $\mathrm{area}(F) = \mathrm{vol}(M, g)$ and $\mathrm{area}(F') \le \mathrm{vol}(M', g')$. Therefore, in order to prove Conjecture 0.2 it suffices to verify the inequality $\mathrm{area}(F) \le \mathrm{area}(F')$. This argument proves one of the implications of Theorem 0.4, namely that the area-minimality of a surface F implies that (M, g) is a minimal filling. The key requirement to the definition of area is the property that nonexpanding maps do not increase area.

1.3. **Lengths of curves.**

Definition 1.10. A *curve* (or *path*) in a topological space X is a continuous map $\gamma : [a, b] \to X$ where $a \le b$.

Note that, if a metric d on X agrees with the topology (in the sense of Definition 1.2) then every curve in X is continuous with respect to the metric d.

Definition 1.11. Let (X, d) be a metric space, $\gamma : [a, b] \to X$ a curve (X, d). A *partition* of γ is a finite sequence of the form $\gamma(t_0), \dots, \gamma(t_n)$, where $\{t_i\}_{i=0}^n$ is a partition of the segment $[a, b]$, that is, $a = t_0 \le t_1 \cdots \le t_n = b$. The *length of a partition* $\gamma(t_1), \dots, \gamma(t_n)$ is the sum $\sum_{i=0}^{n-1} d(\gamma(t_i), \gamma(t_{i+1}))$. The *length* $L_d(\gamma)$ of γ is the supremum of the lengths of all its partitions. We omit the index d in the notation L_d if the metric d is clear from the context. A curve γ is said to be *rectifiable* if $L(\gamma) < \infty$.

Standard properties of length (see e.g. [11, §2.3]) can be trivially generalized to the case of metrics with zero distances. We will need the following elementary facts.

Proposition 1.12. *For every curve $\gamma : [a, b] \to X$ the following holds.*

1. *Additivity*: $L(\gamma) = L(\gamma|_{[a,c]}) + L(\gamma_{[c,b]})$ *for all* $c \in [a, b]$.
2. *Triangle inequality*: $L(\gamma) \ge |\gamma(a)\gamma(b)|$.
3. *The length of a partition of γ converges to $L(\gamma)$ as the mesh goes to zero. By the mesh of a partition $\{t_i\}$ of a segment $[a, b]$ we mean the number* $\max_i |t_i - t_{i+1}|$.

Proposition 1.13. *Let γ be a curve in a metric space (X, d) and let $\{d_n\}$ be a non-decreasing sequence of metrics on X converging to d point-wise (that is, $d_n(x, y) \to d(x, y)$ as $n \to \infty$ for all $x, y \in X$). Then $L_{d_n}(\gamma) \to L_d(\gamma)$ as $n \to \infty$.*

Proof. The inequality $d_n \le d$ implies that γ is continuous with respect to d_n and $L_{d_n}(\gamma) \le L_d(\gamma)$. It suffices to prove that $\underline{\lim}\, L_{d_n}(\gamma) \ge L_d(\gamma)$. Let $[a, b]$ be the domain of γ, $\{t_i\}_{i=0}^N$ a partition of $[a, b]$. Summing up the inequalities $L_{d_n}(\gamma|_{[t_i,t_{i+1}]}) \ge d_n(\gamma(t_i), \gamma(t_{i+1}))$ yields the inequality

$$L_{d_n}(\gamma) \ge \sum_{i=0}^{N-1} d_n(\gamma(t_i), \gamma(t_{i+1})).$$

Passing to the limit as $n \to \infty$ yields that

$$\underline{\lim}\, L_{d_n}(\gamma) \ge \sum_{i=0}^{N-1} d(\gamma(t_i), \gamma(t_{i+1})).$$

Taking the supremum over all partitions $\{t_i\}$ in the right-hand part yields that $\underline{\lim}\, L_{d_n}(\gamma) \ge L_d(\gamma)$, and the proposition follows. $\square$

Definition 1.14. A curve $\gamma : [a, b] \to X$ in a metric space is said to be *absolutely continuous* if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for every finite collection $\{(a_i, b_i)\}_i$ of disjoint intervals contained in $[a, b]$ and satisfying $\sum_i |a_i - b_i| \le \delta$, one has $\sum_i |\gamma(a_i)\gamma(b_i)| < \varepsilon$.

It is easy to check that every absolutely continuous curve is rectifiable. Lipschitz curves are obviously absolutely continuous.

Definition 1.15. Let $\gamma : [a, b] \to X$ be a curve in a metric space and $t \in [a, b]$. Define the *upper metric speed* $\overline{s}_\gamma(t)$ of γ at t by

$$\overline{s}_\gamma(t) = \varlimsup_{t' \to t} \frac{|\gamma(t)\gamma(t')|}{|t - t'|}.$$

The similar lower limit is called the *lower metric speed* and denoted by $\underline{s}_\gamma(t)$.

If the upper speed and the lower speed coincide, that is, if there exists a limit $\lim_{t' \to t} \frac{|\gamma(t)\gamma(t')|}{|t-t'|}$, then this limit is referred to as the *metric speed* of γ at t and denoted by $s_\gamma(t)$.

Proposition 1.16. *If a curve $\gamma : [a, b] \to X$ in a metric space X is absolutely continuous, then the metric speed $s_\gamma(t)$ is defined and finite for almost all $t \in [a, b]$ and furthermore*

$$L(\gamma) = \int_a^b s_\gamma(t)\, dt.$$

Proof. Define a (non-decreasing) function $\lambda : [a, b] \to \mathbf{R}$ by $\lambda(t) = L(\gamma|_{[a,t]})$. Let ε, δ, $\{(a_i, b_i)\}$ be as in Definition 1.14. Subdividing the intervals (a_i, b_i) arbitrarily and substituting the resulting collections of segments into the same definition, one gets the inequality $\sum_i L(\gamma|_{[a_i,b_i]}) \le \varepsilon$. Since $L(\gamma|_{[a_i,b_i]}) = \lambda(b_i) - \lambda(a_i)$, it follows that the function λ is absolutely continuous. Hence it is differentiable almost everywhere on $[a, b]$ and

$$L(\gamma) = \lambda(b) - \lambda(a) = \int_a^b \lambda'(t)\, dt.$$

Hence it suffices to prove that $\underline{s}_\gamma = \overline{s}_\gamma = \lambda'$ almost everywhere on $[a, b]$. The inequality $L(\gamma|_{[t,t']}) \ge |\gamma(t)\gamma(t')|$ implies that $\lambda' \ge \overline{s}_\gamma$ wherever λ' is defined. Thus it suffices to prove that $\underline{s}_\gamma \ge \lambda'$ almost everywhere on $[a, b]$.

Suppose the contrary. Then there is an $\varepsilon > 0$ and a Borel set $T \subset [a, b]$ of positive measure such that $\underline{s}_\gamma(t) < \lambda'(t) - \varepsilon$ for all $t \in T$. By the regularity of the Lebesgue measure, T contains a closed subset of positive measure, hence we may assume that T itself is closed. Choose a $\delta > 0$ such that for every partition of $[a, b]$ of mesh no greater than δ, the length of the corresponding partition of γ differs from $L(\gamma)$ by less than $\frac{1}{3}\varepsilon \cdot m(T)$, where m denotes the Lebesgue measure on $[a, b]$. For every $t \in T$ choose a $t_1 = t_1(t) \in [a, b]$, such that $|t_1 - t| < \delta$ and

$$\frac{|\gamma(t)\gamma(t_1)|}{|t - t_1|} < \frac{|\lambda(t) - \lambda(t_1)|}{|t - t_1|} - \varepsilon$$

(such a t_1 exists since $\underline{s}_\gamma(t) \le \lambda'(t) - \varepsilon$). Then slightly extend the segment $[t, t_1]$ (or $[t_1, t]$), namely include it in an open interval $(a(t), b(t))$ where $a(t)$ and $b(t)$ are so close to t and t_1 that the above inequality hold true for $a(t)$ and $b(t)$, that is, $|a(t) - b(t)| < \delta$ and

$$\frac{|\gamma(a(t)), \gamma(b(t))|}{|a(t) - b(t)|} < \frac{|\lambda(a(t)) - \lambda(b(t))|}{|a(t) - b(t)|} - \varepsilon. \tag{1.17}$$

The interval of the form $(a(t), b(t))$ form an open covering of T; choose a minimal finite sub-covering $\{(a_i, b_i)\}_{i=1}^N$ of this covering. Then change the numeration of the intervals (a_i, b_i) so that $a_1 \le a_2 \le \cdots \le a_N$. The minimality of the covering implies that $(a_i, b_i) \cap (a_j, b_j) = \emptyset$ whenever $|i - j| > 1$, in particular, the intervals indexed by even numbers are disjoint. We may assume that they cover at least half of the measure of T (otherwise use odd indices instead). By (1.17) we have

$$\frac{|\gamma(a_{2i}), \gamma(b_{2i})|}{|a_{2i} - b_{2i}|} < \frac{|\lambda(a_{2i}) - \lambda(b_{2i})|}{|a_{2i} - b_{2i}|} - \varepsilon = \frac{L(\gamma|_{[a_{2i}, b_{2i}]})}{|a_{2i} - b_{2i}|} - \varepsilon,$$

whence

$$\sum_i \left(L(\gamma|_{[a_{2i}, b_{2i}]}) - |\gamma(a_{2i}), \gamma(b_{2i})| \right) > \varepsilon \cdot \sum_i |a_{2i} - b_{2i}| \ge \frac{1}{2}\varepsilon \cdot m(T).$$

Include the collection of segments $\{[a_{2i}, b_{2i}]\}$ to a partition $\{t_j\}$ of $[a, b]$ of mesh less than δ. Then

$$\begin{aligned} L(\gamma) - \sum_j |\gamma(t_j), \gamma(t_{j+1})| &= \sum_j \left(L(\gamma|_{[t_j, t_{j+1}]}) - |\gamma(t_j), \gamma(t_{j+1})| \right) \\ &\ge \sum_i \left(L(\gamma|_{[a_{2i}, b_{2i}]}) - |\gamma(a_{2i}), \gamma(b_{2i})| \right) > \frac{1}{2}\varepsilon \cdot m(T). \end{aligned}$$

(The first inequality here is obtained by removing the summands for which the segment $[t_j, t_{j+1}]$ is not one of the segments $[a_{2i}, b_{2i}]$.) But by the choice of δ the left-hand part is no greater than $\frac{1}{3}\varepsilon \cdot m(T)$. This contradiction completes the proof. $\square$

2. Lipschitz metrics

2.1. **Weak Finsler structures.** In this section we construct an analogue of a tangent cone for an arbitrary Lipschitz metric on a manifold. The construction is similar to that in [15] but differ in some details.

