CENTRALIZERS OF SEMISIMPLE ELEMENTS IN THE FINITE CLASSICAL GROUPS

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1. Introduction

Let G be the group of K-rational points of a connected simple algebraic group over an algebraically closed field K of characteristic p. Let σ be a surjective endomorphism of G such that the group G_σ of σ-stable elements is finite.

The aim of the present paper is to obtain information about the centralizers of semisimple elements in G_σ, for all groups G of classical type. To achieve this we shall use some general results which have been derived in a recent paper [**5**]. We shall also derive explicit formulae for the degrees of the semisimple representations of G_σ when G is of adjoint type.

If x is a semisimple element of G_σ then its *connected centralizer* $C_G(x)^0$ is a σ-stable reductive subgroup of G of maximal rank [**2**, p. 201] and its *connected centralizer* $C_{G_\sigma}(x)^0$ *in* G_σ is the subgroup of σ-stable elements in $C_G(x)^0$. We therefore consider more generally any σ-stable connected reductive subgroup G_1 of maximal rank in G and determine when its subgroup $(G_1)_\sigma$ of σ-stable elements is the connected centralizer of some semisimple element in G_σ.

The set $\mathscr{C}$ of σ-stable conjugates of G_1 in G is acted on by G_σ under conjugation, and the set of orbits $\mathscr{C}/G_\sigma$ is in bijective correspondence with the set of σ-conjugacy classes in $\mathscr{N}_W(W_1)/W_1$, where W is the Weyl group of G and W_1 is the Weyl group of G_1 [**5**]. G_1 factorizes as $G_1 = MS$, where M is semi-simple, S is a torus, and $M \cap S$ is finite. Moreover, the order of $(G_1)_\sigma$ is given by $|(G_1)_\sigma| = |M_\sigma|.|S_\sigma|$. A σ-stable conjugate ${G_1}^g$ is the connected centralizer of some semisimple element in G_σ if and only if a certain finite abelian group Γ, determined by G_1 and the σ-conjugacy class in $\mathscr{N}_W(W_1)/W_1$, has a regular character [**5**].

We shall say that two semisimple elements x, y of G_σ are *of the same genus* if their connected centralizers $C_G(x)^0, C_G(y)^0$ are conjugate under G_σ. The property of belonging to the same genus gives an equivalence relation on the semisimple conjugacy classes in G_σ. The orders of the connected centralizers in G_σ of semisimple elements of the same genus are equal. We shall describe for each group G_σ of classical type the orders of the semisimple and toral parts of the connected centralizers $C_{G_\sigma}(x)^0$.

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Deligne and Lusztig have shown [**6**] how families of irreducible complex representations of the group G_σ can be constructed; these representations might be called the semisimple representations of G_σ. It was proved by Deligne and Lusztig that they are in bijective correspondence with the semisimple conjugacy classes in $\tilde{G}_\sigma$, where $\tilde{G}$ is the dual group of G, if G is an adjoint group [**6**, 10.7]. The degrees of the semisimple representations of G_σ can be written down in terms of the orders of the connected centralizers of the corresponding semisimple elements in the simply-connected group $\tilde{G}_\sigma$. In fact these degrees have the form

$$\left(\frac{|\tilde{G}_\sigma|}{|C_{\tilde{G}_\sigma}(x)^0|}\right)p'.$$

We shall describe these degrees for each adjoint group G_σ of classical type. The results are valid for values of q sufficiently large, where q is the number of elements in the finite field corresponding to G_σ, but for small values of q some of the degrees may be missing in G_σ. If G is not adjoint the degrees of the semisimple representations of G_σ need not be given by the above formula, although they appear to differ from it only in minor respects.

For the general background to the subject under discussion the reader should consult, for example, [**2**] or [**8**].

2. Classical groups over finite fields and their duality

We assume that G is of classical type, i.e. of one of the types A_l, B_l, C_l, D_l in the standard classification. For each of these types there are a number of isogenous simple groups. In type A_l there is the simply-connected group $(A_l)_{\mathrm{sc}}$, the adjoint group $(A_l)_{\mathrm{ad}}$, and a number of other groups which are neither simply-connected nor adjoint. In type B_l the simply-connected group $(B_l)_{\mathrm{sc}}$ and adjoint group $(B_l)_{\mathrm{ad}}$ give the only possibilities, and this is true also in type C_l. In type D_l there is, in addition to $(D_l)_{\mathrm{sc}}$ and $(D_l)_{\mathrm{ad}}$, the special orthogonal group SO_{2l} which we denote by $(D_l)_{\mathrm{so}}$. When l is even, there is a further group of this type called the half-spin group $(D_l)_{\mathrm{HS}}$.

If G is of classical type its dual $\tilde{G}$, as defined by Deligne and Lusztig [**6**], is also of classical type. The duals of the groups G described above are shown on p. 3. For each of the above groups G the Frobenius map of raising matrix coefficients to the qth power gives a surjective endomorphism σ for which G_σ is finite. (Here q is a power of the characteristic p of K.) We obtain in this way the split groups, or Chevalley groups, of classical type over finite fields. However, for the types in which the Dynkin diagram has a symmetry, viz. type A_l when $l \geqslant 2$ and type D_l, we can choose for σ the qth power map combined with the graph automorphism derived from the

G	$\tilde{G}$	
$(A_l)_{\mathrm{sc}}$	$(A_l)_{\mathrm{ad}}$	
$\vdots$	$\vdots$	
$(A_l)_{\mathrm{ad}}$	$(A_l)_{\mathrm{sc}}$	
$(B_l)_{\mathrm{sc}}$	$(C_l)_{\mathrm{ad}}$	
$(B_l)_{\mathrm{ad}}$	$(C_l)_{\mathrm{sc}}$	
$(C_l)_{\mathrm{sc}}$	$(B_l)_{\mathrm{ad}}$	
$(C_l)_{\mathrm{ad}}$	$(B_l)_{\mathrm{sc}}$	
$(D_l)_{\mathrm{sc}}$	$(D_l)_{\mathrm{ad}}$	
$(D_l)_{\mathrm{so}}$	$(D_l)_{\mathrm{so}}$	
$(D_l)_{\mathrm{HS}}$	$(D_l)_{\mathrm{HS}}$	l even
$(D_l)_{\mathrm{ad}}$	$(D_l)_{\mathrm{sc}}$	

symmetry of the Dynkin diagram. In this way we obtain the quasi-split groups, or Steinberg groups, over finite fields. These Steinberg groups will as usual be denoted by ${}^2A_l$ and ${}^2D_l$. (The triality twisted Steinberg groups ${}^3D_4$ may more naturally be regarded as exceptional groups and will not be discussed in the present paper.)

If G_σ is a finite group of classical type its dual $\tilde{G}_\sigma$ is also finite of classical type. The duals $\tilde{G}_\sigma$ of the groups G_σ described above are given in the following list. In this list we have identified each group with its usual description in terms of matrices; the orthogonal and spin groups, labelled

G_σ	$\tilde{G}_\sigma$	
$\mathrm{SL}_{l+1}(q) = (A_l)_{\mathrm{sc}}(q)$,	$(A_l)_{\mathrm{ad}}(q) = \mathrm{PGL}_{l+1}(q)$,	
$\vdots$	$\vdots$	
$\mathrm{PGL}_{l+1}(q) = (A_l)_{\mathrm{ad}}(q)$,	$(A_l)_{\mathrm{sc}}(q) = \mathrm{SL}_{l+1}(q)$,	
$\mathrm{Spin}_{2l+1}(q) = (B_l)_{\mathrm{sc}}(q)$,	$(C_l)_{\mathrm{ad}}(q) = \mathrm{PGSp}_{2l}(q)$,	
$\mathrm{SO}_{2l+1}(q) = (B_l)_{\mathrm{ad}}(q)$,	$(C_l)_{\mathrm{sc}}(q) = \mathrm{Sp}_{2l}(q)$,	
$\mathrm{Sp}_{2l}(q) = (C_l)_{\mathrm{sc}}(q)$,	$(B_l)_{\mathrm{ad}}(q) = \mathrm{SO}_{2l+1}(q)$,	
$\mathrm{PGSp}_{2l}(q) = (C_l)_{\mathrm{ad}}(q)$,	$(B_l)_{\mathrm{sc}}(q) = \mathrm{Spin}_{2l+1}(q)$,	
$\mathrm{Spin}_{2l}(q) = (D_l)_{\mathrm{sc}}(q)$,	$(D_l)_{\mathrm{ad}}(q) = \mathrm{PO}_{2l}(q)$,	
$\mathrm{SO}_{2l}(q) = (D_l)_{\mathrm{so}}(q)$,	$(D_l)_{\mathrm{so}}(q) = \mathrm{SO}_{2l}(q)$,	
$\mathrm{HS}_{2l}(q) = (D_l)_{\mathrm{HS}}(q)$,	$(D_l)_{\mathrm{HS}}(q) = \mathrm{HS}_{2l}(q)$,	l even,
$\mathrm{PO}_{2l}(q) = (D_l)_{\mathrm{ad}}(q)$,	$(D_l)_{\mathrm{sc}}(q) = \mathrm{Spin}_{2l}(q)$,	
$\mathrm{SU}_{l+1}(q) = ({}^2A_l)_{\mathrm{sc}}(q^2)$,	$({}^2A_l)_{\mathrm{ad}}(q^2) = \mathrm{PU}_{l+1}(q)$,	
$\vdots$	$\vdots$	
$\mathrm{PU}_{l+1}(q) = ({}^2A_l)_{\mathrm{ad}}(q^2)$,	$({}^2A_l)_{\mathrm{sc}}(q^2) = \mathrm{SU}_{l+1}(q)$,	
$\mathrm{Spin}^-_{2l}(q) = ({}^2D_l)_{\mathrm{sc}}(q^2)$,	$({}^2D_l)_{\mathrm{ad}}(q^2) = \mathrm{PO}^-_{2l}(q)$,	
$\mathrm{SO}^-_{2l}(q) = ({}^2D_l)_{\mathrm{so}}(q^2)$,	$({}^2D_l)_{\mathrm{so}}(q^2) = \mathrm{SO}^-_{2l}(q)$,	
$\mathrm{PO}^-_{2l}(q) = ({}^2D_l)_{\mathrm{ad}}(q^2)$,	$({}^2D_l)_{\mathrm{sc}}(q^2) = \mathrm{Spin}^-_{2l}(q)$.	

O^-, $Spin^-$, being relative to a non-singular quadratic form which is not of maximal index over the finite field with q elements. We note that all the finite groups G_σ in a given isogeny class and with given σ-action on the Dynkin diagram have the same order [**1**, p. 371].

3. Reductive subgroups and induced symmetries

Let G_1 be a σ-stable reductive subgroup of G of maximal rank. Let T be a σ-stable maximal torus of G_1, so also of G. Let Φ be the root system of G with respect to T and Φ_1 be the root system of G_1. Φ is a subset of the character group X of T. Let W be the Weyl group of G and W_1 the Weyl group of G_1. Let Δ_1 be the Dynkin diagram of G_1.

LEMMA 1. *Let $W_2 = W_1^\perp$ be the orthogonal Weyl subgroup to W_1 in W. (W_2 is generated by the reflections in the hyperplanes orthogonal to roots orthogonal to all roots in Φ_1.) Then $W_1 \times W_2$ is a normal subgroup of $\mathscr{N}_W(W_1)$, and $\mathscr{N}_W(W_1)/W_1 \times W_2$ is isomorphic to* $\mathrm{Aut}_W(\Delta_1)$, *the group of symmetries induced by W on Δ_1.*

Proof. See [**4**, Proposition 28].

