$K(\mathbb{Z}, 2)$ out of circular permutations

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August 17, 2024

Abstract

We discuss $\mathbf{SC}_*$, a simplicial homotopy model of $K(\mathbb{Z}, 2)$ constructed from circular permutations. In any dimension, the number of simplices in the model is finite. The complex $\mathbf{SC}_*$ naturally manifests as a simplicial set representing "minimally" triangulated circle bundles over simplicial bases. On the other hand, existence of the homotopy equivalence $|\mathbf{SC}_*| \approx B(U(1)) \approx K(\mathbb{Z}, 2)$ appears to be a canonical fact from the foundations of the theory of crossed simplicial groups.

Contents

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Research is supported by the Russian Science Foundation grant 19-71-30002.

1. Introduction

This note essentially continues the discussion from [Mnë20]. In that note ([Mnë20, §§ 3.6, 3.7]), we identify circular permutations of $n+1$ ordered elements with "minimal" semi-simplicial triangulations of trivial circle bundles over ordered base n-simplices. Any semi-simplicial triangulation of a circle bundle is non-canonically combinatorially concordant to a minimal triangulation (i.e., having minimal triangulations over all the simplices of the same base complex), and the simplicial set $\mathbf{SC}_*$ of circular permutations naturally represents minimally triangulated circle bundles over semi-simplicial complexes. Such triangulations functorially (via Kan's second derived subdivision Sd_2) have the structure of a classical simplicial PL triangulation. However, the *minimal* triangulations exist only in the semi-simplicial category. The value (if it exists) of the above constructions lies in their very discrete form of the Weil-Kostant correspondence for triangulated circle bundles ([Mnë20, Theorem 1]). Namely, a circle bundle over a given simplicial complex B can be (semi-simplicially) triangulated with base B if and only if its Chern class can be represented by a simplicial 2-cocycle of B having values 0 or 1. The simplicial set of circular permutations is canonically a quotient of the simplicial set of all permutations $\mathbf{S}_*$ by a simplicial equivalence relation induced by *right* actions of cyclic subgroups. The simplicial set of all permutations $\mathbf{S}_*$ has the structure of a *symmetric crossed simplicial group*. We have the simplicial map:

$$\mathbf{S}_* \xrightarrow{\circlearrowleft} \mathbf{SC}_* \tag{1}$$

We aim to prove the following:

Theorem 1.

$$|\mathbf{SC}_*| \approx \mathsf{K}(\mathbb{Z}, 2).$$

To the author's limited knowledge, $\mathbf{SC}_*$ is the first simplicial model of $\mathsf{K}(\mathbb{Z}, 2)$ with a finite number of simplices in every dimension. This fact likely makes the simplicial set $\mathbf{SC}_*$ interesting. The situation is somewhat related to the well-known topic of triangulating $\mathbb{C}\mathrm{P}^n$. See [MY91, AM91] and the new results in [DS24]. There are also interesting computer experiments in [Ser10]. The connections between these results and our construction need further investigation. The connection is probably through the minimal triangulation of the tautological Hopf bundle $\mathrm{U}(1) \to \mathrm{S}^{2n+1} \to \mathbb{C}\mathrm{P}^n$.

Crossed simplicial group theory originated from pioneering works on cyclic homology [Tsy83] and [Con83]. The idea behind the proof of Theorem 1 is to reference the remarkable theorems on geometric realizations of crossed simplicial groups and sets ([Kra87, Theorem 2.3] [FL91, Theorem 5.3, Lemma 5.6], [Lod98, Theorem 7.1.4, Exercise 7.1.4]). The first mention of geometric realization for cyclic sets as $\mathrm{U}(1)$-spaces, and the main ingredient of the construction—the *geometric cyclic cosimplex*, or *twisted shuffle product* $S^1_{\cdot} \times_t \Delta^k \approx S^1_{\cdot} \times \Delta^k$ (here $S^1_{\cdot}$ is a circle composed of one 1-simplex and one point) is found in [Goo85, pp. 208-209] and further extended in [DHK85, §2, Proposition 2.4], [Jon87, Theorem 3.4].

Theorem 1 immediately follows from an inspection of the constructions in the above theorems. The arguments are geometrical. As a result, we will see that the minimally triangulated circle bundles over simplices described in [Mnë20] are nothing more than canonically *order reoriented* twisted shuffle product $S^1_{\cdot} \times_t \Delta^k$, and the map (1) is the universal minimally triangulated circle bundle.

Section 2: In this section, we discuss the basics of crossed simplicial group theory for the case of $\mathbf{C}_* \leq \mathbf{S}_*$, recalling the left crossed action of $\mathbf{C}_*$ on $\mathbf{S}_*$, left crossed cyclic orbits in $\mathbf{S}_*$, and their geometric realizations. Classical left crossed cyclic orbits in $\mathbf{S}_*$ *do not* form a simplicial equivalence relation and have no direct simplicial quotient.

Section 3: In this section, we will explain how to deal with the *right* action of $\mathbf{C}_*$ on $\mathbf{S}_*$ as opposed to the canonical situation of the left action. Right orbits *do* form a simplicial equivalence relation. We obtain $\mathbf{SC}_*$ as the simplicial quotient, which is the set of right cyclic crossed orbits. After geometric realization, $|\mathbf{SC}_*|$ is the cellular structure on the set of right orbits $|\mathbf{S}_*|/|\mathbf{C}_*|$. Here, $|\mathbf{S}_*| \approx *$ is a contractible Hausdorff topological group, and $|\mathbf{C}_*| = \mathrm{U}(1)$ is a Lie group. Therefore, the quotient map $\mathrm{U}(1) \to |\mathbf{S}_*| \xrightarrow{|\circlearrowright|} |\mathbf{SC}_*|$ is a $\mathrm{U}(1)$-fibration, and $|\mathbf{SC}_*| \approx \mathrm{K}(\mathbb{Z}, 2)$, which concludes the proof of Theorem 1.

Author is deeply grateful to Boris Tsygan and André Henriques for pointing the author to the subject of crossed simplicial groups.

2. Preliminaries

The pair of the symmetric crossed simplicial group $\mathbf{S}_*$ and its cyclic subgroup $\mathbf{C}_*$, $\mathbf{C}_* \leq \mathbf{S}_*$ is specially discussed in [Lod98, 6.1].