In the sequel, M denotes a smooth manifold (possibly with boundary) and d_{riem} an arbitrarily chosen auxiliary Riemannian metric on M.

Definition 2.1. We say that a curve $\gamma : [a, b] \to M$ is *Lipschitz* if it is Lipschitz with respect to d_{riem}, that is, there is a $C > 0$ such that

$$d_{\mathrm{riem}}(\gamma(t), \gamma(t')) \le C \cdot |t - t'|$$

for all $t, t' \in [a, b]$.

We say that a metric d on M is *Lipschitz* if it is locally Lipschitz with respect to d_{riem}, that is, for any point $x \in M$ there exists a neighborhood $U \ni x$ and a $C > 0$ such that

$$d(y, z) \le C \cdot d_{\mathrm{riem}}(y, z)$$

for all $y, z \in U$.

Clearly this definition do not depend on the choice of the auxiliary Riemannian metric d_{riem}. If d is a Lipschitz metric and γ is a Lipschitz curve then γ is Lipschitz (and therefore absolutely continuous) with respect to d. By Rademacher's theorem, every Lipschitz curve is differentiable almost everywhere in its domain.

Definition 2.2. A *weak Finsler structure* on a smooth manifold M is a Borel function $\varphi : TM \to \mathbf{R}$ satisfying the following conditions.

1. Non-negativity: $\varphi(v) \geq 0$ for all $v \in TM$.
2. Symmetry and positive homogeneity: $\varphi(\lambda v) = |\lambda|\varphi(v)$ for all $v \in TM$, $\lambda \in \mathbf{R}$.
3. Local boundedness: $\sup(\varphi|_K) < \infty$ for any compact set $K \subset TM$.

Definition 2.3. Let φ be a weak Finsler structure on M, $\gamma : [a,b] \to M$ a Lipschitz curve. The *length* $L_\varphi(\gamma)$ of γ with respect to φ is defined by

$$L_\varphi(\gamma) = \int_a^b \varphi(\gamma'(t))\,dt.$$

Definition 2.4. Let d be a Lipschitz metric on M. For every $v \in TM$ define a real number $\varphi_d(v)$ by

$$\varphi_d(v) = \overline{s}_\gamma(0)$$

where γ is an arbitrary curve of the form $\gamma : (-\varepsilon, \varepsilon) \to M$ such that it is differentiable at zero and $\gamma'(0) = v$. Here $\overline{s}$ denotes the upper metric speed with respect to d (cf. Definition 1.15).

We refer to the so defined function $\varphi_d : TM \to \mathbf{R}$ as the *tangent Finsler structure* of the metric d.

The correctness of the definition is ensured by the following lemma.

Lemma 2.5. *Let d be a Lipschitz metric on M, and let γ_1 and γ_2 be curves differentiable at zero and such that $\gamma_1'(0) = \gamma_2'(0)$. Then $\overline{s}_{\gamma_1}(0) = \overline{s}_{\gamma_2}(0)$ and $\underline{s}_{\gamma_1}(0) = \underline{s}_{\gamma_2}(0)$, where $\overline{s}$ and $\underline{s}$ denote the upper and lower metric speed with respect to d.*

Proof. The definition of a metric speed and the triangle inequality imply that

$$|\overline{s}_{\gamma_1}(0) - \overline{s}_{\gamma_2}(0)| \leq \overline{\lim_{t\to 0}} \frac{d(\gamma_1(t), \gamma_2(t))}{|t|}.$$

Since the metric d is Lipschitz and $\gamma_1'(0) = \gamma_2'(0)$, for some constant C we have

$$d(\gamma_1(t), \gamma_2(t)) \leq C \cdot d_{\mathrm{riem}}(\gamma_1(t), \gamma_2(t)) = o(|t|), \quad t \to 0.$$

Hence the right-hand part of the previous inequality equals zero. The proof for the lower speed is similar. ☐

Remark 2.6. Unlike in the similar definition from [15], the function φ_d from Definition 2.4 is symmetric everywhere on TM since the notion of metric speed that we use (Definition 1.15) is symmetric with respect to the change of a parameter given by $t \mapsto -t$.

Proposition 2.7. *Let d be a Lipschitz metric on M and $\varphi = \varphi_d$ its tangent Finsler structure. Then*

1. *φ is a weak Finsler structure in the sense of Definition* 2.2.
2. *For every $x \in M$, the restriction $\varphi|_{T_xM}$ is Lipschitz.*
3. *For every Lipschitz curve γ one has $L_d(\gamma) = L_\varphi(\gamma)$ where L_d denotes the length with respect to the metric d (cf. Definition* 1.11*), L_φ the length with respect to φ (cf. Definition* 2.3*).*
4. *Let X be a normed vector space, $f : (M,d) \to X$ an isometric map which is differentiable at a point $p \in M$. Then $\varphi|_{T_pM}$ coincides with the semi-norm induced from $\|\cdot\|_X$ by the map d_pf, that is,*

$$\varphi(v) = \|d_pf(v)\|_X$$

for all $v \in T_pM$.

Proof. Assertions 1 and 2 follow trivially from the definitions and the fact that the metric is Lipschitz. Assertion 3 is a re-formulation of Proposition 1.16.

Let us prove the 4th assertion. Let $\gamma : (-\varepsilon, \varepsilon) \to M$ be a curve differentiable at zero, $\gamma(0) = p$, $\gamma'(0) = v$. Then

$$\varphi(v) = \overline{\lim_{t\to 0}} \frac{d(\gamma(t), p)}{t} = \overline{\lim_{t\to\infty}} \frac{\|f(\gamma(t)) - f(p)\|_X}{t} = \|(f \circ \gamma)'(0)\|_X = \|d_p f(v)\|_X.$$

Here the first identity follows from the definition of φ, the second one from the fact that f is isometric, and the last one from the fact that f is differentiable at p. □

Definition 2.8. We say that two functions $\varphi_1, \varphi_2 : TM \to \mathbf{R}$ coincide *almost everywhere on Lipschitz curves* if for every Lipschitz curve $\gamma : [a, b] \to M$, the identity $\varphi_1(\gamma'(t)) = \varphi_2(\gamma'(t))$ is satisfied for almost all $t \in [a, b]$.

If two functions coincide almost everywhere on Lipschitz curves, then they obviously coincide almost everywhere on TM.

Proposition 2.9. *Let d be a Lipschitz metric on M, $\{d_n\}$ a non-decreasing sequence of metrics on M converging to d point-wise. Then*

$$\varphi_d = \lim_{n\to\infty} \varphi_{d_n}$$

almost everywhere on Lipschitz curves.

Proof. Obviously the tangent Finsler structure depend monotonously on the metric, hence $\{\varphi_{d_n}\}$ is a non-decreasing sequence bounded above by the function φ_d. Let $\gamma : [a, b] \to M$ be a Lipschitz curve. Proposition 1.13 implies that $L_{d_n}(\gamma) \to L_d(\gamma)$ as $n \to \infty$. Then Proposition 2.7(3) implies that

$$\int_a^b \varphi_d(\gamma'(t))\, dt = \lim_{n\to\infty} \int_a^b \varphi_{d_n}(\gamma'(t))\, dt = \int_a^b \lim_{n\to\infty} \varphi_{d_n}(\gamma'(t))\, dt$$

(the second identity follows from Levy's theorem since the sequence $\{\varphi_{d_n}\}$ is monotone). Since $\lim_{n\to\infty} \varphi_{d_n} \le \varphi_d$, it follows that the expressions under the integrals are equal for almost all t. □

The next proposition justifies the use of the term "Finsler structure" for a function φ_d.

Proposition 2.10. *Let d be a Lipschitz metric on M, $\varphi = \varphi_d$ its tangent Finsler structure. Then for almost all $x \in M$ the restriction $\varphi|_{T_xM}$ is a semi-norm.*

Proof. First we prove the statement in the case when the metric d admits an isometric map to a finite-dimensional Banach space X. Let $f : (M, d) \to X$ be an isometric map, then it is Lipschitz with respect to an auxiliary Riemannian metric on M, therefore (by Rademacher's theorem) f is differentiable almost everywhere on M. For every point $p \in M$ where f is differentiable, Proposition 2.7(4) implies that $\varphi(v) = \|d_x f(v)\|_X$ for all $v \in T_pM$. Hence $\varphi|_{T_pM}$ is a semi-norm.