COROLLARY 2. *There exists a natural homomorphism*

$$\mathscr{N}_W(W_1)/W_1 \to \mathrm{Aut}_W(\Delta_1).$$

LEMMA 3. *Let Π_1 be a fundamental system in Φ_1 and define*

$$\mathscr{N}_W(\Pi_1) = \{w \in W : w(\Pi_1) = \Pi_1\}.$$

Then we have $\mathscr{N}_W(W_1) = W_1 \mathscr{N}_W(\Pi_1)$ and $W_1 \cap \mathscr{N}_W(\Pi_1) = 1$.

Proof. Let $w \in \mathscr{N}_W(W_1)$. Then $w(\Phi_1) = \Phi_1$. Hence $w(\Pi_1) \subseteq \Phi_1$. $w(\Pi_1)$ is a fundamental system in Φ_1 so there exists $w_1 \in W_1$ such that

$$w(\Pi_1) = w_1(\Pi_1).$$

Hence $w_1^{-1} w(\Pi_1) = \Pi_1$, and so $W = W_1 \mathscr{N}_W(\Pi_1)$. Moreover,

$$W_1 \cap \mathscr{N}_W(\Pi_1) = \mathscr{N}_{W_1}(\Pi_1) = 1,$$

since the only element of the Weyl group which stabilizes a fundamental system is the identity.

COROLLARY 4. *$\mathscr{N}_W(W_1)/W_1$ is isomorphic to $\mathscr{N}_W(\Pi_1)$.*

COROLLARY 5. *W_2 is a normal subgroup of $\mathscr{N}_W(\Pi_1)$, and $\mathscr{N}_W(\Pi_1)/W_2$ is isomorphic to* $\mathrm{Aut}_W(\Delta_1)$.

Proof. Every element of W which stabilizes Π_1 stabilizes the orthogonal system of Π_1, viz. Φ_2, and so fixes the Weyl group W_2 of Φ_2. Hence W_2 is normal in $\mathscr{N}_W(\Pi_1)$. Moreover, we have

$$\mathscr{N}_W(\Pi_1)/W_2 \simeq \mathscr{N}_W(W_1)/W_1 \times W_2 \simeq \mathrm{Aut}_W(\Delta_1),$$

by Lemmas 1 and 3.

Now the G_σ-orbits on the set $\mathscr{C}$ of σ-stable conjugates of G_1 in G are in bijective correspondence with the set of σ-conjugacy classes in $\mathscr{N}_W(W_1)/W_1$. $W_1 w'$ is σ-conjugate to $W_1 w''$ if and only if there exists $w \in W$ such that

$$W_1 w'' = (W_1 w)^\sigma (W_1 w')(W_1 w)^{-1}.$$

This correspondence is derived as follows. If $G_1{}^g$ is σ-stable then $g^\sigma g^{-1} = n$ lies in $N = \mathscr{N}_G(T)$, so under the projection map $\pi\colon N \to W = N/T$ we have $\pi(n) = w$. This element w lies in $\mathscr{N}_W(W_1)$ so gives rise to a σ-conjugacy class in $\mathscr{N}_W(W_1)/W_1$ [5]. We shall use frequently the following result about the structure of the semisimple part M^g of such a σ-stable conjugate of G_1.

Proposition 6. *Let $g \in G$ satisfy $g^\sigma g^{-1} = n \in N$, where*

$$\pi(n) = w \in \mathscr{N}_W(W_1).$$

Let w map to τ under the natural homomorphism $\mathscr{N}_W(W_1) \to \mathrm{Aut}_W(\Delta_1)$. Then $(M^g)_\sigma$ is isomorphic to $M_{\sigma\tau}$. (τ is here interpreted as the graph automorphism of M corresponding to the given symmetry of the Dynkin diagram of M.)

Proof. See [5].

We shall now consider the individual types A_l, B_l, C_l, D_l separately with the aim of determining information about the semisimple part $(M^g)_\sigma$ and toral part $(S^g)_\sigma$ of the group $(G_1{}^g)_\sigma$ of σ-stable elements in the σ-stable conjugate $G_1{}^g$ of G_1.

Type A_l

Suppose G has type A_l. The endomorphism σ of G induces an endomorphism of the character group X of T, also called σ, which has the property that $\sigma = q\sigma_0$, where q is a power of p and σ_0 is an isometry of X. σ_0 has order 1 or 2 depending upon whether G_σ is split or twisted. X contains the set Φ of roots, and Φ can be written conveniently in the form

$$\Phi = \{e_i - e_j\colon i \neq j,\ i,j \in \{0, 1, \ldots, l\}\},$$

where $e_0, e_1, \ldots, e_l$ form an orthonormal basis of an $(l+1)$-dimensional euclidean space. The Weyl group W acts on this space by permuting the basis elements according to the symmetric group S_{l+1}. σ_0 acts on the roots either as the identity or as an element of order 2.

The root system of any σ-stable reductive subgroup of G is equivalent under W to a system Φ_1 of the following type. Let $\lambda = (\lambda_1, \lambda_2, \ldots)$ be a partition of $l+1$ and let $I_1, I_2, \ldots$ be disjoint subsets of $\{0, 1, \ldots, l\}$ with

$$|I_1| = \lambda_1, \quad |I_2| = \lambda_2, \quad \ldots.$$

Let $\Phi_1 = \{e_i - e_j \in \Phi : i,j \in I_\alpha \text{ for some } \alpha\}$. Then Φ_1 is a subsystem of Φ of type $A_{\lambda_1 - 1} \times A_{\lambda_2 - 1} \times \ldots$, and will be σ-stable provided that, when σ_0 has order 2, Φ_1 is stable under the linear map defined by $e_i \to -e_{l-i}$. For each i, let n_i be the number of parts of λ equal to i. Then the orthogonal root system Φ_2 is of type A_{n_1-1} and consists of all $e_i - e_j \in \Phi$ with $i \in I_\alpha$, $j \in I_\beta$, and $|I_\alpha| = |I_\beta| = 1$. $\mathcal{N}_W(W_1)$ consists of all permutations which permute among themselves the I_α of a given size. Thus $\mathcal{N}_W(W_1)/W_1$ is isomorphic to $S_{n_1} \times S_{n_2} \times \ldots$, and this is also isomorphic to $\mathcal{N}_W(\Pi_1)$, by Corollary 4. W_2 is a normal subgroup of $\mathcal{N}_W(\Pi_1)$ isomorphic to S_{n_1}. Thus W_2 has a complement in $\mathcal{N}_W(\Pi_1)$ isomorphic to $\mathrm{Aut}_W(\Delta_1)$, and we have

$$\mathrm{Aut}_W(\Delta_1) \simeq S_{n_2} \times S_{n_3} \times \ldots.$$

Each element of $\mathrm{Aut}_W(\Delta_1)$ gives a permutation of the I_α of size i for $i = 2, 3, 4, \ldots$.

Now σ acts on $\mathcal{N}_W(\Pi_1) \simeq S_{n_1} \times S_{n_2} \times \ldots$, and σ acts on each S_{n_i} individually. Thus two elements $w_1, w_2 \in \mathcal{N}_W(\Pi_1)$ are σ-conjugate if and only if their components in S_{n_i} are σ-conjugate in S_{n_i} for each i. Let τ_1, τ_2 be the symmetries determined from w_1, w_2 as in Corollary 2. Then w_1, w_2 are σ-conjugate in S_{n_i} if and only if $\sigma_0\tau_1$ and $\sigma_0\tau_2$, when regarded as permutations of the components of Φ_1 of type A_{i-1}, have the same cycle type. Let $\tau \in \mathrm{Aut}_W(\Delta_1)$. We describe the structure of the components of the semisimple subgroup $(M^g)_\sigma$ of $(G_1{}^g)_\sigma$ when $G_1{}^g$ is a σ-stable conjugate of G_1 giving rise to the induced symmetry τ. Since $\sigma_0\tau \in S_{n_2} \times S_{n_3} \times \ldots$, $\sigma_0\tau$ determines, by its cycle type on S_{n_i}, a partition $\mu^{(i)}$ of n_i for each $i = 2, 3, 4, \ldots$. For each part $\mu^{(i)}{}_j$ of $\mu^{(i)}$ we have a cyclic permutation of $\mu^{(i)}{}_j$ components A_{i-1}. We now distinguish between the two possible cases for σ_0.

(i) Suppose $\sigma_0 = 1$. Then τ fixes the product of the $\mu^{(i)}{}_j$ components A_{i-1} and by [5] gives rise to the twisted form of a group of type

$$A_{i-1} \times A_{i-1} \times \ldots \times A_{i-1} \quad (\mu^{(i)}{}_j \text{ terms})$$

obtained by combining the cyclic graph automorphism of order $\mu^{(i)}{}_j$ with a field automorphism of the same order over the finite field $\mathrm{GF}(q^{\mu^{(i)}{}_j})$ and taking the fixed points of the product. This procedure gives a group of type $A_{i-1}(q^{\mu^{(i)}{}_j})$. Thus the simple components of the semisimple group $(M^g)_\sigma$ are of the form $A_{i-1}(q^{\mu^{(i)}{}_j})$. Thus we have:

PROPOSITION 7. *Let G be a group of type A_l and let σ be such that G_σ is split. Let G_1 be a reductive subgroup of maximal rank in G corresponding to a partition λ of $l+1$. Let $G_1{}^g$ be a σ-stable reductive subgroup of G obtained by twisting G_1 by an element $w \in W$ defined by $\pi(g^\sigma g^{-1}) = w$. Suppose w maps to τ under the homomorphism $\mathcal{N}_W(W_1) \to \mathrm{Aut}_W(\Delta_1)$. Let n_i be the number of parts of λ equal to i, so that $\mathrm{Aut}_W(\Delta_1) \simeq S_{n_2} \times S_{n_3} \times \ldots$. Suppose τ gives rise to partitions $\mu^{(2)}, \mu^{(3)}, \ldots$ of $n_2, n_3, \ldots$ respectively. Then the* *simple components of the semisimple group* $(M^g)_\sigma$ *are of type* $A_{i-1}(q^{\mu^{(i)}_j})$ *with one component for each* $i = 2, 3\ldots$ *and each part* $\mu^{(i)}_j$ *of* $\mu^{(i)}$.

The order of the toral part $(S^g)_\sigma$ *of* $(G_1{}^g)_\sigma$ *is given by*

$$(q-1)\,|(S^g)_\sigma| = \prod_{i,j} (q^{\mu^{(i)}_j} - 1).$$

Proof. It remains to prove only the formula for the order of the torus $(S^g)_\sigma$. By [5], $(S^g)_\sigma$ is isomorphic to $X/\bar{P}_1/(q\sigma_0 w - 1)X/\bar{P}_1$, where $\bar{P}_1$ consists of these elements of X expressible as rational combinations of roots in Φ_1. It follows from this that the order of $(S^g)_\sigma$ is $\chi(q)$, where $\chi(t)$ is the characteristic polynomial of $\sigma_0 w$ on the vector space $X \otimes Q/\bar{P}_1 \otimes Q$.