(2.1) Simplicial notations. We denote $\mathbf{\Delta}$ the category of finite linear orders $[n] = \{0, 1, 2, ..., n\}$ and non-decreasing maps between them called operators. The category $\mathbf{\Delta}$ is generated by "cofaces" δ_i and "codegeneracies" σ_i :

$$[n-1] \xrightarrow{\delta_i} [n] \xleftarrow{\sigma_j} [n+1], i, j = 0 \ldots n$$

Cofaces δ_i are the only injective order preserving maps "missing i" in the target. Codegeneracies σ_j are the only non-decreasing surjections "hitting j in the target twice", i.e $\sigma_j(j) = \sigma_j(j+1) = j$. Opposite category $\mathbf{\Delta}^{op}$ is generated by faces $d_i = \delta_i^{op}$, and degeneracies $s_i = \sigma_i^{op}$. Simplicial set X is a functor $\mathbf{\Delta}^{op} \xrightarrow{X} \mathbf{Sets}$. Face and degeneracies goes to face and degeneracy maps which are again denoted $X_n \xrightarrow{d_i} X_{n-1}$ and $X_n \xrightarrow{s_i} X_{n+1}$. The category of functors $\mathbf{\Delta}^{op} \to \mathbf{Sets}$ or "presheaves" on $\mathbf{\Delta}$ and natural transformations of those (maps of simplicial sets) is denoted by $\widehat{\mathbf{\Delta}}$.

(2.2) Category $\mathbf{\Delta G}$. Crossed simplicial groups and sets are related to extension of $\mathbf{\Delta}$ and $\widehat{\mathbf{\Delta}}$ by a correct adjoining of automorphism groups $\mathbf{G}_n^{op}$ to $[n]$ in a such way that $\mathbf{G}_n$ will act correctly by automorphisms of sets X_n.

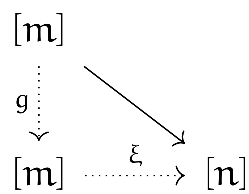
Definition 2. *[FL91, Definition 1.1]*
A sequence of groups $\mathbf{G} = \{\mathbf{G}_n\}, n \geq 0$ *is a crossed simplicial group if it is equipped with the following structure. There is a small category* $\mathbf{\Delta G}$*, which is part of the structure, such that*

(a) *the objects of* $\mathbf{\Delta G}$ *are* $[n]$, $n \geq 0$,

(b) $\mathbf{\Delta G}$ *contains* $\mathbf{\Delta}$ *as a subcategory,*

(c) $\mathrm{Aut}_{\mathbf{\Delta G}}([n]) = \mathbf{G}_n^{\mathrm{op}}$ *(opposite group of* $\mathbf{G}_n$*)* ,

(d) *any morphism* $[m] \to [n]$ *in* $\mathbf{\Delta G}$ *can be uniquely written as a composite* $\xi{\cdot}g$ *where* $\xi \in \mathrm{Hom}_{\mathbf{\Delta}}([m],[n])$ *and* $f \in \mathbf{G}_m^{\mathrm{op}}$ *(whence the notation* $\mathbf{\Delta G}$ *).*



(2.3) $\mathbf{\Delta C} \subset \mathbf{\Delta S}$, $\mathbf{C}_* \leq \mathbf{S}_*$. Here we follow [Lod98, 6.1]. We denote $\mathbf{S}_n$ the group of permutations of $n+1$ (sic!) ordered elements $[n] = \{0, 1, \ldots, n\}$, i.e. $g \in \mathbf{S}_n$ is a one-to one map $[n] \xrightarrow{f} [n]$ represented as permutation $(f(0), \ldots, f(n))$. We have a commutative subgroup $\mathbf{C}_n \leq \mathbf{S}_n$ of cyclic permutations generated by the cycle $\tau = (n, 0, 1, \ldots, n-1)$. We denote $\mathbf{S}_*$, $\mathbf{C}_*$ corresponding graded groups equipped with graded multiplication. They are equipped with simplicial structure interacting with multiplication in a canonical "crossed" way. For this we should pass to category $\mathbf{\Delta S}$.

The category $\mathbf{\Delta S}$ is the category $\mathbf{\Delta}$ enlarged by groups of arbitrary non-monotone automorphisms of ordered sets $[n]$ written as opposite symmetric group $\mathbf{S}_n^{\mathrm{op}}$ (or $\mathbf{S}_n$ acting from the right on $[n]$). Checking and unwinding conditions of Defition 2 is subject of [Lod98, Theorem 6.1.4], see also [FT87, Appendix A10 p. 191].

It is instructive to imagine both permutations and operators of $\mathbf{\Delta}$ as "wire diagrams" of maps between finite linear orders (Fig. 1).

For an element $g \in \mathbf{S}_n^{\mathrm{op}}$ there is associated set map $[n] \xrightarrow{g} [n]$ with the same name $g(i) = \tilde{g}^{-1}(i)$, where $\tilde{g} \in \mathbf{S}_n$ is the corresponding permutation.

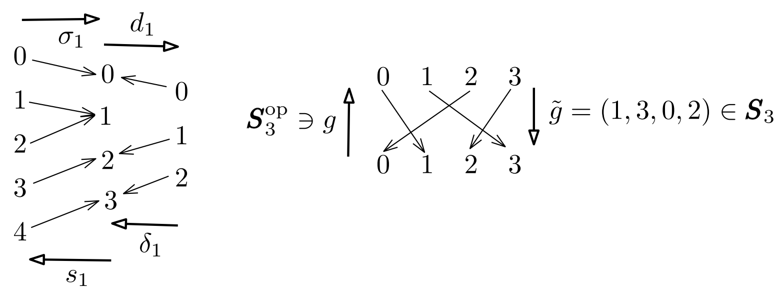


Figure 1: Wire diagrams of operators, permutations and their opposites.