In the general case, by Proposition 1.4 there exists a non-decreasing sequence $\{d_n\}$ of metrics admitting isometric maps to finite-dimensional Banach spaces and converging to d. Applying the above proof to the metrics d_n, we obtain that the associated functions $\varphi_n = \varphi_{d_n}$ are semi-norms on almost all fibers T_xM, $x \in M$. By Proposition 2.9, $\varphi = \lim_{n\to\infty} \varphi_n$ almost everywhere on TM. Thus for almost all $x \in M$ the restriction $\varphi|_{T_xM}$ coincide almost everywhere (on T_xM) with a limit of a sequence of semi-norms. By continuity (cf. Proposition 2.7(2)) this means that $\varphi|_{T_xM}$ itself is a semi-norm. $\square$

Remark 2.11. In general, it is not true that $\varphi|_{T_xM}$ is a semi-norm for all $x \in M$. For example, it is easy to construct a Lipschitz metric d on $\mathbf{R}^2$ such that $d((0,0),(0,x)) = d((0,0),(x,0)) = |x|$ and $d((0,0),(x,x)) = 10|x|$ for all $x \in \mathbf{R}$. Then, for the standard generators e_1 and e_2 of $T_0\mathbf{R}^2$ we have $\varphi_d(e_1) = \varphi_d(e_2) = 1$, $\varphi_d(e_1 + e_2) \geq 10$, hence the restriction of φ_d on $T_0\mathbf{R}^2$ does not satisfy the triangle inequality.

The next proposition shows that the tangent Finsler structure is essentially preserved when a metric is replaced by an associated intrinsic one.

Proposition 2.12. *Let d be a Lipschitz metric on M. Define a metric d^* on M by*

$$d^*(x,y) = \inf\{L_d(\gamma) : \ \gamma \text{ is a Lipschitz curve connecting } x \text{ and } y\}.$$

Then the tangent Finsler structures of d and d^* coincide almost everywhere on Lipschitz curves.

Proof. Denote $\varphi = \varphi_d$, $\varphi^* = \varphi_{d^*}$. Obviously $d^* \geq d$, whence $\varphi^* \geq \varphi$. Let $\gamma : [a,b] \to M$ be a Lipschitz curve. Then $L_{d^*}(\gamma) = L_d(\gamma)$. Indeed, the inequality $d^* \geq d$ implies that $L_{d^*}(\gamma) \geq L_d(\gamma)$. To prove the opposite inequality, it suffices to check that for every partition $\{t_i\}$ of $[a,b]$ one has

$$\sum d^*(\gamma(t_i), \gamma(t_{i+1})) \leq L(\gamma).$$

Observe that

$$d^*(\gamma(t_i), \gamma(t_{i+1})) \leq L_d(\gamma|_{[t_i,t_{i+1}]}),$$

since $\gamma|_{[t_i,t_{i+1}]}$ belongs to the set of curves over which the infimum is taken in the definition of the d^* distance in the left-hand side. Summing up such inequalities over all i, we obtain the desired one.

Thus $L_{d^*}(\gamma) = L_d(\gamma)$, hence $L_{\varphi^*}(\gamma) = L_\varphi(\gamma)$ by Proposition 2.7(3). Since $\varphi^* \geq \varphi$, this implies that $\varphi^*(\gamma'(t)) = \varphi(\gamma(t))$ for almost all $t \in [a,b]$. $\square$

Remark 2.13. The metric d^* from Proposition 2.12 is a length metric since it is defined by a length structure (cf. [11]). Another way to construct a length metric is the following: for $x, y \in M$ let $d'(x,y)$ equal the the infimum of all continuous (not only Lipschitz) curves connecting x and y. The tangent Finsler structure of such a metric d' also coincides with φ_d almost everywhere on Lipschitz curves; this follows from the obvious inequalities $d \leq d' \leq d^*$.

If the metric d is bi-Lipschitz equivalent to a Riemannian one then the metrics d^* and d' coincide since every rectifiable curve admits a Lipschitz parameterization.

2.2. **Smoothening Lipschitz metrics.** Let (M, φ) be a Finsler manifold. The Finsler structure φ defines a metric d_φ on M by $d_\varphi(x,y) = \inf\{L_\varphi(\gamma)\}$ where the infimum is taken over all Lipschitz (or piecewise smooth) curves γ connecting the points x and y in M. A Finsler structure φ (and the associated metric d_φ) is said to be *smooth* if the function $\varphi : TM \to \mathbf{R}$ is smooth (that is, C^∞) outside the zero section. A smooth Finsler structure φ (and the metric d_φ) is said to be *strictly convex* if for every point $x \in M$ the second differential of the function $\varphi^2|_{T_xM}$ is positive definite everywhere on $T_xM \setminus \{0\}$.

Proposition 2.14. *Let M be a compact smooth manifold, d a Lipschitz metric on M, $\varphi = \varphi_d$ its tangent Finsler structure. Then there exists a sequence $\{\varphi_i\}_{i=1}^\infty$ of smooth strictly convex Finsler structures on M such that*

1. $\varphi_i|_{T_xM} \to \varphi|_{T_xM}$ *for almost all $x \in M$.*
2. $d_i(x,y) \ge d(x,y) - \varepsilon_i$ *for all $x, y \in M$, where $d_i = d_{\varphi_i}$, $\{\varepsilon_i\}$ is a sequence of real numbers converging to zero.*
3. $d_i \le C \cdot d_{\mathrm{riem}}$ *for all i, where d_{riem} is an auxiliary Riemannian metric and $C > 1$ is a constant such that $d \le (C-1)d_{\mathrm{riem}}$.*

Proof. Propositions 1.4 and 2.9 reduce the statement to the case when the metric d admits an isometric map to $\mathbf{R}^N_\infty$. Such a map $f : M \to \mathbf{R}^N_\infty$ is Lipschitz with a Lipschitz constant $C-1$ with respect to d_{riem}. Smoothening f with a suitable convolution yields a sequence of smooth maps $f_i : M \to \mathbf{R}^N_\infty$ such that $f_i \rightrightarrows f$, $df_i \to df$ almost everywhere on TM, and $\|df_i\| \le C - \frac{1}{2}$ where the norm $\|df_i\|$ is regarded with respect to the metric d_{riem}. We may assume that $N > 2\dim M$, then the maps f_i can be replaced by smooth embeddings with the same properties. We approximate the norm $\|\cdot\|_\infty$ on $\mathbf{R}^N$ by smooth strictly convex norms $\|\cdot\|_{p_i}$, $p_i \to \infty$ (and all p_i are large enough) where

$$\|(x_1, \dots, x_N)\|_p = (x_1^p + \dots + x_N^p)^{1/p}.$$

Let φ_i be a Finsler structure on M induced by the map f_i from the norm $\|\cdot\|_{p_i}$. Then $\{\varphi_i\}$ is a desired sequence. Indeed the 1st requirement follows from the convergence $df_i \to df$ a.e. and the convergence $\|\cdot\|_p \to \|\cdot\|_\infty$ as $p \to \infty$. The 2nd requirement follows from the relations

$$d_i(x,y) \ge \|f_i(x) - f_i(y)\|_{p_i} \ge \|f_i(x) - f_i(y)\|_\infty \rightrightarrows \|f(x) - f(y)\|_\infty = d(x,y).$$

The 3rd requirement follows from the condition $\|df_i\| \le C - \frac{1}{2}$ and the fact that the norms $\|\cdot\|_{p_i}$ and $\|\cdot\|_\infty$ are close to each other. ☐

Remark 2.15. If the tangent Finsler structure $\varphi = \varphi_d$ in Proposition 2.14 is smooth and strictly convex in a neighborhood of a closed set $K \subset M$, then one can choose approximating Finsler structures φ_i so that they coincide with φ on K. To achieve this, it suffices to combine φ and the structures φ_i constructed in the proposition using a smooth partition of unity.

3. Weak differentiability

3.1. **Rademacher's Theorem.** A classical Rademacher's theorem (cf. [16, Theorem 3.1.6]) asserts that every Lipschitz map $f : \mathbf{R}^m \to \mathbf{R}^n$ is differentiable almost everywhere. This theorem is also true for functions with values in reflexive Banach spaces and, more generally, in Banach spaces having the Radon–Nikodym property (cf. [4, ch. 5]).

However for L^∞ type spaces (which are used as target spaces for Kuratowski's embeddings) a similar assertion is incorrect. For example, consider a map $f : [0,1] \to L^\infty[0,1]$ defined by $f(x)(y) = |x-y|$. It is easy to check that f is Lipschitz with a Lipschitz constant 1 but nowhere differentiable.

To work around this difficulty, we introduce the notion of weak differentiability for maps valued in Banach spaces dual to separable ones. We are going to show that Lipschitz maps are weakly differentiable almost everywhere and their weak differentials are naturally related with tangent Finsler structures of induced metrics.

In the sequel, M denotes a smooth manifold equipped with an auxiliary Riemannian metric d_{riem}, X a Banach space, X^* the dual space (of continuous linear functions $X \to \mathbf{R}$). In the applications we usually set $X = L^1(\mu)$ and $X^* = L^\infty(\mu)$ where μ is a finite measure.

Definition 3.1. Let $f : M \to X^*$ be an arbitrary map. For every $u \in X$ consider the function $f_u : M \to \mathbf{R}$, given by

$$f_u(x) = \langle f(x), u\rangle,$$

where $\langle , \rangle$ denotes the standard coupling of X^* and X. We say that f is *weakly differentiable* at a point $p \in M$ if there exists a linear map $L : T_pM \to X^*$ such that for every $u \in X$ the function f_u is differentiable at p and its differential is given by the identity

$$d_p f_u(v) = \langle L(v), u\rangle \quad \text{for all } v \in T_pM.$$

The map L is referred to as the *weak differential* of f at p and denoted by $d_p^w f$.