Now we have

$$X = \{\sum a_i e_i : a_i \in Z, \sum a_i = 0\}.$$

In $X/\bar{P}_1$ we identify those e_i coming from the same component of G_1. Thus

$$X/\bar{P}_1 = \{\sum a_i \bar{e}_i : a_i \in Z, \sum a_i = 0\},$$

with one term $\bar{e}_i$ for each component of G_1. We then obtain a contribution to the characteristic polynomial $\chi(t)$ from each cycle of $\sigma_0 w$ on the components of G_1. In the present case $\sigma_0 = 1$, and for each cycle of τ of length $\mu^{(i)}_j$ of components of type A_{i-1} we obtain a contribution $t^{\mu^{(i)}_j} - 1$ to $\chi(t)$. Thus, we have

$$\chi(t) = \prod_{i,j} (t^{\mu^{(i)}_j} - 1)/(t-1),$$

dividing by $t-1$ because of the condition $\sum a_i = 0$.

(ii) Now suppose σ_0 has order 2. Given $\tau \in \mathrm{Aut}_W(\Delta_1)$ we again wish to describe the structure of $(M^g)_\sigma$ where $\pi(g^\sigma g^{-1}) = w$ maps to τ. We again consider the cycles of $\sigma_0\tau$ on the components of type A_{i-1}. Now we have

$$(M^g)_\sigma \simeq M_{\sigma\tau} = M_{q\sigma_0\tau},$$

and this has a simple component for each r-cycle which is isomorphic to $A_{i-1}(q^r)$ or ${}^2A_{i-1}(q^{2r})$ depending on whether $(\sigma_0\tau)^r$ twists the component A_{i-1} or not. Calculation shows that if r is even then $(\sigma_0\tau)^r \in W$, so cannot twist A_{i-1}, whereas if r is odd $(\sigma_0\tau)^r \in \sigma_0 W$, so must twist A_{i-1}. Hence one obtains a component $A_{i-1}(q^{\mu^{(i)}_j})$ when $\mu^{(i)}_j$ is even, and ${}^2A_{i-1}(q^{2\mu^{(i)}_j})$ when $\mu^{(i)}_j$ is odd. Thus we have:

PROPOSITION 8. *Let G be a group of type* A_l *and let* G_σ *be the twisted form of G. Let* G_1 *be a* σ*-stable reductive subgroup of maximal rank in G corresponding to a partition* λ *of* $l+1$. *Let* $G_1{}^g$ *be a* σ*-stable reductive subgroup of G obtained by twisting* G_1 *by an element* $w \in W$ *defined by* $\pi(g^\sigma g^{-1}) = w$. *Suppose w maps to* τ *under the homomorphism* $\mathcal{N}_W(W_1) \to \mathrm{Aut}_W(\Delta_1)$. *Let* n_i *be the number of parts of* λ *equal to i, so that* $\mathrm{Aut}_W(\Delta_1) \simeq S_{n_2} \times S_{n_3} \times \ldots$. *Suppose*

$\sigma_0 \tau$ gives rise to partitions $\mu^{(2)}, \mu^{(3)}, \ldots$ of $n_2, n_3, \ldots$ respectively. Then the simple components of the semisimple group $(M^g)_\sigma$ are of type $A_{i-1}(q^{\mu^{(i)}_j})$ for $\mu^{(i)}_j$ even and of type ${}^2A_{i-1}(q^{2\mu^{(i)}_j})$ for $\mu^{(i)}_j$ odd.

The order of the toral part $(S^g)_\sigma$ is given by

$$(q+1)\,|(S^g)_\sigma| = \prod_{\substack{i,j \\ \mu^{(i)}_j \text{ even}}} (q^{\mu^{(i)}_j}-1) \prod_{\substack{i,j \\ \mu^{(i)}_j \text{ odd}}} (q^{\mu^{(i)}_j}+1).$$

Proof. Again we have verified everything except the order of $(S^g)_\sigma$. As in the proof of Proposition 7 we must consider the characteristic polynomial $\chi(t)$ of $\sigma_0 w$ on $X/\bar{P}_1 \otimes Q$. For each r-cycle of $\sigma_0 \tau$ on the components of G, we obtain a contribution to $\chi(t)$ of t^r-1 if r is even and t^r+1 if r is odd. Finally, we must divide the product by $t+1$ to take account of the condition $\sum a_i = 0$ in the definition of X.

Type C_l

Suppose G has type C_l and the characteristic of K is not 2. The root system Φ of G may be written in the form

$$\Phi = \{\pm e_i \pm e_j,\ \pm 2e_i : i \neq j\},$$

with $i,j \in \{1,2,\ldots,l\}$, where $e_1, e_2, \ldots, e_l$ form an orthonormal basis of an l-dimensional euclidean space. The subsystems Φ_1 of Φ which are additively closed (i.e. which satisfy $Z\Phi_1 \cap \Phi = \Phi_1$) may be obtained by the algorithm of Borel and de Siebenthal [3]. Every reductive subgroup $G_1 = \langle T, X_r, r \in \Phi_1 \rangle$ has a root system Φ_1 which is additively closed (although this is false in characteristic 2). The additively closed subsystems Φ_1 are obtained as follows. Let λ, μ be partitions with $|\lambda|+|\mu| = l$. Let $\lambda = (\lambda_1, \lambda_2, \ldots)$, $\mu = (\mu_1, \mu_2, \ldots)$. Let $I_1, I_2, \ldots, J_1, J_2, \ldots$ be subsets of $\{1,2,\ldots,l\}$ such that $|I_\alpha| = \lambda_\alpha$, $|J_\alpha| = \mu_\alpha$, and $\{1,2,\ldots,l\}$ is a disjoint union of $I_1, I_2, \ldots$ and $J_1, J_2, \ldots$. Let Φ_1 be given by

$$\Phi_1 = \bigcup_\alpha \{e_i - e_j : i \neq j,\ i,j \in I_\alpha\} \bigcup_\beta \{\pm e_i \pm e_j,\ \pm 2e_i : i \neq j, i,j \in J_\beta\}.$$

Then Φ_1 is the root system of a semisimple subgroup of type

$$A_{\lambda_1-1} \times A_{\lambda_2-1} \times \ldots \times C_{\mu_1} \times C_{\mu_2} \times \ldots$$

and any additively closed subsystem is equivalent under W to one of these given ones.

Now W has order $2^l l\,!$ and $w(e_i) = \pm e_j$ for all w, i, and for appropriate j. $W_1 = W(\Phi_1)$ has order $\lambda_1!\,\lambda_2!\ldots 2^{\mu_1}\mu_1!\,2^{\mu_2}\mu_2!\ldots$. Consider the subgroup $\mathcal{N}_W(W_1)$. This consists of all elements of W which permute among themselves the components A_α of a given rank and the components C_α of a given rank. Let m_i be the number of parts of λ equal to i and n_i be the number of parts of μ equal to i. Then we have

$$|\mathcal{N}_W(W_1)/W_1| = 2^{m_1}m_1!\,2^{m_2}m_2!\ldots n_1!\,n_2!\ldots.$$

The orthogonal subsystem Φ_2 has type $C_{m_1} \times A_1 \times \ldots \times A_1$ with m_2 factors A_1, thus $|W_2| = 2^{m_1}m_1!\,2^{m_2}$. Hence we have

$$|\mathcal{N}_W(\Pi_1)/W_2| = m_2!\,2^{m_3}m_3!\,2^{m_4}m_4!\ldots n_1!\,n_2!\ldots.$$

Now W_2 is complemented in $\mathcal{N}_W(\Pi_1)$ by the subgroup fixing e_i for all i in a subset I_α with $|I_\alpha| = 1$. This subgroup may therefore be identified with $\mathrm{Aut}_W(\Delta_1)$. We have

$$\mathrm{Aut}_W(\Delta_1) \cong S_{m_2} \times (Z_2 \wr S_{m_3}) \times (Z_2 \wr S_{m_4}) \times \ldots \times S_{n_1} \times S_{n_2} \times \ldots,$$

where $Z_2 \wr S_m$ denotes the wreath product of Z_2 and S_m of order $2^m m!$. Let $\tau \in \mathrm{Aut}_W(\Delta_1)$. τ determines elements of

$$S_{m_2}, Z_2 \wr S_{m_3}, Z_2 \wr S_{m_4}, \ldots, S_{n_1}, S_{n_2} \ldots.$$

Now each element of a symmetric group can be expressed as a product of disjoint cycles, and each element of $Z_2 \wr S_m$ can be expressed as a product of disjoint positive and negative cycles [4, Proposition 24]. A cycle $e_{i_1} \to \pm e_{i_2} \to \ldots \to \pm e_{i_1}$ is positive if the sign of the concluding e_{i_1} is positive and negative otherwise. Thus the component of τ in $Z_2 \wr S_{m_i}$ determines a pair of partitions $\xi^{(i)}, \eta^{(i)}$ with $|\xi^{(i)}|+|\eta^{(i)}| = m_i$, where the parts of $\xi^{(i)}$ give the lengths of the positive cycles and the parts of $\eta^{(i)}$ give the lengths of the negative cycles. Also the component of τ in S_{n_i} determines a partition $\zeta^{(i)}$ of n_i giving the lengths of the cycles.

Now consider the semisimple subgroup $(M^g)_\sigma$ of $(G_1{}^g)_\sigma$, where $\pi(g^\sigma g^{-1}) = w$ and $w \to \tau \in \mathrm{Aut}_W(\Delta_1)$, as in Proposition 6. Proposition 6 implies that for each part $\xi^{(i)}{}_j$ of $\xi^{(i)}$ this group has a component of type $A_{i-1}(q^{\xi^{(i)}{}_j})$ and for each part $\eta^{(i)}{}_j$ of $\eta^{(i)}$ there is a component of type ${}^2A_{i-1}(q^{2\eta^{(i)}{}_j})$. Moreover, for each part $\zeta^{(i)}{}_j$ of $\zeta^{(i)}$ there is a component of type $C_i(q^{\zeta^{(i)}{}_j})$. Thus we have:

PROPOSITION 9. *Let G be a group of type C_l over an algebraically closed field of characteristic $p \neq 2$. Let G_1 be a reductive subgroup of maximal rank in G corresponding to a pair of partitions λ, μ with $|\lambda|+|\mu| = l$. Let m_i be the number of parts of λ equal to i and n_i be the number of parts of μ equal to i. Then* $\mathrm{Aut}_W(\Delta_1) \cong S_{m_2} \times (Z_2 \wr S_{m_3}) \times (Z_2 \wr S_{m_4}) \times \ldots \times S_{n_1} \times S_{n_2} \times \ldots$. *Let $G_1{}^g$ be a σ-stable reductive subgroup of G obtained by twisting G_1 by an element $w \in W$ defined by $\pi(g^\sigma g^{-1}) = w$. Suppose w maps to τ under the natural homomorphism $\mathcal{N}_W(W_1) \to$* $\mathrm{Aut}_W(\Delta_1)$. *Suppose τ gives rise to a pair of partitions $\xi^{(i)}, \eta^{(i)}$ with $|\xi^{(i)}|+|\eta^{(i)}| = m_i$ ($\eta^{(i)}$ is vacuous if $i = 2$) and partitions $\zeta^{(i)}$ with $|\zeta^{(i)}| = n_i$. Then the simple components of the semisimple group $(M^g)_\sigma$ are of type $A_{i-1}(q^{\xi^{(i)}{}_j})$, ${}^2A_{i-1}(q^{2\eta^{(i)}{}_j})$, $C_i(q^{\zeta^{(i)}{}_j})$ with one component for each part of each $\xi^{(i)}, \eta^{(i)}, \zeta^{(i)}$.*