In the language of wire diagrams permutations and their duals in the opposite groups, (co)boundaries (co)degeneracies communicate as depicted in Fig. 2. Inspecting wire diagrams for (co)boundaries and (co)degeneracies we get that simplicial relations produces *for any pair* (g, ξ)*,* $g \in \mathbf{S}_n^{op}, \xi \in \mathrm{Hom}_{\mathbf{\Delta}}([m],[n])$ *unique maps* $\xi^* g, g_* \xi$ *such that*
(i) the following diagram is commutative:

$$\begin{array}{ccc} [m] & \xrightarrow{\xi \in \mathbf{\Delta}} & [n] \\ {\scriptstyle \mathbf{S}_m^{op} \ni \xi^* g} \downarrow & & \downarrow {\scriptstyle g \in \mathbf{S}_n^{op}} \\ [m] & \xrightarrow[g_* \xi \in \mathbf{\Delta}]{} & [n] \end{array} \tag{2}$$

and
(ii) restriction of $\xi^* g$ *to each subset* $\xi^{-1}(i), i = 0...n$ *preserves the order.*
The above statement is the subject of [Lod98, Lemma 6.1.5].

Thus we have a category $\mathbf{\Delta S}$ with objects - finite orders $[n]$ and morphisms - pairs $[m] \xrightarrow{(\xi,g)} [n]$, $\xi \in \mathrm{Hom}_{\mathbf{\Delta}}([m],[n])$, $g \in \mathbf{S}_m^{op}$. Having another morphism $[k] \xrightarrow{(\phi,h)} [m]$ the composition is defined by the

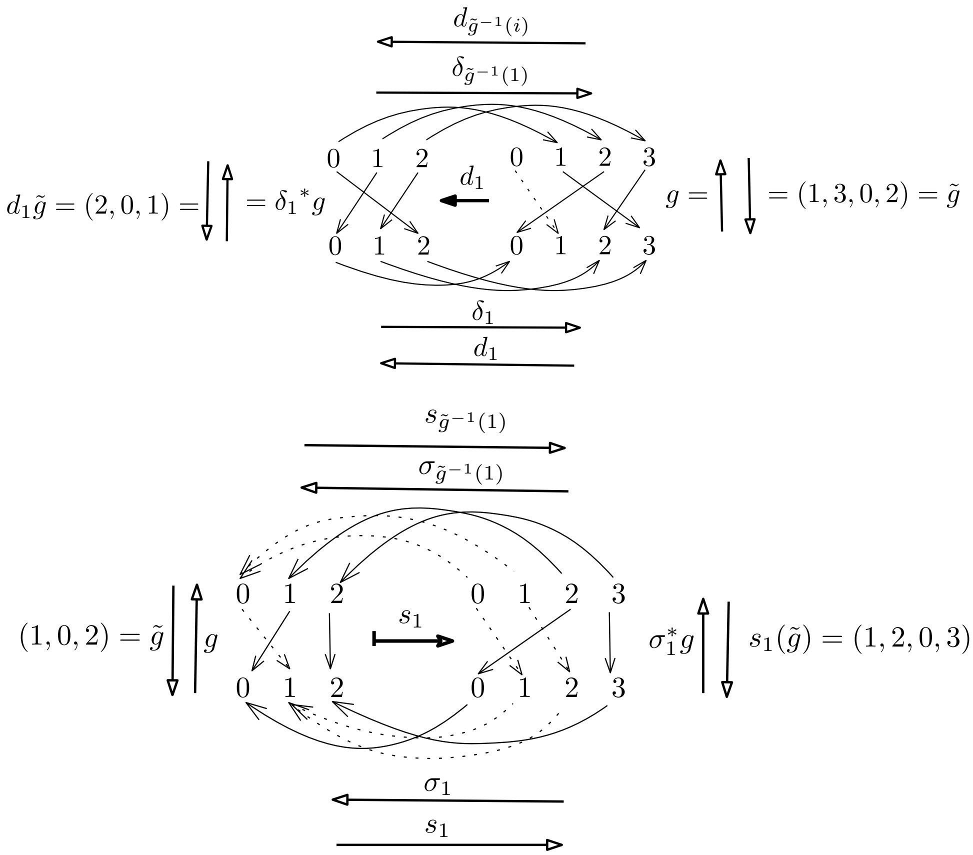


Figure 2: In wire diagrams deletion d_i corresponds to deletion of the arrow with target i and degeneracy s_i corresponds to parallel doubling of the arrow with target i.

rule

$$(\xi, g) \circ (\phi, h) = (\xi \circ g_* \phi, \phi^* \circ h) \tag{3}$$

where the compositions of components are in $\mathbf{\Delta}$ and $\mathbf{S}_k^{op}$. The category $\mathbf{\Delta S}$ satisfies requirements of Definition 2. The opposite category $\mathbf{\Delta S}^{op}$

has decomposition of arrows opposite to (2):

$$\begin{array}{ccc} [m] & \xleftarrow{\alpha\in\mathbf{\Delta}^{op}} & [n] \\ {\scriptstyle \mathbf{S}_m\ni\alpha_* f}\uparrow & & \uparrow{\scriptstyle f\in\mathbf{S}_n} \\ [m] & \xleftarrow[f^*\alpha\in\mathbf{\Delta}^{op}]{} & [n] \end{array} \tag{4}$$

When in (4) we set $\alpha = d_i = \delta_i^{op}$, $m = n-1$, we have $f^* d_i = d_{f^{-1}(i)}$ and we denote $(d_i)_* f$ by $d_i f$. When in (4) we choose $\alpha = s_i = \sigma_i^{op}$, $m = n+1$, we have $f^* s_i = s_{f^{-1}(i)}$ and we denote $(s_i)_* f$ by $s_i f$. Applying opposite to composition rule (3) we got that for $f, h \in \mathbf{S}_n$

$$\begin{aligned} d_i(h \circ f) &= d_i h \circ d_{h^{-1}(i)} f \\ s_i(h \circ f) &= s_i h \circ d_{h^{-1}(i)} f \end{aligned} \tag{5}$$

Replacing symmetric groups $\mathbf{S}_n$ by cyclic subgroups $\mathbf{C}_n$ we obtain subcategory $\mathbf{\Delta C} \subset \mathbf{\Delta S}$.