Proposition 3.2. *Let $f : M \to X^*$ be a Lipschitz map (with respect to the metric d_{riem}) and $p \in M$. Then the weak differentiability of f at p is equivalent to the property that for every $u \in X$ the function f_u from Definition 3.1 is differentiable at p.*

Proof. Let C be a Lipschitz constant for f. Then for every $x, y \in M$ one has $\|f(x) - f(y)\|_{X^*} \le C \cdot d_{\text{riem}}(x, y)$. Then for every $u \in X$,

$$|f_u(x) - f_u(y)| = |\langle f(x) - f(y), u\rangle| \le C \cdot \|u\|_X \cdot d_{\text{riem}}(x, y),$$

that is, the function f_u is Lipschitz with a Lipschitz constant $C\|u\|_X$.

Suppose that f_u is differentiable at p for all $u \in X$. Fix a vector $v \in T_pM$. The Lipschitz continuity of f_u implies the following derivative estimate: $d_p f_u(v) \le C\|u\|_X|v|$. Observe that the function $u \mapsto d_p f_u(v)$ is linear and continuous due to the above estimate. Therefore it represents an element $L(v)$ of X^* such that

$$d_p f_u(v) = \langle L(v), u\rangle \quad \text{for all } u \in X.$$

Thus we have constructed a map $L : T_pM \to X^*$. It is obviously linear and hence is a weak differential of f at p. □

Theorem 3.3. *Let X be a separable Banach space and let $f : M \to X^*$ be a Lipschitz map. Then f is weakly differentiable almost everywhere in M.*

Proof. Since the assertion is local we may assume that M is a region in $\mathbf{R}^n$ and d_{riem} is the standard Euclidean metric. Let U be a countable dense subset of X. For every $u \in U$ the function f_u from Definition 3.1 is Lipschitz and hence is differentiable almost everywhere. Therefore for almost every point $p \in M$ it is true that for all $u \in U$ the function f_u is differentiable at p. We are going to show that for every such a point p the map f is weakly differentiable at p. Let C be a Lipschitz constant for f. Then for every $u \in X$ the function f_u from Definition 3.1 is Lipschitz with a constant $C\|u\|_X$. Fix a $u \in X$ and a sequence $\{u_i\} \subset U$, converging to u. By our assumption every function f_{u_i} is differentiable at p, denote its differential $d_p f_{u_i}$ by L_i ($L_i : T_pM \to \mathbf{R}$). For all i and j the function $f_{u_i} - f_{u_j} = f_{u_i - u_j}$ is Lipschitz with a constant $C\|u_i - u_j\|_X$, whence $\|L_i - L_j\| \le C\|u_i - u_j\| \to 0$ as $i, j \to \infty$. Therefore the sequence $\{L_i\}$ converges to a linear function $L : T_pM \to M$.

Choose an $\varepsilon > 0$ and fix an i such that $\|u - u_i\|_X < \varepsilon$, then $\|L - L_i\| \le C\varepsilon$. Since $L_i = d_p f_{u_i}$, there is a $\delta > 0$ such that

$$|f_{u_i}(q) - f_{u_i}(p) - L_i(q-p)| < \varepsilon|q-p| \tag{3.4}$$

for all $q \in M \subset \mathbf{R}^n$ such that $|q-p| < \delta$. By Lipschitz continuity of f we have $\|f(q) - f(p)\|_{X^*} \le C \cdot |q-p|$, whence

$$\begin{aligned} &|(f_u(q) - f_u(p)) - (f_{u_i}(q) - f_{u_i}(q))| \\ &\qquad = |\langle f(q) - f(p), u - u_i\rangle| \le C \cdot |q-p| \cdot \|u - u_i\|_X \le C\varepsilon|q-p|. \end{aligned} \tag{3.5}$$

The inequality $\|L - L_i\| \le C\varepsilon$ implies that

$$|L(q-p) - L_i(q-p)| \le C|q-p|. \tag{3.6}$$

Adding together (3.4), (3.5) and (3.6), we obtain that

$$|f_u(q) - f_u(p) - L(q-p)| < (2C+1)\varepsilon|q-p|$$

whenever $|q-p| < \delta$. Since ε is arbitrary, this means that the function f_u is differentiable at p and $d_p f_u = L$. Since u is an arbitrary element of X, Proposition 3.2 implies that f is weakly differentiable at p, hence the result. $\square$

3.2. **Weak differential and metric.** As in the previous section, consider a Lipschitz map $f : M \to X^*$ where M is a smooth manifold and X is a separable Banach space. This map induces a metric d on M such that the map is isometric with respect to it (that is, $d(x,y) = \|f(x) - f(y)\|_{X^*}$).

If f is differentiable at p in the usual sense, then its differential induces a semi-norm $\|\cdot\|_p$ on T_pM by $\|v\|_p = \|d_p f(v)\|_{X^*}$. By Proposition 2.7 this semi-norm coincides with the tangent Finsler structure of d at p.

In general, a similar identity for a weak differential is incorrect. For example, consider $M = [-1,1]$ and $X = L^1[0,1]$ and a map $f : [-1,1] \to X^* = L^\infty[0,1]$ given by the equalities

$$f(x)(t) = \begin{cases} x \cdot \max\{1 - t/|x|, 0\}, & x \ne 0, \\ 0, & x = 0. \end{cases}$$

It is easy to check that the map f is isometric and weakly differentiable at zero, however $d_0^w f = 0$.

Nevertheless, the following theorem shows that the tangent Finsler structure and the semi-norm induced by the weak differential agree almost everywhere on M.

Theorem 3.7. *Let d be a Lipschitz metric on M, $\varphi = \varphi_d$ its tangent Finsler structure, X a separable Banach space, $f : M \to X^*$ an isometric map of the space (M,d). Then for almost every point $p \in M$, the identity*

$$\varphi_d(v) = \|d_p^w f(v)\|$$

holds for all $v \in T_pM$.

Proof. Choose a countable set $U = \{u_i\}_{i=1}^\infty$ dense is the unit sphere of X. For every $\xi \in X^*$ one has

$$\|\xi\|_{X^*} = \sup\{|\langle \xi, u\rangle| : u \in X, \|u\|_X = 1\} = \sup_i |\langle \xi, u_i\rangle|$$

where the first identity follows from the definition of the norm $\|\cdot\|_{X^*}$ and the second one from the fact that $\{u_i\}$ is dense in the unit sphere of X.

For every positive integer n consider the map $P_n : X^* \to \mathbf{R}^n_\infty$ given by $P_n(\xi) = (\langle \xi, u_1\rangle, \langle \xi, u_2\rangle, \dots, \langle \xi, u_n\rangle)$, and define a semi-norm $\|\cdot\|_n$ on X^* by $\|\xi\|_n = \|P_n(\xi)\|_\infty$. For every fixed $\xi \in X^*$ one has

$$\|\xi\|_n = \max_{1\le i\le n} |\langle \xi, u_i\rangle|,$$

therefore the sequence $\{\|\xi\|_n\}_{n=1}^\infty$ is non-decreasing and converges to $\|\xi\|_{X^*}$.

For every positive integer n define a Lipschitz metric d_n on M by $d_n(x, y) = \|f(x) - f(y)\|_n$. Then $\{d_n\}$ is a non-decreasing sequence of metrics converging to d point-wise. Let φ_n denote the tangent Finsler structure of d_n, then by Proposition 2.9, $\varphi_n|_{T_pM}$ converges to $\varphi|_{T_pM}$ for almost all $p \in M$.

Let $p \in M$ be a point such that $\varphi_n|_{T_pM}$ converges to $\varphi|_{T_pM}$ and f is weakly differentiable at p. Then for every n the map $P_n \circ f : M \to \mathbf{R}^n_\infty$ is differentiable at p and $d_p(P_n \circ f) = P_n \circ d^w_p f$, since the coordinate functions of the map $P_n \circ f$ are functions f_{u_i} from the definition of weak differentiability, $1 \le i \le n$. Observe that $P_n \circ f$ is an isometric map from (M, d_n) to $\mathbf{R}^n_\infty$. Therefore by Proposition 2.7(4), $\varphi_n|_{T_pM}$ equals the semi-norm induced from the norm $\|\cdot\|_\infty$ by the map $d_p(P_n \circ f)$, or, equivalently, the semi-norm induced from the semi-norm $\|\cdot\|_n$ on X^* by the map $d^w_p f$. Hence for every $v \in T_pM$,

$$\varphi(v) = \lim_{n\to\infty} \varphi_n(v) = \lim_{n\to\infty} \|d^w_p f(v)\|_n = \|d^w_p f(v)\|_{X^*},$$

and the theorem follows. □

4. Finsler volumes

4.1. **Examples.** Unlike in the Riemannian case, there are different (non-equivalent) "natural" definitions of volume used for Finsler manifolds. By a "natural" definition we mean one for which the volume depends monotonously on the metric and agrees with the Riemannian volume on the class of Riemannian manifolds. Recall that a Finsler manifold is a smooth manifold with a *continuous* Finsler structure $\Phi : TM \to \mathbf{R}$ (unlike weak Finsler structures from §2 that are assumed measurable only).

Definition 4.1. Let n be a fixed positive integer. We say that an *n-dimensional Finsler volume functional* is defined if, for every n-dimensional Finsler manifold (M, Φ) a Borel measure vol_Φ on M is associated to it so that the following conditions are satisfied.

1. The measure vol_Φ depends monotonously on Φ, that is, $\mathrm{vol}_{\Phi'} \le \mathrm{vol}_\Phi$ if $\Phi' \le \Phi$.
2. The measure is preserved by isometries, that is, if (M, Φ) and (M', Φ') are n-dimensional Finsler manifolds and $f : M \to M'$ is an injective smooth map such that $\Phi = \Phi' \circ df$, then $\mathrm{vol}_{\Phi'}|_{f(M)} = f_* \mathrm{vol}_\Phi$, where the star denotes the push-forward of the measure by f.
3. If $M = \mathbf{R}^n$ and Φ is the standard Euclidean metric, then vol_Φ is the standard Euclidean volume (i.e., the n-dimensional Lebesgue measure).