The order of the torus $(S^g)_\sigma$ is given by

$$|(S^g)_\sigma| = \prod_{i,j} (q^{\xi^{(i)}{}_j} - 1) \prod_{i,j} (q^{\eta^{(i)}{}_j} + 1).$$

Type D_l

Suppose G has type D_l. Then the root system Φ of G may be written in the form $\Phi = \{\pm e_i \pm e_j : i \neq j,\ i,j \in \{1,2,\ldots,l\}\}$, where $e_1, e_2, \ldots, e_l$ form an orthonormal basis of l-dimensional euclidean space. The Weyl group $W(\Phi)$ has order $2^{l-1} l!$ and consists of all elements w such that $w(e_i) = \pm e_j$ and with an even number of negative signs for $i \in \{1,2,\ldots,l\}$. Let λ, μ be a pair of partitions with $|\lambda|+|\mu| = l$ such that no part of μ is equal to 1. Let $\lambda = (\lambda_1, \lambda_2, \ldots)$, $\mu = (\mu_1, \mu_2, \ldots)$. Let $I_1, I_2, \ldots, J_1, J_2, \ldots$ be subsets of $\{1,2,\ldots,l\}$ such that $|I_\alpha| = \lambda_\alpha$, $|J_\alpha| = \mu_\alpha$, and $\{1,2,\ldots,l\}$ is a disjoint union of $I_1, I_2, \ldots, J_1, J_2, \ldots$. Let Φ_1 be the subset of Φ given by

$$\Phi_1 = \bigcup_\alpha \{e_i - e_j : i \neq j,\ i,j \in I_\alpha\} \bigcup_\beta \{\pm e_i \pm e_j : i \neq j,\ i,j \in J_\beta\}.$$

Then Φ_1 is a subsystem of Φ of type $A_{\lambda_1-1} \times A_{\lambda_2-1} \times \ldots \times D_{\mu_1} \times D_{\mu_2} \times \ldots$. Moreover, any subsystem of Φ is isomorphic to some such Φ_1, and is equivalent to Φ_1 under some automorphism of Φ, although not necessarily under W. It is therefore sufficient for our purpose to consider such systems Φ_1. Now $W_1 = W(\Phi_1)$ has order $\lambda_1! \lambda_2! \ldots 2^{\mu_1-1}\mu_1! \, 2^{\mu_2-1}\mu_2! \ldots$. Consider the subgroup $\mathcal{N}_W(W_1)$. The elements of $\mathcal{N}_W(W_1)$ permute among themselves the components A_α in Φ of a given rank and also the components D_α of a given rank. Moreover, $|\mathcal{N}_W(W_1)/W_1| = |W_2|\,|\mathrm{Aut}_W(\Delta_1)|$. Let m_i be the number of α with $\lambda_\alpha = i$, and let n_i be the number of β with $\mu_\beta = i$. The subsystem Φ_2 orthogonal to Φ_1 has type

$$D_{m_1} \times A_1 \times A_1 \times \ldots \times A_1$$

with m_2 factors A_1 if $m_1 \geqslant 2$ and type $A_1 \times A_1 \times \ldots \times A_1$ if $m_1 = 0$ or 1. Thus

$$|W_2| = \begin{cases} 2^{m_1-1} m_1! \, 2^{m_2} & \text{if } m_1 \geqslant 1, \\ 2^{m_2} & \text{if } m_1 = 0. \end{cases}$$

Now consider the group $\mathrm{Aut}_W(\Delta_1)$ of symmetries of the diagram of Φ_1 induced by W. This group of symmetries contains permutations of isomorphic components and also the non-trivial symmetry of A_α, for $\alpha \geqslant 2$, and D_α, for $\alpha \geqslant 2$ (although not the symmetries of order 3 in D_4). If $m_1 \geqslant 1$ we have

$$\mathrm{Aut}_W(\Delta_1) \cong S_{m_2} \times (Z_2 \wr S_{m_3}) \times (Z_2 \wr S_{m_4}) \times \ldots \times (Z_2 \wr S_{n_2}) \times (Z_2 \wr S_{n_3}) \times \ldots,$$

since the element $w \in W$ inducing such a symmetry can be made to change an even number of signs by choosing its action appropriately in the orthogonal space of Φ_1 in V. If $m_1 = 0$, however, $\mathrm{Aut}_W(\Delta_1)$ is a subgroup of index 2 in the above. In order to describe this subgroup of index 2 we consider the decomposition into positive and negative cycles of an element in $\mathrm{Aut}_W(\Delta_1)$. Each element of S_{m_2} is a product of positive cycles and each element of $Z_2 \wr S_{m_3}, Z_2 \wr S_{m_4}, \ldots, Z_2 \wr S_{n_2}, Z_2 \wr S_{n_3}, \ldots$ is a product of positive or negative cycles. Now a negative cycle for components A_α, with α even, is induced by an element $w \in W$ which changes an odd number of signs of the e_i occurring in the components in the cycle. A negative cycle for components D_α is also induced by an element w which changes an odd number of signs. However, a negative cycle for components A_α, with α odd, is induced by an element $w \in W$ which changes an even number of signs. Now each $w \in W$ changes an even number of signs altogether. Thus, if $m_1 = 0$, $\mathrm{Aut}_W(\Delta_1)$ is the subgroup of

$$S_{m_2} \times (Z_2 \wr S_{m_3}) \times (Z_2 \wr S_{m_4}) \times \ldots \times (Z_2 \wr S_{n_2}) \times (Z_2 \wr S_{n_3}) \times \ldots$$

of index 2 consisting of all elements whose components in

$$(Z_2 \wr S_{m_3}) \times (Z_2 \wr S_{m_5}) \times (Z_2 \wr S_{m_7}) \times \ldots \times (Z_2 \wr S_{n_2}) \times (Z_2 \wr S_{n_3}) \times \ldots$$

have an even number of negative cycles. We have

$$|\mathcal{N}_W(W_1)/W_1| = \tfrac{1}{2}(2^{m_1}m_1!\,2^{m_2}m_2!\ldots 2^{n_2}n_2!\,2^{n_3}n_3!\ldots)$$

in all cases.

Let $\tau \in \mathrm{Aut}_W(\Delta_1)$. Then $\sigma_0\tau$ has components in

$$S_{m_2}, Z_2 \wr S_{m_3}, Z_2 \wr S_{m_4}, \ldots, Z_2 \wr S_{n_2}, Z_2 \wr S_{n_3}, \ldots.$$

The component of $\sigma_0\tau$ in $Z_2 \wr S_{m_i}$ determines a pair of partitions $\xi^{(i)}, \eta^{(i)}$ with $|\xi^{(i)}|+|\eta^{(i)}| = m_i$ (a single partition $\xi^{(i)}$ when $i = 2$). The parts of $\xi^{(i)}$ give the lengths of the positive cycles, and the parts of $\eta^{(i)}$ give the lengths of the negative cycles. Also the component of $\sigma_0\tau$ in $Z_2 \wr S_{n_i}$ determines likewise a pair of partitions $\zeta^{(i)}, \omega^{(i)}$ with $|\zeta^{(i)}|+|\omega^{(i)}| = n_i$, the parts of $\zeta^{(i)}, \omega^{(i)}$ giving the lengths of the positive and negative cycles respectively.

Consider the semisimple subgroup $(M^g)_\sigma$ of $(G_1{}^g)_\sigma$, where $\pi(g^\sigma g^{-1}) = w$ and w maps to $\tau \in \mathrm{Aut}_W(\Delta_1)$. For each part $\xi^{(i)}{}_j$ of $\xi^{(i)}$ this group has a component of type $A_{i-1}(q^{\xi^{(i)}{}_j})$ and for each part $\eta^{(i)}{}_j$ of $\eta^{(i)}$ there is a component of type ${}^2A_{i-1}(q^{2\eta^{(i)}{}_j})$. Moreover, for each part $\zeta^{(i)}{}_j$ of $\zeta^{(i)}$ there is a component of type $D_i(q^{\zeta^{(i)}{}_j})$ and for each part $\omega^{(i)}{}_j$ of $\omega^{(i)}$ there is a component of type ${}^2D_i(q^{2\omega^{(i)}{}_j})$.

Thus we have:

PROPOSITION 10. *Let G be a group of type D_l over an algebraically closed field of characteristic p and let G_1 be a σ-stable reductive subgroup of maximal rank in G determined by a pair of partitions λ, μ with $|\lambda|+|\mu| = l$. Let m_i be the number of parts of λ equal to i, and let n_i be the number of parts of μ equal to i ($n_1 = 0$). Then*

$$\mathrm{Aut}_W(\Delta_1) \cong \begin{cases} S_{m_2} \times (Z_2 \wr S_{m_3}) \times (Z_2 \wr S_{m_4}) \times \ldots \times (Z_2 \wr S_{n_2}) \times (Z_2 \wr S_{n_3}) \times \ldots \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } m_1 > 0, \\ \text{a subgroup of index 2 in this} \quad \text{if } m_1 = 0. \end{cases}$$

Let $G_1{}^g$ be a σ-stable reductive subgroup of G obtained by twisting G_1 by an element $w \in W$ defined by $\pi(g^\sigma g^{-1}) = w$. Suppose w maps to $\tau \in \mathrm{Aut}_W(\Delta_1)$. Suppose $\sigma_0\tau$ gives rise to pairs of partitions $\xi^{(i)}, \eta^{(i)}$ with $|\xi^{(i)}|+|\eta^{(i)}| = m_i$ (where $\eta^{(i)}$ is vacuous if $i = 2$), and pairs of partitions $\zeta^{(i)}, \omega^{(i)}$ with $|\zeta^{(i)}|+|\omega^{(i)}| = n_i$. Then the simple components of the semisimple group $(M^g)_\sigma$ are of type $A_{i-1}(q^{\xi^{(i)}_j})$, ${}^2A_{i-1}(q^{2\eta^{(i)}_j})$, $D_i(q^{\zeta^{(i)}_j})$, ${}^2D_i(q^{2\omega^{(i)}_j})$ with one component for each part of each $\xi^{(i)}, \eta^{(i)}, \zeta^{(i)}, \omega^{(i)}$.

If $m_1 = 0$ the total number of components of type ${}^2A_{i-1}$, with i odd, and type ${}^2D_i$ is even if $\sigma_0 = 1$, and odd if $\sigma_0 \neq 1$.

The order of the torus $(S^g)_\sigma$ is given by $|(S^g)_\sigma| = \prod_{i,j}(q^{\xi^{(i)}_j}-1)\prod_{i,j}(q^{\eta^{(i)}_j}+1)$.