Now crossed simplicial group $\mathbf{S}_*$ can be canonically identified with representable $\mathbf{Sets}$-valued Yoneda presheaf

$$\mathbf{Y}_{\mathbf{\Delta S}}([0]) = \mathrm{Hom}_{\mathbf{\Delta S}}(-, [0]) = \mathbf{S}_* \tag{6}$$

By decomposition rules it is simplicial set structure on graded set of groups $\mathbf{S}_*$ with boundaries and degeneracies defined by (4) and communicating with multiplication in a "crossed" way by rules (5) (see [FL91, Proposition 1.7]). On $\mathbf{C}_*$ we have induced structure

$$\mathbf{Y}_{\mathbf{\Delta C}}([0]) = \mathrm{Hom}_{\mathbf{\Delta C}}(-, [0]) = \mathbf{C}_* \leq \mathbf{S}_* \tag{7}$$

thus the pair $\mathbf{C}_* \leq \mathbf{S}_*$ is defined.

(2.4) $\mathbf{C}_* \leq \mathbf{S}_*$ in terms of permutations. Here we rephrase the resulting from canonical $\mathbf{\Delta S}$-construction §(2.3) structure of $\mathbf{S}_*$ in terms of permutations. So, denote $\mathbf{S}_n$ the group of permutations of $n+1$ ordered elements $[n] = \{0, 1, \ldots, n\}$, i.e. $f \in \mathbf{S}_n$ is a one-to one map $[n] \xrightarrow{f} [n]$ represented as permutation $(f(0), \ldots, f(n))$. The graded set of permutations $\mathbf{S}_* = \mathbf{S}_0, \mathbf{S}_1 \ldots$ forms a simplicial set. The i-th boundary map $\mathbf{S}_n \xrightarrow{d_i} \mathbf{S}_{n-1}, i = 0, ..., n$ is deleting i-th element of permutation and reordering other elements monotonically, i.e. elements from 0 to $i-1$ preserves their numbers. Elements from $i+1$ to n got the numbers $i \ldots n-1$ (see (8)). The i-th degeneracy $\mathbf{S}_n \xrightarrow{s_i} \mathbf{S}_{n+1}, i = 0, ..., n$ inserts element with number $i+1$ *next* to the element i and reorders other elements monotonically. Elements from 0 to i preserves numbers and the old elements $i+1...n$ of the permutation got shifted by one numbers $i+2...n+1$ correspondently.

$$
\begin{aligned}
(d_i f)(j) &= \begin{cases} f(j) & \text{if } j = 0 \ldots i-1 \\ f(j-1) & \text{if } j = i \ldots n \end{cases} \\
(s_i f)(j) &= \begin{cases} f(j) & \text{if } f(j) = 0, ..., i \\ i+1 & \text{if } j = f^{-1}(i)+1 \\ f(j)+1 & \text{if } f(j) = i+1, ..., n \end{cases}
\end{aligned} \tag{8}
$$

Additionally in $\mathbf{S}_*$ we have crossed multiplication $(f_n, g_n) \mapsto f_n g_n$ communicating with boundaries and degeneracies by rules (5). This crossed multiplication will became canonically functorial in §(2.6.3) We have a crossed simplicial subgroup $\mathbf{C}_n \leq \mathbf{S}_n$ of cyclic permutations generated by cycles $\tau_n = (n, 0, 1, \ldots, n-1)$.

(2.5) $\mathbf{S}_* \xrightarrow{\circlearrowleft} \mathbf{SC}_*$ We recall the simplicial map $\mathbf{S}_* \xrightarrow{\circlearrowleft} \mathbf{SC}_*$ from [Mnë20]. The group $\mathbf{C}_n$ acts from the *right* on permutations by shifts $f_n \tau_n = (f_n(n), f_n(0), \ldots f_n(n-1))$. The orbits of the right action of $\mathbf{C}_n$ on $\mathbf{S}_n$ are numbered by *circular permutations*, i.e. oriented circular necklaces with $n+1$ beads coloured by $[n]$. We denote this set of right orbits or $n+1$ circular permutations $\mathbf{S}_n/\mathbf{C}_n$ by $\mathbf{SC}_n$. The rules (8) induces the simplicial set structure on the graded set of circular permutations $\mathbf{SC}_* = \mathbf{SC}_0, \mathbf{SC}_1 \ldots$ – we can delete a bead i (this provides d_i) and we can insert a bead $i+1$ right after the bead i since luckily the relation "right after" exist in circular order (this provides s_i). Thus we got simplicial set of circular permutations $\mathbf{SC}_*$ together with the simplicial factor-map $\mathbf{S}_* \xrightarrow{\circlearrowleft} \mathbf{SC}_*$ sending a permutation to its right cyclic orbit.

(2.6) Simplicial, cyclic and symmetric sets, base change adjacency and left crossed cyclic orbit of a permutation.

Symmetric or cyclic set is a **Sets**-valued presheaf on $\mathbf{\Delta S}$ or $\mathbf{\Delta C}$. In the following we use $\mathbf{G}$ for definitions and statements which are equivalent for $\mathbf{S}$ and $\mathbf{C}$. For example $\mathbf{G}_*$ (6),(7) is a $\mathbf{G}$-set.

The important point for us is that due to embedding $\mathbf{\Delta C} \subset \mathbf{\Delta S}$ we got that canonically $\mathbf{S}_*$ is a $\mathbf{C}$-set.

The categories of of $\mathbf{G}$-sets with morphisms - natural transformations are denoted by $\widehat{\mathbf{\Delta G}}$. By construction these are simplicial sets X with fixed left actions of groups $\mathbf{G}_n$ by automorphisms of X_n. The action are explicitly described by "base change adjunction".

(2.6.1) Adjunction data. We recall (see [Mac98, Chapter X]) that adjunction $\langle F, G, \varphi \rangle$ between two small categories A, B is a pair of functors $A \xrightarrow{F} B, B \xrightarrow{G} A$ and bifunctorial isomorphism of Hom sets $B(F(X), Y) \xrightarrow{\varphi} A(X, G(Y))$, where X is running over A and Y over B. Functor F called left adjoint to G, G called right adjoint to F, and adjunction sometimes denoted by $F \dashv G$. Adjunction defines and is defined by "monad of adjunction": a natural transformation of A-endofunctors $\mathrm{Id}_A \xrightarrow{\iota} GF$ called "unit of adjunction" and a natural transformation B-endofunctors $FG \xrightarrow{\varepsilon} \mathrm{Id}_B$ called counit of adjunction satisfying "triangular identities"

$$
\begin{array}{ccc}
F & \xrightarrow{F\cdot\iota} & F\circ G\circ F \\
 & \searrow^{\mathrm{id}} & \downarrow{\scriptstyle \varepsilon\cdot F} \\
 & & F
\end{array}
\qquad
\begin{array}{ccc}
G & \xrightarrow{\iota\cdot G} & G\circ F\circ G \\
 & \searrow^{\mathrm{id}} & \downarrow{\scriptstyle G\cdot\varepsilon} \\
 & & G
\end{array}
\tag{9}
$$