Example 4.2 (Busemann volume)**.** The n-dimensional Hausdorff measure obviously satisfies the above requirements. In Finslerian context, it is referred to as the *Busemann volume.* Busemann [12] proved that this is the only volume functional satisfying the following: the volume of a unit ball in an n-dimensional normed vector space equals ω_n (that is, does not depend on the norm). Despite being geometrically natural, this volume turn out to be inconvenient in many respects and does not have some properties expected from volume in differential and integral geometry (cf., for example, [2, 24]).

Example 4.3 (Holmes–Thompson volume)**.** Let (M, Φ) be an n-dimensional Finsler manifold. Consider the co-tangent bundle T^*M. In every fiber $T^*_x M$ there is a norm Φ^*_x dual to Φ_x. The union of these norms is a continuous function $\Phi^* : T^*M \to \mathbf{R}$. Consider the set

$$B^*(M, \Phi) = \{w \in T^*M : \Phi^*(w) \leq 1\}$$

(i.e. the union of the unit balls of the norms Φ^*_x over all $x \in M$).

A canonical $2n$-dimensional (symplectic) volume is defined on the co-tangent bundle, we denote it by V_{symp}. Now define a Finsler volume vol^s_Φ by the formula

$$\mathrm{vol}^s_\Phi(M) = \frac{1}{\omega_n} V_{\mathrm{symp}}(B^*(M, \Phi)).$$

More precisely, define a measure vol^s_Φ as the push-forward of the measure $\frac{1}{\omega_n} V_{\mathrm{symp}}$ by the projection map $B^*(M, \Phi) \to M$.

This volume functional is referred to as the *Holmes–Thompson volume* or the *symplectic Finsler volume*. It satisfies the requirements of Definition 4.1 since the construction is invariant and Φ^* depends on Φ anti-monotonously.

This volume was introduced by Holmes and Thompson [18] in 1979, but problems related to it were studied even before that. One of the reasons why it is convenient in differential geometry is the fact that the symplectic volume on the set $B^*(M, \Phi)$ corresponds to the Liouville measure on the unit tangent bundle and the latter is invariant under the geodesic flow.

Example 4.4 (inscribed Riemannian volume)**.** Let (M, Φ) be a Finsler manifold. For every measurable $U \subset M$ define

$$\mathrm{vol}^e_\Phi(U) = \inf\{\mathrm{vol}_g(U) : g \text{ is a Riemannian metric on } M,\ g \geq \Phi^2\}, \tag{4.5}$$

where vol_g denoted the Riemannian volume with respect to g. In the inequality $g \geq \Phi^2$, the Riemannian metric g is regarded as a function on TM that yields the square of the length of a tangent vector. We refer to the resulting measure vol^e_Φ as the *inscribed Riemannian volume* of the Finsler metric Φ. Obviously this volume satisfies the requirements from Definition 4.1. Moreover it is the maximal functional satisfying these requirements. This follows from the fact that the volume functional is uniquely defined within the class of Riemannian metrics (cf. Proposition 4.6).

There exists a unique Riemannian metric g at which the infimum in (4.5) is attained. In every fiber T_xM, the unit ball of this metric g is the maximal-volume ellipsoid contained in the unit ball of the norm Φ_x (the *John ellipsoid* [20]).

This volume is not convenient for the purposes of Finslerian geometry but it proves useful for Riemannian volume estimates requiring auxiliary Finslerian constructions. It particular, it is this definition of volume that yields the equality of filling volumes within Riemannian metrics and within Finsler metrics, proved below in Theorem 5.2.

It is easy to verify that the above three volume functionals are different. For example, consider the set $[-1, 1]^2$ in the normed vector space $\mathbf{R}^2_\infty$. Its Busemann volume equals π (since it is the unit ball of the norm), the Holmes–Thompson volume equals $\frac{8}{\pi}$ and the inscribed Riemannian volume equals 4.

Proposition 4.6. 1. *The n-dimensional Finsler volume functional is uniquely determined by its values on n-dimensional Banach spaces.*

2. *Any Finsler volume functional coincides with the Riemannian volume on the class of Riemannian manifolds.*

Proof. The second requirement of Definition 4.1 allows us to limit ourselves by Finsler structures in $\mathbf{R}^n$. Let Φ be a Finsler structure in $\mathbf{R}^n$. Identifying all the tangent spaces $T_x\mathbf{R}^n$ with $\mathbf{R}^n$, we may regard Φ as a family of norms $\{\Phi_x\}_{x\in\mathbf{R}^n}$ in $\mathbf{R}^n$ where $\Phi_x = \Phi|_{T_x\mathbf{R}^n}$. Fix an $x_0 \in \mathbf{R}^n$ and denote $\|\cdot\| = \Phi_{x_0}$. Let $\varepsilon > 0$. Then, by continuity of Φ, there is a neighborhood U of x_0 such that

$$(1-\varepsilon)\|\cdot\| \le \Phi_x \le (1+\varepsilon)\|\cdot\|$$

for all $x \in U$. For every $\lambda > 0$ the space $(\mathbf{R}^n, \lambda\|\cdot\|)$ is isometric to $(\mathbf{R}^n, \|\cdot\|)$ via a λ-homothety. This and the second requirement of Definition 4.1 implies that

$$\mathrm{vol}_{\lambda\|\cdot\|}(A) = \mathrm{vol}_{\|\cdot\|}(\lambda A)$$

for every measurable $A \subset \mathbf{R}^n$. The measure $\mathrm{vol}_{\|\cdot\|}$ is proportional to the Lebesgue measure since it is locally finite and translation-invariant, hence $\mathrm{vol}_{\|\cdot\|}(\lambda A) = \lambda^n \mathrm{vol}_{\|\cdot\|}(A)$. Thus $\mathrm{vol}_{\lambda\|\cdot\|} = \lambda^n \mathrm{vol}_{\|\cdot\|}$. Substituting $\lambda = 1 \pm \varepsilon$ and using the monotonicity of measure with respect to the metric, we obtain that

$$(1-\varepsilon)^n \mathrm{vol}_{\|\cdot\|} \le \mathrm{vol}_\Phi \le (1+\varepsilon)^n \mathrm{vol}_{\|\cdot\|}$$

within U. Sending ε to zero yields that the density of vol_Φ with respect to the Lebesgue measure at x_0 equals the density of $\mathrm{vol}_{\|\cdot\|}$. Thus the measure vol_Φ has a density whose value at every point $x \in \mathbf{R}^n$ is determined by the volume functional on the Banach space $(\mathbf{R}^n, \Phi_x)$. The first assertion of the proposition follows.

If the Finsler structure Φ is actually Riemannian, then all the norms Φ_x, $x \in \mathbf{R}^n$, are Euclidean ones. Since all n-dimensional Euclidean spaces are isometric, the second and third requirements from Definition 4.1 uniquely determine the density of vol_Φ and it equals the density of the standard Riemannian volume. The second assertion of the proposition follows. □

4.2. **Finsler volume densities.** Proposition 4.6 shows that, in order to define an n-dimensional Finsler volume functional it suffices to define it on n-dimensional Banach spaces. In this section we give the respective definitions following the approach from [3] and define a Finsler volume for arbitrary Lipschitz metrics.

Assume that we have fixed an n-dimensional Finsler volume functional $\Phi \mapsto \mathrm{vol}_\Phi$. In particular, for every n-dimensional normed vector space $(V, \|\cdot\|)$ there is an associated translation-invariant locally finite Borel measure $\mathrm{vol}_{\|\cdot\|}$ on V. All such measures on V are proportional to one another. They are in a natural 1-to-1 correspondence with norms on the nth exterior power $\Lambda^n V$, namely the norm of an n-vector $v_1 \wedge v_2 \wedge \cdots \wedge v_n$ equals the measure of the parallelotope spanned by the vectors $v_1, v_2, \dots, v_n$. We define the so defined norm of an n-vector σ by $\mathrm{vol}_{\|\cdot\|}(\sigma)$.

Definition 4.1 implies that this map has the following properties.

(4.7) If $(V, \|\cdot\|)$ and $(V', \|\cdot\|')$ are n-dimensional normed vector spaces and $f : V \to V'$ is a nonexpanding linear map, then $\mathrm{vol}_{\|\cdot\|}(\sigma) \le \mathrm{vol}_{\|\cdot\|'}(f_*(\sigma))$ for all $\sigma \in \Lambda^n V$ where the star denotes the natural action of an isomorphism on n-forms.

In particular, if $\|\cdot\|$ and $\|\cdot\|'$ are norms on V and $\|\cdot\| \le \|\cdot\|'$, then $\mathrm{vol}_{\|\cdot\|} \le \mathrm{vol}_{\|\cdot\|'}$.

(4.8) If $|\cdot|$ is a Euclidean norm then $\mathrm{vol}_{|\cdot|}$ is the corresponding Euclidean volume.

Definition 4.9. We say that an *n-dimensional Banach volume functional* is defined if, for every n-dimensional normed vector space $(V, \|\cdot\|)$ there is a norm $\mathrm{vol}_{\|\cdot\|}$ on $\Lambda^n V$ associated to it, so that the properties (4.7) and (4.8) are satisfied.

The property (4.7) implies that the volume is preserved by isometries. Therefore it suffices to define an n-dimensional Banach volume $\mathrm{vol}_{\|\cdot\|}$ only for the norms $\|\cdot\|$ on $\mathbf{R}^n$. For a fixed vector space V the volume functional $\|\cdot\| \mapsto \mathrm{vol}_{\|\cdot\|}$ defines a map from $\mathcal{N}(V)$ to $\mathcal{N}(\Lambda^n(V))$. (Recall that $\mathcal{N}(V)$ denotes the set of all norms on V. The space $\mathcal{N}(\Lambda^n(V))$ is one-dimensional and can be identified with $\mathbf{R}_+$, however such identification is not invariant.) This map can be naturally extended to the set of all semi-norms $\mathcal{N}_0(V)$, namely we define $\mathrm{vol}_{\|\cdot\|} = 0$ if the semi-norm $\|\cdot\|$ is degenerate (i.e., not a norm).

