$K(\mathbb{Z}, 2)$ out of circular permutations

Nikolai Mnëv*
mnev@pdmi.ras.ru

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Abstract

We briefly discuss **SC**, a simplicial homotopy model of $K(\mathbb{Z}, 2)$ constructed from circular permutations. In any dimension, the number of simplices in the model is finite. The complex **SC** naturally manifests as a simplicial set representing “minimally” triangulated circle bundles over simplicial bases. On the other hand, the homotopy $|\mathbf{SC}| \approx B(U(1)) \approx K(\mathbb{Z}, 2)$ appears to be a canonical fact from the foundations of crossed simplicial groups theory.

This micro-note essentially continues the note [Mnë20]. In that note ([Mnë20, §§ 3.6, 3.7]), we identify circular permutations of $k + 1$ ordered elements with ”minimal” semi-simplicial triangulations of trivial circle bundles over ordered base k-simplices. Any semi-simplicial triangulation of a circle bundle is non-canonically combinatorially concordant to a minimal triangulation (i.e., having minimal triangulations over all the simplices of the same base complex), and the simplicial set **SC** of circular permutations naturally represents

*PDMI RAS; Chebyshev Laboratory, SPbSU. Research is supported by the Russian Science Foundation grant 19-71-30002.

minimally triangulated circle bundles over semi-simplicial complexes. Such triangulations functorially (by Kan's second derived subdivision Sd_2) have a structure of a classical simplicial PL triangulation. However, the *minimal* triangulations exist only in the semi-simplicial category.

Our simple observation is that the complex $\mathbf{SC}$ appears canonically as the simplicial *right* coset complex of the cyclic crossed simplicial subgroup $\mathbf{C}$ in the symmetric crossed simplicial group $\mathbf{S}$, providing the sequence

$$\mathbf{C} \to \mathbf{S} \xrightarrow{\circlearrowleft} \mathbf{SC} \tag{1}$$

The natural conjecture is that $\mathbf{SC}$ is a homotopy model of $\mathrm{K}(\mathbb{Z}, 2)$. To the author's limited knowledge, $\mathbf{SC}$ is the first simplicial model of $\mathrm{K}(\mathbb{Z}, 2)$ having a finite number of simplices in every dimension. This fact likely makes the simplicial set $\mathbf{SC}$ interesting. The situation is a non-direct relative of the well-known subject of triangulating $\mathbb{C}\mathrm{P}^n$. See [MY91, AM91] and the new results [DS24]. There are interesting computer experiments [Ser10]. The connections of these achievements with our construction have to be investigated. Probably the connection is through minimal triangulation of the tautological Hopf bundle $\mathrm{U}(1) \to \mathrm{S}^{2n+1} \to \mathbb{C}\mathrm{P}^n$. The fact

$$|\mathbf{SC}| \approx \mathrm{K}(\mathbb{Z}, 2)$$

can be deduced from the very basics of crossed simplicial groups theory ([FT87, Kra87, FL91, Lod98]).

Theorem 1.

$$|\mathbf{SC}| \approx \mathrm{K}(\mathbb{Z}, 2).$$

Proof. The miracle of geometric realization of crossed simplicial groups makes $|\mathbf{C}| = \mathrm{U}(1)$, $|\mathbf{S}|$ a contractible topological group (see [FL91, 1.5 Example 6]), and $|\mathbf{SC}|$ a right coset space of the subgroup

$U(1)$ in $|\mathbf{S}|$. We are in the situation of a Borel construction of the classifying space. The geometric realization of the sequence (1) becomes a principal fibration. Therefore $|\mathbf{SC}| \approx BU(1) \approx K(\mathbb{Z}, 2)$. $\square$

We deduce that the sequence (1) is a combinatorial form of the Hopf fibration. One may verify that minimally triangulated circle bundles over simplices from [Mnë20] correspond to pullbacks of the simplicial map $\mathbf{S} \xrightarrow{\circlearrowleft} \mathbf{SC}$ over base simplices. Those combinatorics is the combinatorics of the twisted shuffle product $|\mathbf{C}| \times_t \Delta^k$ of $|\mathbf{C}|$ – the circle composed of one point and one 1-simplex – and the base simplex (see [DHK85], [Jon87]).

References

[AM91] Pierre Arnoux and Alexis Marin. The Kühnel triangulation of the complex projective plane from the view point of complex crystallography. II. *Mem. Fac. Sci., Kyushu Univ., Ser. A*, 45(2):167–244, 1991.

[DHK85] W.G. Dwyer, M.J. Hopkins, and D.M. Kan. The homotopy theory of cyclic sets. *Trans. Am. Math. Soc.*, 291:281–289, 1985.

[DS24] Basudeb Datta and Jonathan Spreer. Simplicial cell decompositions of $\mathbb{CP}^n$, 2024.

[FL91] Zbigniew Fiedorowicz and Jean-Louis Loday. Crossed simplicial groups and their associated homology. *Trans. Am. Math. Soc.*, 326(1):57–87, 1991.

[FT87] B. L. Feigin and B. L. Tsygan. Additive K-theory. K-theory, arithmetic and geometry, Semin., Moscow Univ. 1984-86, Lect. Notes Math. 1289, 67-209 (1987)., 1987.

[Jon87] John D. S. Jones. Cyclic homology and equivariant homology. *Invent. Math.*, 87:403–423, 1987.

[Kra87] R. Krasauskas. Skew-simplicial groups. *Lith. Math. J.*, 27(1):47–54, 1987.

[Lod98] Jean-Louis Loday. *Cyclic homology. 2nd ed.* Berlin: Springer, 2nd ed. edition, 1998.

[Mnë20] N. Mnëv. Minimal triangulations of circle bundles, circular permutations, and the binary Chern cocycle. *J. Math. Sci., New York*, 247(5):696–710, 2020. arXiv:1908.04029 [math.GT]

[MY91] Bernard Morin and Masaaki Yoshida. The Kühnel triangulation of the complex projective plane from the view point of complex crystallography. I. *Mem. Fac. Sci., Kyushu Univ., Ser. A*, 45(1):55–142, 1991.

[Ser10] Francis Sergeraert. Triangulations of complex projective spaces. In *Contribuciones científicas en honor de Mirian Andrés Gómez.*, pages 507–519. Logroño: Universidad de La Rioja, Servicio de Publicaciones, 2010.