Embedding of (skeletal) categories $\mathbf{\Delta} \subset \mathbf{\Delta G}$ creates embedding of those duals $\mathbf{\Delta}^{\mathrm{op}} \xrightarrow{\mathcal{P}} \mathbf{\Delta G}^{\mathrm{op}}$. On presheaves we got forgetful functor $\widehat{\mathbf{\Delta}} \xleftarrow{\overline{*}} \widehat{\mathbf{\Delta G}}$ making simplicial set $\overline{Y}$ from $\mathbf{G}$-set Y. Functor $\overline{*}$ has left adjoint which we denote[1] $\mathbf{G}_* \times_t *$: $(\mathbf{G}_* \times_t *) \dashv \overline{*}$. The left adjoint is computed as pointwise left Kan extension of simplicial set X along $\mathcal{P}$ [Mac98, X.3 Theorem 1]. This is a specially simple situation of *"base change adjunction"*.

(2.6.2) Crossed left action. The left Kan extension of X along $\mathcal{P}$ produces the following element-wise formulas for simplicial and $\mathbf{G}_*$-structure on $\mathbf{G}_* \times_t X$:

$$
\begin{aligned}
(\mathbf{G}_* \times_t X)_n &= \{(h_n, x_n)\}_{h_n \in \mathbf{G}_n, x_n \in X_n} \\
d_i(h_n, x_n) &= (d_i h_n, d_{h^{-1}(i)} x_n) \\
s_i(h_n, x_n) &= (s_i h_n, s_{h^{-1}(i)} x_n) \\
f_n \cdot (h_n, x_n) &= (f_n h_n, x_n)
\end{aligned}
\tag{10}
$$

The adjunction $(\mathbf{G}_* \times_t *) \dashv \overline{*}$ defines monad with the unit

$$\mathrm{id}_{\widehat{\mathbf{\Delta}}} \xrightarrow{\iota} \overline{(\mathbf{G}_* \times_t *)}$$

and counit

$$(\mathbf{G}_* \times_t \overline{*}) \xrightarrow{ev} \mathrm{id}_{\widehat{\mathbf{\Delta G}}}$$

[1] The functor is denoted by G in [Kra87] and F in [FL91]

satisfying "triangular identities" (9). The unit of the adjunction is computed on elements as follows. For an element $x_n \in X_n$ of simplicial set X we got

$$\iota(x_n) = (1_{\mathbf{G}_n}, x_n)$$

Counit of the adjunction defines *the crossed left action* of $\mathbf{G}_*$ on $\mathbf{G}$-set Y, namely for $y_n \in Y_n$ we got

$$ev(g_n, y_n) = g_n \cdot y_n$$

Triangular identities (9) of the monad ensures that the action is correct action of $\mathbf{G}_*$ in a crossed way:

$$\begin{array}{ccc} \mathbf{G}_* \times_t X & \xrightarrow{(h,x)\mapsto(h,(1,x))} & \mathbf{G}_* \times_t \overline{(\mathbf{G}_* \times_t X)} \\ & \searrow^{\mathrm{id}} & \downarrow{\scriptstyle (h,(f,x))\mapsto(hf,x)} \\ & & \mathbf{G}_* \times_t X \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{x\mapsto(1,x)} & \overline{\mathbf{G}_* \times_t X} \\ & \searrow^{\mathrm{id}} & \downarrow{\scriptstyle (h,x)\mapsto h\cdot x} \\ & & X \end{array}$$

(2.6.3) Crossed product. If $X = \mathbf{G}_*$ then the counit

$$\mathbf{G}_* \times_t \mathbf{G}_* \xrightarrow{ev} \mathbf{G}_*$$

represents "crossed product" in crossed simplicial group $\mathbf{G}_*$.

(2.6.4) Yoneda Lemma and left crossed orbits. Categories of presheaves $\widehat{\mathbf{\Delta}}$, $\widehat{\mathbf{\Delta}G}$ has representable (Yoneda) objects – cosimplices

$$\mathbf{\Delta}[n] = \mathbf{\Delta}(-, [n]) = \mathbf{\Delta}^{op}([n], -)$$

and $\mathbf{G}$-cosimplices

$$\mathbf{\Delta G}[n] = \mathbf{\Delta G}(-, [n]) = \mathbf{\Delta G}^{op}([n], -)$$

There is the key isomorphism $\mathbf{\Delta G}[n] \approx \mathbf{G}_* \times_t \mathbf{\Delta}[n]$ ([FL91, Exersise 4.5]). (Co)Yoneda Lemma states that every presheaf is canonically colimit of representables. For simplicial set X and $x_n \in X_n$ this creates colimit cone structure map

$$\mathbf{\Delta}[n] \xrightarrow{\mathbf{y}_{\mathbf{\Delta}}(x_n)} X$$

sending $\mathrm{id}[n]$ to x_n. Analogously for $\mathbf{G}$-set Y and $y_n \in Y_n$ this creates colimit cone structure map

$$\mathbf{\Delta G}[n] \approx \mathbf{G}_* \times_t \mathbf{\Delta}[n] \xrightarrow{\mathbf{y}_{\mathbf{\Delta}G}(y_n)} Y$$

sending $(1_{\mathbf{G}_n}, \mathrm{id}[n])$ to y_n. Relation between unit-counit of adjunction and bifunctorial isomorphism of Hom-sets

$$\widehat{\mathbf{\Delta G}}(\mathbf{G}_* \times_t X, Y) \xrightarrow{\varphi} \widehat{\mathbf{\Delta}}(X, \overline{Y})$$

connects the two Yoneda maps. For a $\mathbf{G}$-set Y and element $y_n \in Y_n$ the isomorphism φ sends $\mathbf{G}_* \times_t \mathbf{\Delta}[n] \xrightarrow{\mathbf{y}_{\mathbf{\Delta}G}(y_n)} Y$ to $\mathbf{\Delta}[n] \xrightarrow{\mathbf{y}_{\mathbf{\Delta}}(\overline{y}_n)} \overline{Y}$. In the inverse direction φ sends $\mathbf{y}(\overline{y}_n)$ to $\mathbf{y}_{\mathbf{\Delta}G}(y_n)$ by the following commutative diagram

$$\begin{array}{ccc} \mathbf{G}_* \times_t \overline{Y} & \xrightarrow{ev} & Y \\ {\scriptstyle \mathbf{G}_* \times_t \mathbf{y}_{\mathbf{\Delta}}(\overline{y}_n)} \uparrow & \nearrow {\scriptstyle \mathbf{y}_{\mathbf{\Delta}G}(y_n)} & \\ \mathbf{G}_* \times_t \mathbf{\Delta}[n] & & \end{array} \qquad (11)$$