Lemma 4.10. *Let $\|\cdot\| \mapsto \mathrm{vol}_{\|\cdot\|}$ be an n-dimensional Banach volume functional and V an n-dimensional vector space. Then*

1. $\mathrm{vol}_{\|\cdot\|}$ *is a homogeneous of degree n function of $\|\cdot\|$, that is,* $\mathrm{vol}_{\lambda\|\cdot\|} = \lambda^n \mathrm{vol}_{\|\cdot\|}$ *for every norm $\|\cdot\|$ on V and every $\lambda > 0$.*
2. *The map $\|\cdot\| \mapsto \mathrm{vol}_{\|\cdot\|} : \mathcal{N}_0(V) \to \mathcal{N}_0(\Lambda^n V)$ is continuous.*

Proof. 1. Follows from the fact that the spaces $(V, \lambda\|\cdot\|)$ and $(V, \|\cdot\|)$ are isometric via a λ-homothety (cf. the proof of Proposition 4.6).

2. First let us prove the continuity on the set of norms $\mathcal{N}(V)$. Consider a sequence $\{\|\cdot\|_i\}_{i=1}^\infty$ of norms converging to a norm $\|\cdot\|$. The ratio $\frac{\|\cdot\|_i}{\|\cdot\|}$ uniformly converges to 1, that is,

$$(1-\varepsilon_i)\|\cdot\| \le \|\cdot\|_i \le (1+\varepsilon_i)\|\cdot\|$$

for some sequence $\varepsilon_i \to 0$. Hence by homogeneity,

$$(1-\varepsilon_i)^n \mathrm{vol}_{\|\cdot\|} \le \mathrm{vol}_{\|\cdot\|_i} \le (1+\varepsilon_i)^n \mathrm{vol}_{\|\cdot\|},$$

therefore $\mathrm{vol}_{\|\cdot\|_i} \to \mathrm{vol}_{\|\cdot\|}$ and the continuity on $\mathcal{N}(V)$ follows.

Now let $\|\cdot\|$ be a degenerate semi-norm, $\{\|\cdot\|_i\}_{i=1}^\infty$ a sequence of semi-norms converging to $\|\cdot\|$. Let V_0 be a null subspace of $\|\cdot\|$, i.e. $V_0 = \{v \in V : \|v\| = 0\}$, V_1 a subspace complementary to V_0. We may assume that $V_0 = \mathbf{R}^k$ and $V_1 = \mathbf{R}^{n-k}$ where $1 \le k \le n$, $V = \mathbf{R}^k \times \mathbf{R}^{n-k} = \mathbf{R}^n$. For every $\varepsilon > 0$ consider a norm $\|\cdot\|_{(\varepsilon)}$ on $\mathbf{R}^n$, given by

$$\|(x,y)\|_{(\varepsilon)} = \varepsilon|x| + 2\|y\|, \quad x \in \mathbf{R}^k, y \in \mathbf{R}^{n-k},$$

where $|\cdot|$ is the standard Euclidean norm. Since the restriction of $\|\cdot\|$ on V_1 is a norm, the ration $\frac{\|\cdot\|_i}{\|\cdot\|}$ uniformly converges to 1 on $V_1 \setminus \{0\}$, hence $\|\cdot\|_i \le 2\|\cdot\|$ on V_1 for all sufficiently large i. We may assume that this inequality is satisfied for all i. Define

$$\varepsilon_i = \max\{\|v\|_i : v \in V_0, |v| = 1\}.$$

Then $\varepsilon_i \to 0$ since the restrictions of the semi-norms $\|\cdot\|_i$ on V_0 converge to zero. The inequalities $\|\cdot\|_i \le 2\|\cdot\|$ on V_1 and $\|\cdot\|_i \le \varepsilon_i|\cdot|$ on V_0 imply that $\|\cdot\|_i \le \|\cdot\|_{(\varepsilon_i)}$ everywhere on V.

Observe that all norms of the form $\|\cdot\|_{(\varepsilon)}$ are isometric; an isometry between $\|\cdot\|_{(1)}$ and $\|\cdot\|_{(\varepsilon)}$ is given by the map $(x,y) \mapsto (\varepsilon x, y)$, $x \in \mathbf{R}^k$, $y \in \mathbf{R}^{n-k}$. Therefore $\mathrm{vol}_{\|\cdot\|_{(\varepsilon)}} = \varepsilon^k \mathrm{vol}_{(1)}$ where $\mathrm{vol}_{(1)} = \mathrm{vol}_{\|\cdot\|_{(1)}}$. This and the monotonicity of volume with respect to the norm imply that $\mathrm{vol}_{\|\cdot\|_i} \le \varepsilon_i^k \mathrm{vol}_{(1)}$. Therefore $\mathrm{vol}_{\|\cdot\|_i} \to 0 = \mathrm{vol}_{\|\cdot\|}$. □

Definition 4.11. By a *density* on an n-dimensional smooth manifold M we mean a nonnegative measurable function $\nu : \Lambda^n TM \to \mathbf{R}$ whose restriction to each fiber is a semi-norm, i.e. is symmetric and positive definite. ($\Lambda^n TM$ is a one-dimensional vector bundle over M whose fiber over a point $x \in M$ is the nth exterior power $\Lambda^n T_x M$.)

One can integrate such a structure over a manifold the same way as a differential form, furthermore (unlike in the case of a differential form) the integral does not depend on manifold's orientation. Thus the function ν defines a measure μ on M. We indicate this relation between μ and ν by the formula $\mu = \int \nu$.

Definition 4.12. Let an n-dimensional Banach volume functional $\|\cdot\| \mapsto \mathrm{vol}_{\|\cdot\|}$ be fixed. Let d be a Lipschitz metric on an n-dimensional smooth manifold M and $\varphi = \varphi_d$ its tangent Finsler structure. Consider a density $\nu = \nu_d$ on M whose value at an n-vector $\sigma \in \Lambda^n T_x M$ equals $\mathrm{vol}_{\varphi|_{T_x M}}(\sigma)$ if $\varphi|_{T_x M}$ is a semi-norm, and 0 otherwise. We refer to the measure $\mathrm{vol}_d = \int \nu_d$ as the *Finsler volume* of d.

Proposition 4.13. *The measure* vol_d *from Definition* 4.12 *is correctly defined and has the following properties.*

1. *Homogeneity*: $\mathrm{vol}_{\lambda d} = \lambda^n \mathrm{vol}_d$ *for every Lipschitz metric* d *and every* $\lambda > 0$.
2. *If* (M, d) *and* (M', d') *are manifolds with Lipschitz metrics and* $f : (M, d) \to (M', d')$ *is a nonexpanding map, then* $\mathrm{vol}_{d'}(f(A)) \le \mathrm{vol}_d(A)$ *for every measurable* $A \subset M$.

   In particular, if $M' = M$ *and* $d' \le d$ *then* $\mathrm{vol}_{d'} \le \mathrm{vol}_d$.
3. *If* $\{d_i\}$ *is a non-decreasing sequence of metrics on* M *point-wise converging to a Lipschitz metric* d, *then* $\mathrm{vol}_{d_i}(A) \to \mathrm{vol}(A)$ *for every measurable* $A \subset M$.

Proof. In order to verify that the definition is correct, one has to prove that the density ν_d in Definition 4.12 is measurable. This follows from the facts that the function $\varphi_d : TM \to \mathbf{R}$ is measurable and the volume continuously depends on the semi-norm (cf. Lemma 4.10(2)).

The homogeneity follows from the respective property of a Banach volume (cf. Lemma 4.10(1)).

To prove the second assertion, it suffices to verify that, at every point $x \in M$ where the map f is differentiable, its Jacobian is no greater than 1. Here by Jacobian we mean the ratio of densities

$$\frac{(d_x f)_* \nu_d(x)}{\nu_{d'}(f(x))}.$$

This follows from the property (4.7) of a Banach volume.

The third assertion follows from the facts that the tangent Finsler structures converge almost everywhere (cf. Proposition 2.9) and and the volume continuously depends on the semi-norm (cf. Lemma 4.10(2)). □

Corollary 4.14. *Definition* 4.12 *restricted to the class of Finsler manifolds defines a Finsler volume functional in the sense of Definition* 4.1. *Conversely, every Finsler volume functional can be obtained this way.*

Proof. The first statement follows from Proposition 4.13(2). The converse one follows from Proposition 4.6. □

Examples. Let Φ be a Finsler metric in a region $U \subset \mathbf{R}^n$. Identifying $T_xU \simeq \mathbf{R}^n$ for all $x \in U$ we may regard Φ as a family of norms $\{\Phi_x\}_{x\in U}$ on $\mathbf{R}^n$. Denote by B_x the unit ball of Φ_x. Every Finsler volume has a continuous density ρ with respect to the Lebesgue measure m_n, it is related to the above coordinate-free density by

$$\rho(x) = \nu_{\Phi_x}(e_1 \wedge e_2 \wedge \cdots \wedge e_n),$$

where $(e_1, e_2, \dots, e_n)$ is the standard basis of $\mathbf{R}^n$.

In the case of Busemann volume the density equals

$$\rho(x) = \frac{\omega_n}{m_n(B_x)},$$

since the volume of the norm's unit ball equals ω_n.

For the Holmes–Thompson volume the density equals

$$\rho(x) = \frac{m_n(B_x^\circ)}{\omega_n},$$

where B_x° is the polar set of B_x. This follows from the definition of the Holmes–Thompson volume and the fact that B_x° is the unit ball of the dual norm $\|\cdot\|_x^*$ (modulo the standard identification $(\mathbf{R}^n)^* \simeq \mathbf{R}^n$).