Type B_l

Suppose G has type B_l and the characteristic of K is not 2. The root system Φ of G has form $\Phi = \{\pm e_i \pm e_j, \pm e_i : i \neq j, i,j \in \{1,2,\dots,l\}\}$. The additively closed subsystems Φ_1 of Φ are obtained as follows. Let λ, μ be partitions with $|\lambda|+|\mu| \leqslant l$ and let $\nu = l-|\lambda|-|\mu|$. Suppose no part of μ is equal to 1. Let $I_1, I_2, \dots, J_1, J_2, \dots$ be subsets of $\{1,2,\dots,l\}$ such that $|I_\alpha| = \lambda_\alpha$, $|J_\alpha| = \mu_\alpha$, and let K be a subset with $|K| = \nu$. Suppose $\{1,2,\dots,l\}$ is a disjoint union of $I_1, I_2, \dots, J_1, J_2, \dots, K$. Let Φ_1 be defined by

$$\Phi_1 = \bigcup_\alpha \{e_i - e_j : i \neq j,\ i,j \in I_\alpha\} \bigcup_\beta \{\pm e_i \pm e_j : i \neq j,\ i,j \in J_\beta\}$$
$$\bigcup \{\pm e_i \pm e_j,\ \pm e_i : i \neq j,\ i,j \in K\}.$$

Then Φ_1 is an additively closed subsystem of type

$$A_{\lambda_1-1} \times A_{\lambda_2-1} \times \dots \times D_{\mu_1} \times D_{\mu_2} \times \dots \times B_\nu$$

and every additively closed subsystem of Φ is equivalent to some such Φ_1 under W. Let $W_1 = W(\Phi_1)$ and consider the subgroup $\mathcal{N}_W(W_1)$. The elements of $\mathcal{N}_W(W_1)$ permute among themselves the components A_α of a given rank and also the components D_α of a given rank. Let m_i be the number of parts of λ equal to i, and n_i be the number of parts of μ equal to i. We have $|\mathcal{N}_W(W_1)/W_1| = |W_2|.|\mathrm{Aut}_W(\Delta_1)|$. The subsystem Φ_2 orthogonal to Φ_1 has type $B_{m_1} \times A_1 \times A_1 \times \dots \times A_1$ with m_2 components A_1. Thus $|W_2| = 2^{m_1}m_1!\,2^{m_2}$. Now $\mathrm{Aut}_W(\Delta_1)$ contains permutations of isomorphic components and also the non-trivial symmetries of A_α, where $\alpha \geqslant 2$, and D_α, with $\alpha \geqslant 2$ (although not the symmetries of order 3 in D_4). We have

$$\mathrm{Aut}_W(\Delta_1) \cong S_{m_2} \times (Z_2 \wr S_{m_3}) \times (Z_2 \wr S_{m_4}) \times \dots \times (Z_2 \wr S_{n_2}) \times (Z_2 \wr S_{n_3}) \times \dots .$$

The situation is then similar to the above case of type D_l and we obtain the following result.

PROPOSITION 11. *Let G be a group of type B_l over an algebraically closed field of characteristic $p \neq 2$. Let G_1 be a reductive subgroup of maximal rank in G determined by a triple (λ, μ, ν), where λ, μ are partitions, ν is a non-negative integer, and $|\lambda|+|\mu|+\nu = l$. Let m_i be the number of parts of λ equal to i, and let n_i be the number of parts of μ equal to i ($n_1 = 0$). Then*

$$\mathrm{Aut}_W(\Delta_1) \simeq S_{m_2} \times (Z_2 \wr S_{m_3}) \times (Z_2 \wr S_{m_4}) \times \ldots \times (Z_2 \wr S_{n_2}) \times (Z_2 \wr S_{n_3}) \times \ldots.$$

Let $G_1{}^g$ be a σ-stable reductive subgroup of G obtained by twisting G_1 by an element $w \in W$ defined by $\pi(g^\sigma g^{-1}) = w$. Suppose w maps to $\tau \in \mathrm{Aut}_W(\Delta_1)$. Suppose τ gives rise to a pair of partitions $\xi^{(i)}, \eta^{(i)}$ with $|\xi^{(i)}|+|\eta^{(i)}| = m_i$ and a pair of partitions $\zeta^{(i)}, \omega^{(i)}$ with $|\zeta^{(i)}|+|\omega^{(i)}| = n_i$ such that the parts of these partitions give the lengths of the positive and negative cycles in the components of τ. Then the simple components of the semisimple group $(M^g)_\sigma$ are of type $A_{i-1}(q^{\xi^{(i)}_j})$, ${}^2A_{i-1}(q^{2\eta^{(i)}_j})$, $D_i(q^{\zeta^{(i)}_j})$, ${}^2D_i(q^{2\omega^{(i)}_j})$, $B_\nu(q)$ with one component for each part of each $\xi^{(i)}, \eta^{(i)}, \zeta^{(i)}, \omega^{(i)}$.

The order of the torus $(S^g)_\sigma$ is given by $|(S^g)_\sigma| = \prod_{i,j}(q^{\xi^{(i)}_j}-1)\prod_{i,j}(q^{\eta^{(i)}_j}+1)$.

Type B_l in characteristic 2

Suppose G has type B_l and K has characteristic 2. Then G is isomorphic as an abstract group to a group of type C_l over K. It will therefore be sufficient to consider the reductive subgroups of a group of type B_l in this case. The root system Φ of G has form

$$\Phi = \{\pm e_i \pm e_j,\ \pm e_i : i \neq j,\ i,j \in \{1,2,\ldots,l\}\}.$$

Since K has characteristic 2 we must consider all subsystems Φ_1 of Φ, not merely those which are additively closed, since all such subsystems give rise to reductive subgroups $G_1 = \langle T, X_r : r \in \Phi_1\rangle$ of G. The subsystems Φ_1 are obtained as follows. Let λ, μ, ν be partitions with $|\lambda|+|\mu|+|\nu| = l$, where no part of μ is equal to 1. Let $\lambda = (\lambda_1, \lambda_2, \ldots)$, $\mu = (\mu_1, \mu_2, \ldots)$, $\nu = (\nu_1, \nu_2, \ldots)$, and let $I_1, I_2, \ldots, J_1, J_2, \ldots, K_1, K_2, \ldots$ be disjoint subsets of $\{1, 2, \ldots, l\}$ with $|I_\alpha| = \lambda_\alpha$, $|J_\alpha| = \mu_\alpha$, $|K_\alpha| = \nu_\alpha$. Let Φ_1 be defined by

$$\Phi_1 = \bigcup_\alpha \{e_i - e_j : i \neq j,\ i,j \in I_\alpha\} \bigcup_\alpha \{\pm e_i \pm e_j : i \neq j,\ i,j \in J_\alpha\}$$
$$\bigcup_\alpha \{\pm e_i \pm e_j,\ \pm e_i : i \neq j,\ i,i \in K_\alpha\};$$

Φ_1 is a subsystem of type $A_{\lambda_1-1} \times A_{\lambda_2-1} \times \ldots \times D_{\mu_1} \times D_{\mu_2} \times \ldots \times B_{\nu_1} \times B_{\nu_2} \times \ldots$. Moreover, any subsystem of Φ is equivalent to some such Φ_1 under W. We have

$$|W_1| = \lambda_1!\,\lambda_2!\ldots 2^{\mu_1-1}\mu_1!\,2^{\mu_2-1}\mu_2!\ldots 2^{\nu_1}\nu_1!\,2^{\nu_2}\nu_2!\ldots.$$

The elements of $\mathscr{N}_W(W_1)$ permute among themselves the components A_α of a given rank in Φ, also the components D_α of a given rank and the components B_α of a given rank. Let m_i be the number of α with $\lambda_\alpha = i$, n_i be the number of α with $\mu_\alpha = i$, and p_i be the number of α with $\nu_\alpha = i$. The subsystem Φ_2 orthogonal to Φ_1 has type $B_{m_1} \times A_1 \times A_1 \times \ldots \times A_1$ with m_2 components A_1, thus $|W_2| = 2^{m_1} m_1!\, 2^{m_2}$. Consider the group $\mathrm{Aut}_W(\Delta_1)$. This contains permutations of isomorphic components and also the non-trivial symmetries of A_α, with $\alpha \geqslant 2$, and of D_α, with $\alpha \geqslant 2$ (but not the symmetries of order 3 in D_4). We have

$$\mathrm{Aut}_W(\Delta_1) \cong S_{m_2} \times (Z_2 \wr S_{m_3}) \times (Z_2 \wr S_{m_4}) \times \ldots \times (Z_2 \wr S_{n_2}) \times (Z_2 \wr S_{n_3}) \times \ldots \times S_{p_1} \times S_{p_2} \times \ldots .$$

We then obtain the following result.

PROPOSITION 12. *Let G be a group of type B_l over an algebraically closed field of characteristic 2. Let G_1 be a reductive subgroup of maximal rank in G determined by a triple of partitions λ, μ, ν with $|\lambda|+|\mu|+|\nu| = l$. Let m_i, n_i, p_i be the number of parts of λ, μ, ν respectively equal to i ($n_1 = 0$). Then*

$$\mathrm{Aut}_W(\Delta_1) \cong S_{m_2} \times (Z_2 \wr S_{m_3}) \times (Z_2 \wr S_{m_4}) \times \ldots \times (Z_2 \wr S_{n_2}) \times (Z_2 \wr S_{n_3}) \times \ldots \times S_{p_1} \times S_{p_2} \times \ldots .$$

Let $G_1{}^g$ be a σ-stable reductive subgroup of G obtained by twisting G_1 by an element $w \in W$ defined by $\pi(g^\sigma g^{-1}) = w$. Suppose w maps to $\tau \in \mathrm{Aut}_W(\Delta_1)$. Suppose τ gives rise to pairs of partitions $\xi^{(i)}, \eta^{(i)}$ with $|\xi^{(i)}|+|\eta^{(i)}| = m_i$, pairs of partitions $\zeta^{(i)}, \omega^{(i)}$ with $|\zeta^{(i)}|+|\omega^{(i)}| = n_i$, and partitions $\gamma^{(i)}$ with $|\gamma^{(i)}| = p_i$ such that the parts of these partitions give the lengths of the positive and negative cycles in the components of τ. Then the simple components of the semisimple group $(M^g)_\sigma$ are of type $A_{i-1}(q^{\xi^{(i)}_j})$, ${}^2A_{i-1}(q^{2\eta^{(i)}_j})$, $D_i(q^{\zeta^{(i)}_j})$, ${}^2D_i(q^{2\omega^{(i)}_j})$, $B_i(q^{\gamma^{(i)}_j})$, with one component for each part of each $\xi^{(i)}$, $\eta^{(i)}$, $\zeta^{(i)}$, $\omega^{(i)}$, $\gamma^{(i)}$.

The order of the torus $(S^g)_\sigma$ is given by

$$|(S^g)_\sigma| = \prod_{i,j} (q^{\xi^{(i)}_j} - 1) \prod_{i,j} (q^{\eta^{(i)}_j} + 1).$$

4. Criteria for being a connected centralizer

We now consider the following question: given a reductive subgroup G_1 of maximal rank in G and a twisting element $w \in \mathcal{N}_W(W_1)$, when is a corresponding subgroup $(G_1{}^g)_\sigma$ of G_σ (determined to within conjugacy in G_σ by § 2) the connected centralizer of a semisimple element of G_σ for values of q sufficiently large? For this to be true it is clearly necessary (though not in general sufficient) that G_1 should be the connected centralizer of some semisimple element of G.

We shall answer this question when G is a group of classical type A_l, B_l, C_l, D_l, and the answer will in general depend upon the isogeny type of G. We begin by giving a summary of results, which will be justified in the subsequent sections of the paper. These results are given in two tables. Table 1 gives the condition for a connected reductive subgroup G_1 of maximal rank in G to be the connected centralizer of some semisimple element of G. (We assume $p \neq 2$.)