The Yoneda map $\mathbf{y}_{\mathbf{\Delta G}}(y_n)$ and its image in $\mathbf{G}$-set Y we call *left crossed* $\mathbf{G}$-*orbit of* $y_n \in Y_n$.

(2.7) Geometric realization. Here we in situation of [Kra87, Theorem 2.3], [FL91, Teorem 5.3]. Geometric realization $|X|$ of a $\mathbf{G}$-set X is the geometric realization $|\overline{X}|$ of the underground simplicial set. The core of geometric realization theorems states that there is a canonical functorial homeomorphism[2]

$$|\mathbf{G}_*| \times |X| \xrightarrow{\Psi} |\mathbf{G}_* \times_t X|$$

such that in induced from geometric realization metric the composite

$$|\mathbf{G}_*| \times |\mathbf{G}_*| \xrightarrow{\Psi} |\mathbf{G}_* \times_t \mathbf{G}_*| \xrightarrow{|ev|} |\mathbf{G}_*|$$

is a topological group. For any $\mathbf{G}$-set Y the composite

$$|\mathbf{G}_*| \times |Y| \xrightarrow{\Psi} |\mathbf{G}_* \times Y| \xrightarrow{|ev|} |Y|$$

makes $|X|$ left topological $|\mathbf{G}_*|$-space.

In our situations $|\mathbf{C}_*|$ is an oriented circle $S^1_\cdot$ made from one vertex and one non-degenerate 1-simplex (and oriented by its orientation). In induced metric the composite map

$$|\mathbf{C}_*| \times |\mathbf{C}_*| \xrightarrow{\Psi} |\mathbf{C}_* \times_t \mathbf{C}_*| \xrightarrow{|ev|} |\mathbf{C}_*|$$

is exactly $\mathbb{R}/\mathbb{Z} \approx U(1)$ group structure on $S^1_\cdot$ with the unit in the vertex of $S^1_\cdot$.

$$|\mathbf{S}_*| \times |\mathbf{S}_*| \xrightarrow{\Psi} |\mathbf{S}_* \times_t \mathbf{S}_*| \xrightarrow{|ev|} |\mathbf{S}_*|$$

is a contractible topological group ([FL91, Example 6]) and since $\mathbf{S}_*$ is a $\mathbf{C}$-set the induced composed map

$$U(1) \times |\mathbf{S}_*| \xrightarrow{\Psi} |\mathbf{C}_* \times_t \mathbf{S}_*| \xrightarrow{|ev|} |\mathbf{S}_*|$$

is a free left action of Lie subgroup $U(1) \leq |\mathbf{S}_*|$ on Hausdorff contractible space $|\mathbf{S}_*|$.

[2] The homeomorphism Ψ is the homeomorphism $\Phi(X)^{-1}$ in [Kra87, Theorem 2.3] and $(p_1, p_2)^{-1}$ in [FL91, Teorem 5.3].

We are specially interested in orbits of the action. Cyclic cosimplex $\mathbf{y}_{\Delta C} = \Delta\mathbf{C}_*[n] = \mathbf{C}_* \times_t \mathbf{\Delta}[n]$ is a cyclic set. Its geometric realization has cellular structure of "twisted shuffle product" $S^1_{\cdot} \times_t \Delta^n$ ([Goo85, pp. 208-209],[DHK85, §2, Proposition 2.4], [Jon87, Theorem 3.4]). Applying geometric realization to (11) we get a comutative diagram of spaces:

$$\begin{array}{ccccc} U(1) \times |\mathbf{S}_*| & \xrightarrow{\Psi} & |\mathbf{C}_* \times_t \mathbf{S}_*| & \xrightarrow{|ev|} & |\mathbf{S}_*| \\ \uparrow {\scriptstyle id \times |\mathbf{y}_{\Delta}(g_n)|} & & \uparrow {\scriptstyle |\mathbf{C}_* \times_t \mathbf{y}_{\Delta}(g_n)|} & \nearrow {\scriptstyle |\mathbf{y}_{\Delta C}(g_n)|} & \\ U(1) \times \Delta^n & \xrightarrow{\Psi} & |\mathbf{C}_* \times_t \mathbf{\Delta}[n]| & & \end{array} \qquad (12)$$

Thus the composit map

$$U(1) \times \Delta^n \xrightarrow{\Psi} S^1 \times_t \Delta^n = |\mathbf{C}_* \times_t \mathbf{\Delta}[n]| \xrightarrow{|\mathbf{y}_{\Delta C}(g_n)|} |\mathbf{S}_*| \qquad (13)$$

provides as its image a continuous trivial family of left $U(1)$ orbits on $|\mathbf{S}_*|$ parametrized by cell which is the image of characteristic map $\Delta^n \xrightarrow{|\mathbf{y}_{\Delta}|} |\mathbf{S}_*|$. In this way in geometric realization the crossed left $\mathbf{C}_*$-orbit became a trivial family of true $U(1)$ left orbits.

3. Proof of Theorem 1.

(3.1) Left and right orbit spaces of topological subgroup. For a group G and a subgroup $H \leq G$, there are left and right actions of H on G, and these actions are free. The left action of H on G creates the set of left orbits $G\backslash H = \{Hg\}_{g \in G}$, while the right action of H on G creates the set of right orbits $G/H = \{gH\}_{g \in G}$. The group G acts from the right on $G\backslash H$ and from the left on G/H, with stabilizer H.