For the inscribed Riemannian volume the density equals

$$\rho(x) = \frac{\omega_n}{m_n(E_x)},$$

where E_x is the John ellipsoid of the norm Φ_x.

5. Filling volumes

Let an n-dimensional Finsler volume functional be fixed. Then, by Definition 4.12, for every n-dimensional manifold M with a Lipschitz metric d there is an associated measure vol_d on M.

Definition 5.1. Let $\mathfrak{M}$ be a class of compact n-dimensional manifolds with Lipschitz metrics. Let S be an $(n-1)$-dimensional manifold and d_0 a Lipschitz metric on S. By the *filling volume* of d_0 within the class $\mathfrak{M}$ (with respect to the given volume functional) we mean the quantity

$$\inf\{\mathrm{vol}_d(M) : (M,d) \in \mathfrak{M}, \partial M = S, d|_{S\times S} \geq d_0\}.$$

We say that a manifold $(M,d) \in \mathfrak{M}$ is a *minimal filling* within $\mathfrak{M}$ if $\mathrm{vol}_d(M)$ equals the filling volume of the metric $d|_{\partial M\times\partial M}$ within $\mathfrak{M}$.

The definition of the filling volume given in the introduction corresponds to the filling volume within the class of Riemannian manifolds.

Theorem 5.2. *Let M be an n-dimensional manifold, $S = \partial M$, and let d_0 be a Lipschitz metric on S. Then*

1. *the filling volume of d_0 within the class of all Lipschitz metrics on M equals its filling volume within the class of smooth strictly convex Finsler metrics*;
2. *if the volume functional is the inscribed Riemannian volume, then the filling volume of d_0 within the class of all Lipschitz metrics on M equals its filling volume within the class of Riemannian metrics.*

Proof. 1. Let d be a Lipschitz metric on M such that $d|_{S\times S} \geq d_0$. It suffices to prove that, for every $\varepsilon > 0$ there is a smooth strictly convex Finsler metric $\tilde{d}$ on M such that $\tilde{d}|_{S\times S} \geq d|_{S\times S}$ and $\mathrm{vol}_{\tilde{d}}(M) \leq \mathrm{vol}_d(M) + \varepsilon$. We may assume that d is a length metric since replacing a metric by an induced length one does not decrease the distances and preserves the volume (by Proposition 2.12).

Attach to M a "collar" $M^+ = S \times [0,1]$ by identifying the sets $S \subset M$ and $S \times \{0\} \subset S \times [0,1]$. Construct a smooth Riemannian metric d^+ on M^+ such that

(1) $\mathrm{vol}_{d^+}(M^+) < \varepsilon$;

(2) $d^+((x,t)(x',t')) > d(x,x')$ for all $x, x' \in S$, $t, t' \in [0,1]$ such that $(x,t) \neq (x',t')$.

For instance, one can define d^+ as the product metric of $(S, d_0^+) \times [0,\theta]$, where d_0^+ is a Riemannian metric on S, $d_0^+ \geq 2d|_{S\times S}$, $0 < \theta < \varepsilon/\mathrm{vol}(S, d_0^+)$.

The enlarged manifold $M' = M \cup M^+$ is equipped by a metric d' obtained by gluing the metrics d on M and d^+ on M^+. The gluing procedure is described in [11, §3.1]; here we use only the fact that the result is a length metric which is isometric to the metrics being glued in the interiors of the two manifolds. Observe that

$$d'((x,t),(x',t')) \geq d(x,y) \tag{5.3}$$

for all $x, x' \in S$, $t, t' \in [0,1]$, since this inequality holds for both metrics before gluing.

Let $\delta > 0$ be such that $d^+((x,1),(x',\frac{1}{2})) \geq d(x,x') + \delta$ for all $x, x' \in S$. For the metric d', construct a sequence $\{\varphi_i\}$ of smooth Finsler structures as in Proposition 2.14, in addition, choose φ_i so that they agree with the Riemannian structure of d^+ on the set $S \times [\frac{1}{2}, 1]$. (cf. Remark 2.15). Let $d_i = d_{\varphi_i}$. By properties 1 and 3 from Proposition 2.14, the densities of Finsler volumes determined by the metrics d_i are uniformly bounded and point-wise converge to the density of $\mathrm{vol}_{d'}$. Therefore $\mathrm{vol}_{d_i}(M') \to \mathrm{vol}_{d'}(M')$, hence for all sufficiently large i one has $\mathrm{vol}_{d_i}(M') < \mathrm{vol}_d(M) + \varepsilon$.

For all sufficiently large i one has $d_i \geq d' - \delta$ everywhere on M' (cf. property 2 from Proposition 2.14). It follows that $d_i((x,1),(y,1)) \geq d(x,y)$ for all $x, y \in S$. Indeed, let γ be a shortest curve of the metric d_i connecting $(x,1)$ and $(y,1)$. If this curve lies entirely in the set $S \times [\frac{1}{2}, 1]$, then its length is no less than $d^+((x,1),(y,1)) \geq d(x,y)$. Otherwise let $(x', \frac{1}{2})$ and $(y', \frac{1}{2})$ be the points of intersection of γ with the set $S \times \{\frac{1}{2}\}$, nearest to x and y respectively. Then

$$\begin{aligned}
d_i((x,1),(y,1)) &= d^+((x,1),(x',\tfrac{1}{2})) + d_i((x',\tfrac{1}{2}),(y',\tfrac{1}{2})) + d^+((y,1),(y',\tfrac{1}{2})) \\
&\geq (d(x,x') + \delta) + (d'((x',\tfrac{1}{2}),(y',\tfrac{1}{2})) - \delta) + (d(y,y') + \delta) \\
&\geq (d(x,x') + \delta) + (d(x',y') - \delta) + (d(y,y') + \delta) > d(x,y).
\end{aligned}$$

Here the second identity follows from (5.3) and the last one from the triangle inequality for d.

Let $f : M \to M'$ be a diffeomorphism mapping $S \subset M$ to $S \times \{1\} \subset M'$ in the natural way. Let $\tilde{d}_i = f^* d_i$ be the metric on M corresponding to d_i under this diffeomorphism. Then for a sufficiently large i the metric $\tilde{d} = \tilde{d}_i$ satisfies the conditions $\tilde{d}|_{S\times S} \geq d|_{S\times S}$ and $\mathrm{vol}_{\tilde{d}}(M) < \mathrm{vol}_d(M) + \varepsilon$, hence the result.

2. For each point $x \in M$ consider the norm on T_xM defined as the restriction of the Finsler structure of the metric $\tilde{d}$ constructed above. Let E_x be the John ellipsoid of this norm (cf. 4.4). Let d_r be a Riemannian metric on M generated by the family of ellipsoids $\{E_x\}_{x\in M}$. Then $d_r \geq \tilde{d}$, whence $d_r|_{S\times S} \geq d|_{S\times S}$. Furthermore, since

the chosen Finsler volume definition is the inscribed Riemannian volume, one has $\mathrm{vol}_{d_r}(M) = \mathrm{vol}_{\tilde{d}}(M) < \mathrm{vol}_d(M) + \varepsilon$. Due to uniqueness of the John ellipsoid, the Riemannian structure of d_r is continuous; it can be smoothened so that the inequalities $d_r|_{S\times S} \ge d|_{S\times S}$ and $\mathrm{vol}_{d_r}(M) < \mathrm{vol}_d(M) + \varepsilon$ persist. Thus the filling volume is realized by smooth Riemannian metrics, hence the result. □

Definition 5.4. Let M be a smooth manifold, X a Banach space, $f : M \to X$ a Lipschitz map. Let a Finsler volume functional be fixed. By the *area* of a map (or surface) f we mean the quantity $\mathrm{area}(f) = \mathrm{vol}_d(M)$ where d is a metric on M given by $d(x, y) = \|f(x) - f(y)\|$, vol_d is the measure defined by this Lipschitz metric and the chosen Finsler volume functional (cf. Definition 4.12).

If the Banach space X is dual to a separable one (for example, has the form $L^\infty(\mu)$ where μ is a finite measure), then the area of a map can be expressed in terms of its weak differential in the standard way. Namely the following holds.

Proposition 5.5. *Let X be a Banach space which is dual to a separable one, M a smooth manifold, $f : M \to X$ a Lipschitz map. Then* $\mathrm{area}(f) = \int_M \nu_f$ *where ν_f is a density on M whose value at an n-vector $\sigma \in T_xM$ equals*

$$\nu_f(\sigma) = \mathrm{vol}_{(d^w_x f)^*\|\cdot\|_X}$$

if f is weakly differentiable at x. Here the star denotes the pull-back of a semi-norm by a linear map, that is, $(d^w_x f)^*\|\cdot\|_X$ is a semi-norm $\|\cdot\|$ on T_xM, given by $\|v\| = \|d^w_x f(v)\|_X$.

Proof. Follows from the definition of area and Theorem 3.7. □

Theorem 5.6. *For every Finsler volume functional the following holds. Let $X = L^\infty(\mu)$ where μ is a finite measure on an arbitrary set, M a smooth manifold, $S = \partial M$.*

1. *Let d_0 be a Lipschitz metric on S and $f : (S, d_0) \to X$ an isometric map. Then the filling volume of (S, d_0) within the class of Lipschitz* (*or, equivalently, smooth strictly convex Finsler*) *metrics on M equals*

$$\inf\{\mathrm{area}(F) : F : M \to X \text{ is Lipshitz, } F|_S = f\}.$$

2. *Let d be a Lipschitz metric on M. Then for every isometric map $F : (M, d) \to X$ the following holds: (M, d) is a minimal filling within the class of all manifolds with Lipschitz metrics if and only if F realizes a minimum of the area among all Lipschitz surfaces in X having the same boundary.*

Proof. 1. Let $F : M \to X$ be a Lipschitz map, $F|_S = f$. Consider a Lipschitz metric d on M induced by F (i.e. given by $d(x, y) = \|F(x) - F(y)\|$). By the definition of area, $\mathrm{area}(F) = \mathrm{vol}(M, d)$. Since the map $f = F|_S$ is isometric with respect to d_0, one has $d|_{S\times S} = d_0$. Therefore the filling volume of (S, d_0) is no greater than $\mathrm{area}(F)$. Since F is arbitrary, the filling volume is no greater than the infimum of such areas.