TABLE 1

Type of G	Condition for G_1 to be the connected centralizer of a semisimple element
A_l	None
C_l	G_1 has at most two components of type C
B_l	G_1 has at most one component of type D
D_l	G_1 has at most two components of type D

Table 2 gives the additional condition (i.e. in addition to the necessary condition given in Table 1) for the finite group $(G_1{}^g)_\sigma$ to be the connected centralizer of a semisimple element in G_σ when q is sufficiently large. This additional condition will depend on the isogeny class of G and on the twisting element $w \in W$ given by $\pi(g^\sigma g^{-1}) = w$.

TABLE 2

Type of G	Isogeny class of G	Additional condition for $(G_1{}^g)_\sigma$
A_l	All	None
C_l	Adjoint	None
C_l	Simply-connected	If there are two components of type C these cannot be interchanged by w
B_l	Adjoint	None
B_l	Simply-connected	If there is a component of type D then $(G_1{}^g)_\sigma$ is not a critical subgroup of G_σ
D_l	Adjoint	None
D_l	SO_{2l}	If there are two components of type D these cannot be interchanged by w
D_l	Half-spin group	If there are two components of type D which are not interchanged by w then $(G_1{}^g)_\sigma$ is not a critical subgroup of G_σ of the first kind. If there are two components of type D which are interchanged by w then $(G_1{}^g)_\sigma$ is not a critical subgroup of G_σ of the second kind
D_l	Simply-connected	If there are two components of type D, these cannot be interchanged by w and $(G_1{}^g)_\sigma$ is not a critical subgroup of G_σ

In Table 2 a critical subgroup $(G_1{}^g)_\sigma$ of G_σ is defined as follows.

(a) If G is a simply-connected group of type B_l the critical subgroups of G_σ have form $(G_1{}^g)_\sigma$ where

(i) G_1 has type $A_{\lambda_1-1} \times A_{\lambda_2-1} \times \ldots \times D_\mu \times B_\nu$ with $\lambda_1 + \lambda_2 + \ldots + \mu + \nu = l$, $\mu \neq 0$, $\lambda_1, \lambda_2, \ldots$ all even, and

(ii) $(G_1{}^g)_\sigma$ has an untwisted component $D_\mu(q)$ if $\frac{1}{2}(q-1)\mu$ is odd, and a twisted component ${}^2D_\mu(q^2)$ if $\frac{1}{2}(q-1)\mu$ is even.

(b) If G is a simply-connected group of type D_l the critical subgroups of G_σ have form $(G_1{}^g)_\sigma$ where

(i) G_1 has type $A_{\lambda_1-1}\times A_{\lambda_2-1}\times \dots \times D_{\mu_1}\times D_{\mu_2}$ with

$$\lambda_1+\lambda_2+\dots+\mu_1+\mu_2 = l,$$

$\mu_1 \neq 0$, $\mu_2 \neq 0$, $\lambda_1, \lambda_2, \dots$ all even, and

(ii) $(G_1{}^g)_\sigma$ has an untwisted component $D_{\mu_i}(q)$ if $\frac{1}{2}(q-1)\mu_i$ is odd, and a twisted component ${}^2D_{\mu_i}(q^2)$ if $\frac{1}{2}(q-1)\mu_i$ is even, for $i = 1, 2$.

(c) If G is a half-spin group of type D_l the critical subgroups of G_σ of the first kind have form $(G_1{}^g)_\sigma$ where

(i) G_1 has type $A_{\lambda_1-1}\times A_{\lambda_2-1}\times \dots \times D_{\mu_1}\times D_{\mu_2}$ with

$$\lambda_1+\lambda_2+\dots+\mu_1+\mu_2 = l,$$

$\mu_1 \neq 0$, $\mu_2 \neq 0$, $\lambda_1, \lambda_2, \dots$ all even, and

(ii) $(G_1{}^g)_\sigma$ has an untwisted component $D_{\mu_i}(q)$ if $\frac{1}{2}(q-1)\mu_i$ is odd, and a twisted component ${}^2D_{\mu_i}(q^2)$ if $\frac{1}{2}(q-1)\mu_i$ is even, for $i = 1, 2$.

(d) If G is a half-spin group of type D_l the critical subgroups of G_σ of the second kind have form $(G_1{}^g)_\sigma$ where

(i) G_1 has type $A_{\lambda_1-1}\times A_{\lambda_2-1}\times \dots \times D_\mu \times D_\mu$ with $\lambda_1+\lambda_2+\dots+\mu+\mu = l$, $\mu \neq 0$, $\lambda_1, \lambda_2, \dots$ all even, and

(ii) $(G_1{}^g)_\sigma$ has a component $D_\mu(q^2)$; and $M \equiv \nu \bmod 2$ if $\frac{1}{2}(q-1)\mu$ is odd and $M \not\equiv \nu \bmod 2$ if $\frac{1}{2}(q-1)\mu$ is even.

Here M, ν are defined as follows. M is the number of simple components of form $A_{\lambda_i-1}(q^e)$ or ${}^2A_{\lambda_i-1}(q^{2e})$, where $\lambda_i \equiv 2 \bmod 4$ and e is odd. ν is 0 if w induces the positive graph automorphism on $D_\mu + D_\mu$ and 1 if w induces the negative graph automorphism. (For the definitions of the positive and negative graph automorphism, see Proposition 18.)

Finally, we explain the situation when K has characteristic 2 by giving the versions of Tables 1 and 2 which are valid in this case.

TABLE 1 (characteristic 2)

Type of G	Condition for G_1 to be the connected centralizer of a semisimple element
A_l, B_l, C_l, D_l	G_1 is the reductive part of some parabolic subgroup of G

TABLE 2 (characteristic 2)

Type of G	Isogeny class of G	Additional condition for $(G_1{}^g)_\sigma$
A_l, B_l, C_l, D_l	All	None

5. Groups of type C_l

In §§ 5, 6, and 7 we shall justify the statements made in Tables 1 and 2 of § 4. Throughout these sections G_1 will be a connected reductive subgroup of maximal rank in G, W_1 will be the Weyl group of G_1, w will be an element of $\mathcal{N}_W(W_1)$, and $G_1{}^g$ will be a σ-stable subgroup of G obtained by twisting G_1 by the element w, where $\pi(g^\sigma g^{-1}) = w$. T will be a σ-stable maximal torus of G_1, so also of G, and X will be the character group of T. P_1 will be the subgroup of X generated by Φ_1 and $\bar{P}_1/P_1$ will be the torsion subgroup of X/P_1. $\bar{\Phi}_1$ is defined by $\bar{\Phi}_1 = \Phi \cap \bar{P}_1$.

The results from [5] which we shall use are as follows.

[**5**, Proposition 11] *G_1 is the connected centralizer of some semisimple element of G if and only if $\bar{P}_1/P_1$ has a regular character of order prime to p (a regular character being one which does not annihilate $P_1 + r$ for any root $r \in \bar{\Phi}_1 - \Phi_1$).*

[**5**, Propositions 17, 19] *$(G_1{}^g)_\sigma$ is the connected centralizer of a semisimple element of G_σ for q sufficiently large if and only if the group*

$$\Gamma = X/(P_1 + (\sigma w - 1)X)$$

has a character which does not annihilate any root in $\bar{\Phi}_1 - \Phi_1$.

If G is of type A_l, then $(G_1{}^g)_\sigma$ is always the connected centralizer of a semisimple element of G_σ for q sufficiently large, since G_1 is a Levi subgroup of a parabolic subgroup of G and so $\bar{\Phi}_1 - \Phi_1$ is empty.

We now consider the situation when G has type C_l and when K has characteristic $p \neq 2$. Let $\Phi = \{\pm e_i \pm e_j,\ \pm 2e_i : i \neq j,\ i,j \in \{1, 2, \ldots, l\}\}$ be the root system of G, and let G_1 be the reductive subgroup with root system Φ_1 given by

$$\Phi_1 = \bigcup_\alpha \{e_i - e_j : i \neq j,\ i,j \in I_\alpha\} \bigcup_\alpha \{\pm e_i \pm e_j,\ \pm 2e_i : i \neq j,\ i,j \in J_\alpha\},$$

as in § 3. Here $|I_\alpha| = \lambda_\alpha$, $|J_\alpha| = \mu_\alpha$, and $\lambda = (\lambda_1, \lambda_2, \ldots)$, $\mu = (\mu_1, \mu_2, \ldots)$ satisfy $|\lambda| + |\mu| = l$. Φ_1 is a root system of type

$$A_{\lambda_1 - 1} \times A_{\lambda_2 - 1} \times \ldots \times C_{\mu_1} \times C_{\mu_2} \times \ldots.$$

We use the result of [**5**, Proposition 11]. Since $\bar{P}_1/P_1$ is the torsion subgroup of X/P_1 we have $\bar{P}_1/P_1 \cong Z_2 \oplus Z_2 \oplus \ldots \oplus Z_2$ with one component Z_2 for each part of μ. If μ has at least three parts, let $\bar{e}_1, \bar{e}_2, \bar{e}_3$ be images in $\bar{P}_1/P_1$ of elements $e_1, e_2, e_3 \in X$ in the components of type C corresponding to three parts of μ. Let ψ be any character of $\bar{P}_1/P_1$. Then $\psi(\bar{e}_1 - \bar{e}_2) = \pm 1$, $\psi(\bar{e}_2 - \bar{e}_3) = \pm 1$, $\psi(\bar{e}_1 - \bar{e}_3) = \pm 1$. Moreover, if

$$\psi(\bar{e}_1 - \bar{e}_2) = -1 \quad \text{and} \quad \psi(\bar{e}_2 - \bar{e}_3) = -1,$$

then $\psi(\bar{e}_1-\bar{e}_3) = 1$. Thus each character of $\bar{P}_1/P_1$ annihilates some element of $\bar{\Phi}_1-\Phi_1$. Thus $\bar{P}_1/P_1$ has no regular character in this case.

If μ has less than two parts then $\Phi_1 = \bar{\Phi}_1$. If μ has exactly two parts then $\bar{\Phi}_1-\Phi_1$ consists of a single non-zero element of $\bar{P}_1/P_1$. Thus there is some character of $\bar{P}_1/P_1$ which does not annihilate it. Thus if μ has less than three parts $\bar{P}_1/P_1$ has a regular character. This justifies the entry in Table 1 for type C_l.

In considering the corresponding problem for $(G_1{}^g)_\sigma$ we can thus restrict attention to the case when μ has at most two parts. If μ has less than two parts we have $\Phi_1 = \bar{\Phi}_1$ and G_1 is a Levi subgroup of a parabolic subgroup of G. The situation is clear in this case by [**5**, Theorem 21]. We may therefore assume that μ has exactly two parts. The result may depend upon the isogeny type of G, and we therefore consider separately the cases when G is simply-connected and adjoint.