The involution

$$G \xrightarrow{\upsilon} G : \upsilon(g) = g^{-1} \qquad (14)$$

switches between left and right H-orbits of g and g^{-1}. It is standard to define the *opposite group* G^{op} with the same elements as G but with multiplication $g_1 * g_2 = g_2 g_1$. Then υ is a group isomorphism $G \xrightarrow{\upsilon} G^{op}$ that maps left H orbits in G to left H^{op} orbits in G^{op} (which were right H-orbits in G), thereby inducing a one-to-one correspondence

$$G\backslash H \overset{\tilde{\upsilon}}{\approx} G/H \tag{15}$$

between the sets of right and left H-orbits in G.

In the topological category, where H and G are topological groups, the sets $G\backslash H$ and G/H become orbit spaces with quotient topology, and $\tilde{\upsilon}$ in (15) is a homeomorphism between left and right orbit spaces. In good situations, for example, if H is a Lie group and G is Hausdorff, the map

$$H \to G \to G\backslash H \quad [G/H] \tag{16}$$

is locally trivial with fiber H (see [Pal61, 4.1 on page 315]). Therefore, $G\backslash H \quad [G/H]$ is Hausdorff (see [Mun00, Theorem 31.2 (a) on page 196]), and (16) is a principal Serre fibration (see [tDKP70, Satz 5.14]).

In the simplicial category, where H and G are simplicial groups, $G\backslash H$ and G/H have the structure of simplicial sets, and $H \to G \to G\backslash H \quad [G/H]$ is a principal Kan fibration (see [May68, Definition 18.1, Lemma 18.2]).

(3.2) Left vs. right crossed action problem. In the crossed-simplicial setting, a cyclic crossed simplicial group $\mathbf{C}_* \leq \mathbf{S}_*$ is a subgroup of a symmetric crossed-simplicial group, acting on $\mathbf{S}_*$ from the *left* in a twisted manner. This twist disappears in geometric realization (see §(2.7)). The geometric realization theorems for crossed simplicial groups imply that $U(1) \approx |\mathbf{C}_*| \leq |\mathbf{S}_*|$, where $|\mathbf{S}_*|$ is a contractible topological group. This provides, according to §(3.1), a principal $U(1)$ fibration

$$U(1) \approx |\mathbf{C}_*| \to |\mathbf{S}| \to |\mathbf{S}_*|\backslash|\mathbf{C}_*| \approx K(\mathbb{Z}, 2) \tag{17}$$

However, for the *left* crossed action of $\mathbf{C}_*$ on $\mathbf{S}_*$, there is *no simplicial structure* on the left orbit set, since the left crossed orbits (§(2.6.4)) *are not* classes of simplicial equivalence relations. Therefore, something like $\mathbf{S}_*\backslash\mathbf{C}_*$ does not exist simplicially.

On the other hand, for a crossed simplicial group, the opposite group is not well-defined as a crossed simplicial group, so the switch between left and right actions is not entirely trivial in the crossed simplicial setting. We need the *right* action to handle $\mathbf{S}_* \overset{\circlearrowleft}{\longrightarrow} \mathbf{SC}_*$ (§(2.5)).

(3.3) Universal order reorientation Υ of simplicial sets. A permutation can be identified with a simplex Δ^n having *two* total orientation orders on vertices: source order and target order (see Fig.3).

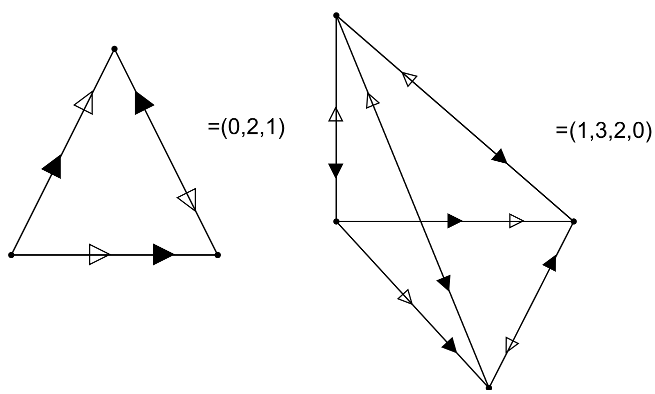


Figure 3: Permutation as a double ordered simplex. Here $\triangleright$ denotes the source order, and $\blacktriangleright$ denotes the target order.

If one has a finite simplicial complex K, a local order orientation of its simplices can be obtained by fixing a total order of its vertices, resulting in a semi-simplicial (or "Δ") set, denoted $\mathsf{K}_{\text{source}}$. This is a standard method. Not all local order orientations of K can be obtained this way. A different total order on the vertices will yield different local orders on simplices, resulting in a semi-simplicial set structure $\mathsf{K}_{\text{target}}$ on the same complex K. The two "source" and "target" orders will produce source and target orders on the vertices of every simplex of K, both comparable with boundaries. Hence, every n-simplex x_n of K has a permutation $p(x_n)$, providing a simplicial map $K_{\text{target}} \xrightarrow{p} \underline{\mathbf{S}}_*$ (where $\underline{\mathbf{S}}_*$ is the semi-simplicial set obtained from $\mathbf{S}_*$ by forgetting degeneracies). There is a non-simplicial involution $\underline{\mathbf{S}}_* \xrightarrow{\Upsilon} \underline{\mathbf{S}}_*$ that switches between the source and target permutations, i.e., sending a permutation f to f^{-1}. The involution induces a simplicial map $K^{\Upsilon p} = K_{\text{source}} \xrightarrow{p^{-1}} \underline{\mathbf{S}}_*$. Thus, together with the involution Υ, the semi-simplicial set $\underline{\mathbf{S}}_*$ "represents" a representable functor of the "double orientation ordering" of a semi-simplicial set, along with the operation of order reorientation.