Conversely, if d is a Lipschitz metric on M such that $d|_{S\times S} \ge d_0$, then by Proposition 1.6 there exists a nonexpanding map $F : (M, d) \to X$. Since F is nonexpanding, the metric on M induced by it is no greater than d, therefore $\mathrm{area}(F) \le \mathrm{vol}(M, d)$. Taking the infimum over (M, d) yields the desired result.

2. Trivially follows from the first assertion of the theorem. □

With the inscribed Riemannian volume used as the definition of volume, Theorem 5.6 and the second assertion of Theorem 5.2 yield

Corollary 5.7. *Let $X = L^\infty(\mu)$ where μ is a finite measure on an arbitrary set, M a smooth manifold, $S = \partial M$. Define the area of a surface in X as the area defined by the inscribed Riemannian volume functional.*

1. *Let d_0 be a Lipschitz metric on S, $f : (S, d_0) \to X$ an isometric map. Then the filling volume of (S, d_0) within the class of Riemannian metrics on M equals*

$$\inf\{\mathrm{area}(F) : F : M \to X \text{ is Lipschitz, } F|_S = f\}.$$

2. *Let d be a Riemannian metric on M. Then for every isometric map $F : (M, d) \to X$ the following holds: (M, d) is a minimal filling within the class of all Riemannian manifolds if and only if F realizes the infimum of the area among all Lipschitz surfaces in X having the same boundary.*

6. Semi-ellipticity

Definition 6.1. We say that an n-dimensional Finsler volume functional is *semi-elliptic over* $\mathbf{Z}$ if the associated n-dimensional area functional in every finite-dimensional Banach space V satisfies the following: the image $L(D^n)$ of the standard disc $D^n \subset \mathbf{R}^n$ under any injective linear map $L : \mathbf{R}^n \to V$, minimizes the area among all oriented Lipschitz surfaces in V having the same boundary.

We say that an n-dimensional Finsler volume functional is *topologically semi-elliptic* if a similar minimality holds in the class of all Lipschitz surfaces parameterized by the disc D^n, and *semi-elliptic over* $\mathbf{R}$ if it holds in the class of all Lipschitz chains with real coefficients.

In other words, a volume functional is semi-elliptic if the area integrand defined by it is semi-elliptic in any finite-dimensional Banach space. It is plausible that in this case the area integrand is elliptic in spaces with strictly convex norms. Ellipticity of area integrands plays an important role in the theory of minimal surfaces (cf. [1; 16, Chapter 5]). In [8] relations between semi-ellipticity and filling volumes, as well as asymptotic volumes of periodic metrics, are shown.

Obviously semi-ellipticity over $\mathbf{R}$ implies semi-ellipticity over $\mathbf{Z}$. The converse is not true, a counter-example is constructed in [9]. Semi-ellipticity over $\mathbf{Z}$ obviously implies topological semi-ellipticity and for all $n \geq 3$ these two properties are equivalent [17, App. 2, Proposition A′].

Semi-ellipticity of the Busemann and Holmes–Thompson volumes is a widely open question. In co-dimension 1 both volumes are semi-elliptic (this is equivalent to convexity of section bodies and projection bodies, respectively, cf. [13, 23]). The two-dimensional Holmes–Thompson volume is topologically semi-elliptic but is not semi-elliptic over $\mathbf{R}$ [8]. The question whether the Busemann volume is semi-elliptic in co-dimensions greater than 1, is completely open by now.

The purpose of this section is to prove the following theorem.

Theorem 6.2. *Let X be a normed vector space and $Y \subset X$ an n-dimensional linear subspace. Then there exists a linear map $P : X \to Y$ such that*

1. *P is a projector onto Y, that is, $P|_Y = id_Y$.*
2. *For every Lipshitz map $f : M \to X$ where M is a smooth n-dimensional manifold, one has* $\mathrm{area}(P \circ f) \leq \mathrm{area}(f)$ *where* area *is the area defined by the inscribed Riemannian volume functional.*

Consequently, the inscribed Riemannian volume functional is semi-elliptic over $\mathbf{R}$.

Proof. The proof is based on ideas from [7]. First assume that the space X is dual to a separable one. Then by Proposition 5.5 it suffices to construct a projector which does not increase the n-dimensional area on n-dimensional linear subspaces of X. We need the following lemma.

Lemma 6.3 (cf. [7, Lemma 1.3]). *Let $(V, \|\cdot\|)$ be an n-dimensional normed vector space. Then there exists a finite collection $\{L_i\}_{i=1}^N$ of nonexpanding linear functions $L_i : (V, \|\cdot\|) \to \mathbf{R}$ and a collection $\{\lambda_i\}_{i=1}^N$ of positive numbers such that $\sum \lambda_i = n$ and $\sum \lambda_i L_i^2(v) \geq \|v\|$ for all $v \in V$.*

Applying this lemma to a subspace Y regarded with the restriction of the norm $\|\cdot\|_X$ we get a collection of nonexpanding linear maps $L_i : Y \to \mathbf{R}$ and coefficients $\lambda_i \geq 0$, $i = 1, \dots, N$, such that $\sum \lambda_i = n$ and $\sum \lambda_i L_i^2 \geq \|\cdot\|_X^2$ everywhere on Y. By the Hahn–Banach theorem the functions L_i have nonexpanding linear extensions $\tilde{L}_i : X \to \mathbf{R}$. Consider a quadratic form Q on X given by $Q(x) = \sum \lambda_i \tilde{L}_i^2$. Denote by area_Q the n-dimensional area with respect to the Euclidean semi-norm $\sqrt{Q}$.

Let us prove that $\mathrm{area}_Q \leq \mathrm{area}$ on every n-dimensional linear subspace $W \subset X$. Recall that here by the area we mean the inscribed Riemannian volume of the induced metric. Let $|\cdot|_W$ be the Euclidean norm on W whose unit ball is the John ellipsoid of the restriction of the norm $\|\cdot\|_X$ to W. Then on W one has $|\cdot|_W \geq \|\cdot\|_X$ and $\mathrm{vol}_{|\cdot|_W} = \mathrm{vol}_{\|\cdot\|_X}$ by definition. The inequality $|\cdot|_W \geq \|\cdot\|_X$ implies that the restrictions $\tilde{L}_i|_W$ are nonexpanding with respect to $|\cdot|_W$, therefore

$$\mathrm{trace}_{|\cdot|_W}(L_i^2|_W) \leq 1,$$

whence

$$\mathrm{trace}_{|\cdot|_W}(Q|_W) \leq \sum \lambda_i = n.$$

By the Cauchy inequality this implies that

$$\det_{|\cdot|_W}(Q|_W) \leq \left(\tfrac{1}{n}\mathrm{trace}_{|\cdot|_W}(Q|_W)\right)^n \leq 1.$$

(Here $\mathrm{trace}_{|\cdot|_W}$ and $\det_{|\cdot|_W}$ denote respectively the trace and the determinant of a quadratic form in the Euclidean space $(W, |\cdot|_W)$.) Therefore the volume form on W defined by the quadratic form $Q|_W$ is no greater that the volume form defined by the norm $|\cdot|_W$. Thus $\mathrm{area}_Q \leq \mathrm{area}$ on W.

The inequality $Q(x) \geq \|x\|_X^2$ for all $x \in Y$ implies that, on the subspace Y, one has $\mathrm{area}_Q \geq \mathrm{area}$, whence $\mathrm{area}_Q = \mathrm{area}$ on Y.

Let $P : X \to Y$ be the orthogonal projection of X onto Y with respect to the quadratic form Q. Since an orthogonal projection does not increase lengths of vectors, it does not increase the area with respect to Q, hence

$$\mathrm{area}(P(A)) = \mathrm{area}_Q(P(A)) \leq \mathrm{area}_Q(A) \leq \mathrm{area}(A)$$

for every measurable A contained in an n-dimensional linear subspace $W \subset X$. Therefore the map P is a desired one. Thus we have proved the theorem in the case when the space X is dual to a separable one.

Now consider the general case. Let $S = \{s_i\}_{i=1}^\infty$ be a countable dense subset of the unit sphere of Y. For each i, by the Hahn–Banach theorem there exists a nonexpanding linear map $f_i : X \to \mathbf{R}$ such that $f_i(s_i) = 1$. Define a linear map $f : X \to \ell_\infty$ by

$$f(x) = (f_1(x), f_2(x), \dots).$$

This map is nonexpanding since so are all the maps f_i. Futhermore the restriction $f|_Y$ is isometric. Indeed, for every $x \in Y$ such that $\|x\|_X = 1$ one has

$$\|f(x)\|_\infty = \sup_i f_i(x) \geq \sup_i (f_i(s_i) - \|x - s_i\|_X) = \sup_i (1 - \|x - s_i\|_X) = 1$$

since f is nonexpanding and S is dense in the unit sphere of Y. The space ℓ_∞ is dual to a separable one, hence by the already proven case there exists a projector $P_0 : \ell_\infty \to f(Y)$ which does not increase n-dimensional areas. Then the map $P = (f|_Y)^{-1} \circ P_0 \circ f$ is a desired one. □

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