The simply-connected case

Suppose G is simply-connected. Then the group X of rational characters of a maximal torus T of G is generated by the fundamental weights $q_1, q_2, \ldots, q_l$, which are defined by

$$(q_i, 2p_j/(p_j, p_j)) = \delta_{ij},$$

where $\Pi = \{p_1, p_2, \ldots, p_l\}$ is the set of fundamental roots. In the present context, since $e_1, e_2, \ldots, e_l$ form an orthonormal system, we may take

$$p_1 = e_1-e_2, \quad p_2 = e_2-e_3, \quad \ldots, \quad p_{l-1} = e_{l-1}-e_l, \quad p_l = 2e_l,$$

and we then have

$$q_1 = e_1, \quad q_2 = e_1+e_2, \quad \ldots, \quad q_{l-1} = e_1+e_2+\ldots+e_{l-1}, \quad q_l = e_1+e_2+\ldots+e_l.$$

Hence $X = \{\sum_{i=1}^{l} a_i e_i \colon a_i \in Z\}$. Now we must consider the quotient $\Gamma = X/P_1+(qw-1)X$ according to [**5**, Proposition 17]. We have

$$\Gamma \cong \frac{X/P_1}{(P_1+(qw-1)X)/P_1},$$

and we first consider X/P_1. Let $e_i \to \bar{e}_i$ under the natural homomorphism $X \to X/P_1$. Then $\bar{e}_i = \bar{e}_j$ when $i,j \in I_\alpha$, and $\bar{e}_i = \bar{e}_j$, $2\bar{e}_i = 0$ when $i,j \in J_\alpha$. Thus X/P_1 is generated by elements $\bar{e}_i$, one for each I_α, and by elements $\bar{e}_j$ satisfying $2\bar{e}_j = 0$, one for each J_α. Since μ has exactly two parts we have two elements $\bar{e}_{j_1}, \bar{e}_{j_2}$ of the latter type and so

$$X/P_1 \cong Z \oplus Z \oplus \ldots \oplus Z_2 \oplus Z_2,$$

with one component Z for each part of λ.

Let $w \in \mathcal{N}_W(W_1)$ and let w map to $\tau \in \mathrm{Aut}_W(\Delta_1)$. By Proposition 9, $\mathrm{Aut}_W(\Delta_1) \cong S_{m_2} \times (Z_2 \wr S_{m_3}) \times (Z_2 \wr S_{m_4}) \times \ldots \times S_{n_1} \times S_{n_2} \times \ldots$, and τ gives rise to pairs of partitions $\xi^{(i)}, \eta^{(i)}$ with $|\xi^{(i)}|+|\eta^{(i)}| = m_i$ and partitions $\zeta^{(i)}$ with $|\zeta^{(i)}| = n_i$. The parts of these partitions correspond to the positive or negative cycles induced by w on the components of a given type.

In order to pass from X/P_1 to Γ we impose additional relations $(qw-1)x = 0$ for all $x \in X$. If $w(e_i) = \pm e_j$ we have $\bar{e}_i = \pm q\bar{e}_j$ in Γ. Suppose $\bar{e}_i$ is one of the generators of X/P_1 of infinite order which lies in a positive cycle of length k under w. Then we have $q^k\bar{e}_i = \bar{e}_i$ in Γ. If $\bar{e}_i$ lies in a negative cycle of length k we have $q^k\bar{e}_i = -\bar{e}_i$ in Γ. Since q is odd we do not obtain additional relations on the generators $\bar{e}_{j_1}, \bar{e}_{j_2}$ of order 2 by passing from X/P_1 to Γ, except that we shall have $\bar{e}_{j_1} = \bar{e}_{j_2}$ in Γ if the two components of type C are interchanged by w. Thus each positive cycle of length k on components of type A contributes a factor Z_{q^k-1} to Γ and each negative cycle of length k on components of type A contributes a factor Z_{q^k+1} to Γ. Moreover, Γ has one or two factors Z_2 depending upon whether w interchanges the two components of type C or not. Thus Γ is a direct product of cyclic groups of order $q^{\xi^{(i)}_j}-1$ for all $\xi^{(i)}{}_j$, $q^{\eta^{(i)}_j}+1$ for all $\eta^{(i)}{}_j$, and either one or two copies of Z_2.

Now we must consider the images in Γ of the roots in $\bar{\Phi}_1 - \Phi_1$. $\bar{\Phi}_1$ is a root system of type $A_{\lambda_1-1} \times A_{\lambda_2-1} \times \ldots \times C_{\mu_1+\mu_2}$ and $\bar{\Phi}_1 - \Phi_1$ consists of roots $\pm e_i \pm e_j$, where $i \in J_\alpha$, $j \in J_\beta$, and $\alpha \neq \beta$. The required condition, by [**5**, Corollary 20], is that there is a character of Γ which does not annihilate the image in Γ of any element of $\bar{\Phi}_1 - \Phi_1$. This certainly means that J_α, J_β, with $\alpha \neq \beta$, cannot be in the same w-orbit. For otherwise, taking $j_1 \in J_\alpha$, $j_2 \in J_\beta$, we would have $\pm\bar{e}_{j_1} \pm \bar{e}_{j_2} = 0$ in Γ. Thus all roots in $\bar{\Phi}_1 - \Phi_1$ would map to zero in Γ.

Suppose, on the other hand, that w leaves invariant each of the two components of type C in G_1. Then the image of $\bar{\Phi}_1 - \Phi_1$ in Γ is $\bar{e}_{j_1} + \bar{e}_{j_2}$, where $j_1 \in J_\alpha$, $j_2 \in J_\beta$. Thus the image of $\bar{\Phi}_1 - \Phi_1$ in Γ is $\bar{e}_{j_1} + \bar{e}_{j_2}$, where $j_1 \in J_\alpha$, $j_2 \in J_\beta$. Thus the image of $\bar{\Phi}_1 - \Phi_1$ in Γ consists of a single element of Γ of order 2. It is certainly possible to find a character of Γ which does not annihilate this element. This justifies the entry in Table 2 for simply-connected groups of type C_l.

The adjoint case

Now suppose G is an adjoint group of type C_l. Then the group X of rational characters of the maximal torus T is generated by the root system Φ. Thus $X = \{\sum_{i=1}^l a_i e_i : a_i \in Z, \sum_{i=1}^l a_i \text{ is even}\}$. As before, we have $\bar{e}_i - \bar{e}_j = 0$ in X/P_1 when $i,j \in I_\alpha$, and $\bar{e}_i - \bar{e}_j = 0$, $2\bar{e}_i = 0$ when $i,j \in J_\alpha$. Thus X/P_1 is the set of elements $\sum a_i \bar{e}_i$ with one $\bar{e}_i$ for each component of Φ_1 such that $\sum a_i$ is even and $a_i \in \{0, 1\}$ if $\bar{e}_i$ corresponds to a component of type C. The torsion subgroup $\bar{P}_1/P_1$ of X/P_1 is the set of $\sum a_i\bar{e}_i$ such that each $\bar{e}_i$ corresponds to a component of type C and $\sum a_i$ is even. Now we are assuming that there are exactly two components of type C—suppose they give rise to elements $\bar{e}_{j_1}, \bar{e}_{j_2}$ of X/P_1. Thus the torsion subgroup $\bar{P}_1/P_1$ is generated by $\bar{e}_{j_1}+\bar{e}_{j_2}$.

We consider the additional relations needed to pass from X/P_1 to its quotient group Γ. These relations have the form $(qw-1)x = 0$ for all $x \in X$. Detailed calculation in the group Γ shows:

LEMMA 13. *Suppose Φ_1 has exactly two components of type C which are interchanged by w. Then Γ is isomorphic to the abelian group with generators c_i, d_i (one pair for each w-orbit of type A), and $\bar{e}_i-\bar{e}_j$ (one term for each pair of distinct w-orbits), subject to the following relations:*

(i) *when $k = \xi^{(i)}{}_j$,*

$$\tfrac{1}{2}(q^k-1)d_i = \begin{cases} 0 & \text{if } k \text{ is even,} \\ c_i & \text{if } k \text{ is odd;} \end{cases}$$

(ii) *when $k = \eta^{(i)}{}_j$,*

$$\tfrac{1}{2}(q^k+1)d_i = \begin{cases} 0 & \text{if } k \text{ is even,} \\ c_i & \text{if } k \text{ is odd;} \end{cases}$$

(iii) *$c_i = c_j$ for all pairs of w-orbits;*

(iv) *$2c_i = 0$ for all orbits;*

(v) *$(\bar{e}_i-\bar{e}_j)+(\bar{e}_j-\bar{e}_k) = (\bar{e}_i-\bar{e}_k)$;*

(vi) *$2(\bar{e}_i-\bar{e}_j) = \varepsilon_i d_i-\varepsilon_j d_j$, for certain integers $\varepsilon_i, \varepsilon_j = \pm 1$.*

We write $c = c_i$ for all i.

Now the image of $\bar{\Phi}_1-\Phi_1$ in Γ consists of the single element $\bar{e}_{j_1}+\bar{e}_{j_2}$, which is identified with the element c. We must therefore decide whether or not $c = 0$ in Γ. The following lemma simplifies the discussion.

LEMMA 14. *We have $c = 0$ in Γ if and only if $c_i = 0$ in some two generator subgroup $\langle c_i, d_i\rangle$ subject to the relations* (i) *or* (ii) *respectively of Lemma* 13 *and also subject to the relation $2c_i = 0$.*

Proof. Consider the subgroup Γ_0 of Γ generated by elements c_i, d_i (one pair for each w-orbit of type A). Γ_0 is generated by two-generator subgroups $\langle c_i, d_i\rangle$, and Lemma 13 shows that the only relations in Γ_0 involving generators from distinct subgroups $\langle c_i, d_i\rangle$ are those identifying c_i in the different two-generator subgroups. Γ_0 is therefore isomorphic to the direct product of the two-generator subgroups with c_i amalgamated. Hence $c_i = 0$ in Γ_0 if and only if $c_i = 0$ in some two-generator subgroup $\langle c_i, d_i\rangle$.

We now observe that in the two-generator groups $\langle c_i, d_i\rangle$, with relations (i) or (ii) holding together with $2c_i = 0$, we have in all cases $c_i \neq 0$. It follows from Lemma 14 that $c \neq 0$ in Γ.

This completes the discussion of the adjoint group and we have justified the entry in Table 2 for this group.

We note that the simple components of $(G_1{}^g)_\sigma$ can be of type $A_{i-1}(q^{\xi^{(i)}_j})$, ${}^2A_{i-1}(q^{2\eta^{(i)}_j})$, $C_j(q)$, or $C_j(q^2)$.