The same involution acts on $\mathbf{S}_*$, respecting degeneracies. By inspecting wire diagrams (see Fig 2), we can see that the diagram of permutation f_n^{-1} is obtained from the diagram of f_n by reversing the direction of arrows. In this process, boundaries map to boundaries, and degeneracies to degeneracies in a canonical but non-simplicial way. Thus, Υ is a non-simplicial automorphism of the simplicial set $\mathbf{S}_*$, sending f_n to

$$\Upsilon(f_n) = f_n^{-1},$$

boundary $d_i f_n$ to boundary

$$\Upsilon(d_i f_n) = d_{f_n^{-1}(i)} f_n^{-1},$$

degeneracy $s_i f_n$ to degeneracy

$$\Upsilon(s_i f_n) = s_{f_n^{-1}(i)} f_n^{-1}$$

providing a coordinate change on geometric realization. Also, we have the left-right multiplication involution

$$\Upsilon(f_n h_n) = h_n^{-1} f_n^{-1}.$$

If one has a simplicial map $X \xrightarrow{a} \mathbf{S}_n$, this means that simplices of X are decorated by permutations in a way compatible with boundaries and degeneracies. We can reorient simplices by changing the source and target orders, i.e., by using the non-simplicial map $X \xrightarrow{a} \mathbf{S}_* \xrightarrow{\Upsilon} \mathbf{S}_*$, resulting in a new simplicial set $X^{\Upsilon a}$ on the same set of simplices, with canonically homeomorphic geometric realization. Together with the involution Υ, the simplicial set $\mathbf{S}_*$ represents functor of double orientation ordering of simplicial sets, along with the operation of order reorientation.

(3.4) Right crossed $\mathbf{C}_*$-orbits in $\mathbf{S}_*$. We don't know exactly what the opposite of a crossed simplicial group is (since it is not a crossed simplicial group), but we can define a *right* $\mathbf{C}_*$*-orbit* of $g \in \mathbf{S}_n$. For this, we define a simplicial set denoted by $E(\circlearrowright g)$. We follow notations (4) for $\mathbf{\Delta S}^{op}$. Define

$$E(\circlearrowright g)_m = \{([n] \xrightarrow{\alpha} [m], \alpha_* g \cdot h) \mid h \in \mathbf{C}_m\}$$

$$d_i([n] \xrightarrow{\alpha} [m], \alpha_* g \cdot h) = (d_i \alpha, d_i(\alpha_* g \cdot h))$$
$$s_i([n] \xrightarrow{\alpha} [m], \alpha_* g \cdot h) = (s_i \alpha, s_i(\alpha_* g \cdot h))$$

It has simplicial projections

$$E(\circlearrowright g) \xrightarrow{q_1} \mathbf{\Delta}[n]$$

$$E(\circlearrowright g) \xrightarrow{q_2(g)} \mathbf{S}_*$$

The right-$\mathbf{C}_*$ orbit of g is by definition the image of $q_2(g)$ in $\mathbf{S}_n$.

Tautological computations provide the following lemma:

Lemma 3.
(i) Let $\circlearrowright g_n \in \mathbf{SC}_n$ *and* $\Delta[n] \xrightarrow{\mathbf{y}_{\Delta}(\circlearrowright g_n)} \mathbf{SC}$ *be the Yoneda simplex of* $\circlearrowright g_n$ *in* $\mathbf{SC}$*. Then* q_1 *and* $q_2(g_n)$ *are the components of the*

pullback diagram

$$\begin{array}{ccc} E(\circlearrowright g_n) & \overset{q_2(g_n)}{\dashrightarrow} & \mathbf{S}_* \\ {\scriptstyle q_1}\downarrow & & \downarrow{\scriptstyle \circlearrowright} \\ \mathbf{\Delta}[n] & \xrightarrow{y(\circlearrowright g_n)} & \mathbf{SC} \end{array}$$

(ii) $\Upsilon(\mathbf{y}_{\mathbf{\Delta C}}(g_n)) = q_2(g_n^{-1})$, $E(\circlearrowright g_n^{-1}) = (\mathbf{C}_* \times_t \mathbf{\Delta}[n])^{\Upsilon \mathbf{y}_{\mathbf{\Delta C}}(g_n)}$. *This means that order orientation involution* Υ *turns left crossed cyclic orbit of permutation* $\mathbf{C}_* \times_t \mathbf{\Delta}[n] \xrightarrow{\mathbf{y}_{\mathbf{\Delta C}}(g_n)} \mathbf{S}_*$ *into pullback*

$$\begin{array}{ccc} E(\circlearrowright g_n^{-1}) & \overset{q_2(g_n^{-1})}{\dashrightarrow} & \mathbf{S}_* \\ {\scriptstyle q_1}\downarrow & & \downarrow{\scriptstyle \circlearrowright} \\ \mathbf{\Delta}[n] & \xrightarrow{y(\circlearrowright g_n^{-1})} & \mathbf{SC} \end{array}$$

It follows that crossed right $\mathbf{C}_*$ orbits *form* a simplicial equivalence relations on $\mathbf{S}_*$ (unlike the left orbits). Its factor set is $\mathbf{SC}_* \approx \mathbf{S}_*/\mathbf{C}_*$. The space $|E(\circlearrowright G)| \xrightarrow{p_1} \Delta^n$ is a minimally triangulated circle bundle associated with $\circlearrowright g_n$. It is just the Υ-reoriented geometric twisted shuffle product

$$S^1_{\cdot} \times_t \Delta^n = |\mathbf{C}_* \times_t \mathbf{\Delta}[n]| \approx U(1) \times \Delta^n.$$

(3.5) Order reorientation $|\Upsilon|$ on geometric realization $|\mathbf{S}_*|$ is the canonical group involution υ. It follows from classical constructions (see [Kra87, the map χ in the proof of Theorem 2.3 on page 52]) that the geometric realization $|\Upsilon|$ of the order reorientation involution Υ is exactly the involution υ ((14) §(3.1)):

$$\begin{array}{ccc} \mathbf{C}_* \le \mathbf{S}_* & \xrightarrow{\Upsilon} & \mathbf{C}_* \le \mathbf{S}_* \\ {\scriptstyle |*|}\downarrow & & \downarrow{\scriptstyle |*|} \\ |\mathbf{C}_*| \le |\mathbf{S}_*| & \xrightarrow{\upsilon} & |\mathbf{C}_*| \le |\mathbf{S}_*| \end{array} \tag{18}$$

extending the chain (17) by

$$|\mathbf{S}_*|\backslash|\mathbf{C}_*| \overset{\tilde{\upsilon}}{\approx} |\mathbf{S}_*|/|\mathbf{C}_*| \approx |\mathbf{SC}_*| \approx \mathsf{K}(\mathbb{Z},2) \tag{19}$$

This completes the proof of Theorem 1.

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