6. Groups of type B_l

Suppose G is a group of type B_l over an algebraically closed field K of characteristic $p \neq 2$. Let $\Phi = \{\pm e_i \pm e_j,\ \pm e_i\colon i \neq j,\ i,j \in \{1,2,\ldots,l\}\}$ be the root system of G and let G_1 be a reductive subgroup with root system Φ_1 given by

$$\Phi_1 = \bigcup_\alpha \{e_i - e_j\colon i \neq j,\ i,j \in I_\alpha\} \bigcup_\alpha \{\pm e_i \pm e_j\colon i \neq j,\ i,j \in J_\alpha\}$$
$$\bigcup \{\pm e_i \pm e_j,\ \pm e_i\colon i \neq j,\ i,j \in K\},$$

where $\{1,2,\ldots,l\}$ is the disjoint union of subsets $I_1, I_2, \ldots, J_1, J_2, \ldots, K$ as in § 3. Here $|I_\alpha| = \lambda_\alpha, |J_\alpha| = \mu_\alpha, |K| = \nu$, and $\lambda = (\lambda_1, \lambda_2, \ldots), \mu = (\mu_1, \mu_2, \ldots)$, and ν satisfy $|\lambda| + |\mu| + \nu = l$. Φ_1 is a root system of type

$$A_{\lambda_1-1} \times A_{\lambda_2-1} \times \ldots \times D_{\mu_1} \times D_{\mu_2} \times \ldots \times B_\nu.$$

We use the result of [5, Proposition 11]. The torsion subgroup $\bar{P}_1/P_1$ of X/P_1 satisfies $\bar{P}_1/P_1 \cong Z_2 \oplus Z_2 \oplus \ldots$ with one component for each part of μ. The roots in $\bar{\Phi}_1 - \Phi_1$ have the form $\pm e_i \pm e_j$, where i,j lie in distinct sets $J_1, J_2, \ldots, K$, and also $\pm e_i$ where i lies in a component of type J. Thus the images of $\bar{\Phi}_1 - \Phi_1$ in X/P_1 are the elements $\bar{e}_i$ corresponding to components J_α and the elements $\bar{e}_i - \bar{e}_j$ corresponding to pairs of distinct components J_α, J_β. For each character ψ of $\bar{P}_1/P_1$ we have $\psi(\bar{e}_i) = \pm 1$. If $\psi(\bar{e}_i) = -1$ and $\psi(\bar{e}_j) = -1$, then $\psi(\bar{e}_i - \bar{e}_j) = 1$. Thus if there are two distinct components of type J, then $\bar{P}_1/P_1$ has no regular character. If there is just one component J_α the image of $\bar{\Phi}_1 - \Phi_1$ in $\bar{P}_1/P_1$ contains a single element $\bar{e}_j$ and this element has order 2. Thus $\bar{P}_1/P_1$ has a character of order prime to p which does not annihilate this element. Finally, if there is no component J_α we have $\bar{\Phi}_1 = \Phi_1$. This justifies the entry in Table 1.

We now turn to the finite group $(G_1{}^g)_\sigma$. If μ has no parts then $\Phi_1 = \bar{\Phi}_1$ and G_1 is a Levi subgroup of a parabolic subgroup of G. Theorem 21 of [5] then gives the required information. If μ has more than one part we have just seen that $(G_1{}^g)_\sigma$ cannot be the connected centralizer of a semisimple element. We shall therefore assume from now on that μ has exactly one part, i.e. that G_1 has just one component of type D. We must now distinguish between the different isogeny types for G, since G can be either adjoint or simply-connected.

The adjoint case

Suppose G is of adjoint type. Then the group X of rational characters of a maximal torus T is generated by the fundamental roots. In the present context the fundamental roots $p_1, p_2, \ldots, p_l$ and fundamental weights $q_1, q_2, \ldots, q_l$ are given by

$$p_1 = e_1 - e_2, \quad p_2 = e_2 - e_3, \quad \ldots, \quad p_{l-1} = e_{l-1} - e_l, \quad p_l = e_l,$$

$$q_1 = e_1, \quad q_2 = e_1 + e_2, \quad \ldots, \quad q_{l-1} = e_1 + e_2 + \ldots + e_{l-1}, \quad q_l = \tfrac{1}{2}(e_1 + e_2 + \ldots + e_l).$$

Hence $X = \{\sum_{i=1}^{l} a_i e_i : a_i \in Z\}$. Let $e_i \to \bar{e}_i$ be the natural homomorphism $X \to X/P_1$. Then $\bar{e}_i = \bar{e}_j$ when $i,j \in I_\alpha$, and $\bar{e}_i = \bar{e}_j$, $2\bar{e}_i = 0$ when $i,j \in J_\alpha$. Also $\bar{e}_i = 0$ if $i \in K$. Hence X/P_1 is generated by elements $\bar{e}_i$, one for each I_α, and by elements $\bar{e}_j$ satisfying $2\bar{e}_j = 0$, one for each J_α. We are assuming that there is only one J_α. Hence $X/P_1 \cong Z \oplus Z \oplus \ldots \oplus Z_2$, with one component Z for each part of λ.

Let $w \in \mathcal{N}_W(W_1)$ and let w map to $\tau \in \mathrm{Aut}_W(\Delta_1)$. By Proposition 11 we have

$$\mathrm{Aut}_W(\Delta_1) \cong S_{m_2} \times (Z_2 \wr S_{m_3}) \times (Z_2 \wr S_{m_4}) \times \ldots \times Z_2,$$

where m_i is the number of parts of λ equal to i. Suppose τ gives rise to a pair of partitions $\xi^{(i)}, \eta^{(i)}$ with $|\xi^{(i)}| + |\eta^{(i)}| = m_i$, where the parts of the partitions give the lengths of the positive and negative cycles on the components of type A_{i-1}.

In order to pass from X/P_1 to Γ we impose additional relations $(qw-1)x = 0$ for all $x \in X$. If $w(e_i) = \pm e_j$ we have $\bar{e}_i = \pm q\bar{e}_j$ in Γ. If $\bar{e}_i$ is one of the generators of type A which lies in a positive cycle of length k under w then we have $q^k\bar{e}_i = \bar{e}_i$ in Γ. If $\bar{e}_i$ lies in a negative cycle of length k we have $q^k\bar{e}_i = -\bar{e}_i$ in Γ. Thus Γ is generated by elements $\bar{e}_i$, one for each w-orbit of type A, and by an element $\bar{e}_j$ subject to relations

$$(q^k - 1)\bar{e}_i = 0 \quad \text{for positive } k\text{-cycles},$$

$$(q^k + 1)\bar{e}_i = 0 \quad \text{for negative } k\text{-cycles},$$

$$2\bar{e}_j = 0.$$

Thus Γ is a direct product of cyclic groups of order $q^{\xi^{(i)}_j} - 1$ for all $\xi^{(i)}_j$, $q^{\eta^{(i)}_j} + 1$ for all $\eta^{(i)}_j$, and one copy of Z_2. The image of $\overline{\Phi}_1 - \Phi_1$ in Γ is given by the single element $\bar{e}_j$, and this element is non-zero in Γ. Thus there is a character of Γ which does not annihilate it. This justifies the entry in Table 2 for adjoint groups of type B_l.

The simply-connected case

Suppose G is a simply-connected group of type B_l. Then X is generated by the fundamental weights $q_1, q_2, \ldots, q_l$. Since we have

$$q_1 = e_1,\quad q_2 = e_1+e_2,\quad \ldots,\quad q_{l-1} = e_1+e_2+\ldots+e_{l-1},\quad q_l = \tfrac{1}{2}(e_1+e_2+\ldots+e_l),$$

it follows that X is generated by $e_1, e_2, \ldots, e_l, q_l$. Moreover, X/P_1 is generated by elements $\bar{e}_i$, one for each I_α, by an element $\bar{e}_j$ satisfying $2\bar{e}_j = 0$, and $\bar{q}_l$.

Let $w \in \mathscr{N}_W(W_1)$ and let w map to $\tau \in \mathrm{Aut}_W(\Delta_1)$. By Proposition 11 we have

$$\mathrm{Aut}_W(\Delta_1) \cong S_{m_2} \times (Z_2 \wr S_{m_3}) \times (Z_2 \wr S_{m_4}) \times \ldots \times Z_2.$$

Suppose τ gives rise to a pair of partitions $\xi^{(i)}, \eta^{(i)}$ with $|\xi^{(i)}|+|\eta^{(i)}| = m_i$, where the parts of the partitions give the lengths of the positive and negative cycles on the components of type A_{i-1}. In order to pass from X/P_1 to Γ we impose additional relations $(qw-1)x = 0$ for all $x \in X$, and it is convenient to distinguish between two cases depending on whether w induces the trivial or the non-trivial automorphism on the single component D_μ of type D.

Case 1. Suppose w induces the trivial graph automorphism on D_μ. Then Γ is generated by elements $\bar{e}_i$ corresponding to the w-orbits on the components of type A, by $\bar{e}_j$ coming from the component D_μ, and by $\bar{q}_l$. Also $2\bar{q}_l = \overline{\sum_{i=1}^{l} e_i}$. If μ is even, $2\bar{q}_l$ is a linear combination of the $\bar{e}_i$ of type A, whereas if μ is odd, $2\bar{q}_l = (\bar{e}_j + \text{a linear combination of } \bar{e}_i \text{ of type } A)$. We obtain a complete set of relations by taking $(qw-1)\bar{e}_i = 0$, $(qw-1)\bar{q}_l = 0$, since $(qw-1)\bar{e}_j = 0$ is a consequence of $2\bar{e}_j = 0$.

The image of $\overline{\Phi}_1 - \Phi_1$ in Γ consists of the single element $\bar{e}_j$. We must therefore decide whether the relations imply $\bar{e}_j = 0$. It is clear from the above system of relations that the only relation which could possibly imply $\bar{e}_j = 0$ is $(qw-1)\bar{q}_l = 0$. Now $w(\bar{q}_l) - \bar{q}_l$ is a linear combination of $\bar{e}_i$'s and does not involve $\bar{e}_j$, since w induces the trivial graph automorphism on D_μ. Thus the relation $(qw-1)\bar{q}_l = 0$ has the form

$$(q-1)\bar{q}_l = \text{linear combination of } \bar{e}_i\text{'s},$$

or, alternatively,

$$\tfrac{1}{2}(q-1)(2\bar{q}_l) = \text{linear combination of } \bar{e}_i\text{'s}.$$

If μ is even, $2\bar{q}_l$ is a linear combination of $\bar{e}_i$'s and does not involve $\bar{e}_j$. Thus in this case the relation $(qw-1)\bar{q}_l = 0$ does not involve $\bar{e}_j$. In the quotient group of Γ obtained by imposing the additional relations $\bar{e}_i = 0$ we thus have $\bar{e}_j \neq 0$. Thus if μ is even we have $\bar{e}_j \neq 0$ in Γ.

Suppose μ is odd. Then

$$2\bar{q}_l = \bar{e}_j + \text{a linear combination of } \bar{e}_i\text{'s}.$$

If $q \equiv 1 \bmod 4$ the relation

$$\tfrac{1}{2}(q-1)(2\bar{q}_l) = \text{linear combination of } \bar{e}_i\text{'s}$$

does not involve $\bar{e}_j$ since $\frac{1}{2}(q-1)$ is even and $2\bar{e}_j = 0$. Thus if μ is odd and $q \equiv 1 \bmod 4$ we have $\bar{e}_j \neq 0$ in Γ.

Thus we may concentrate on the remaining case when μ is odd and $q \equiv -1 \bmod 4$. Here it is more difficult to decide whether $\bar{e}_j = 0$ in Γ. The relation $(qw-1)\bar{q}_l = 0$ has the form

$$\bar{e}_j = \text{a linear combination of } \bar{e}_i\text{'s},$$

and we must decide whether the right-hand side is zero. Γ is generated by elements $\bar{e}_i$, one for each w-orbit of type A, by $\bar{e}_j$, and by $\bar{q}_l$, subject to relations

$$(qw-1)\bar{e}_i = 0, \quad 2\bar{e}_j = 0, \quad 2\bar{q}_l = (\overline{\textstyle\sum_{i=1}^{l} e_i}), \quad (qw-1)\bar{q}_l = 0.$$

The relations $(qw-1)\bar{e}_i = 0$ imply that $(q^k-1)\bar{e}_i = 0$ if $\bar{e}_i$ is associated with a positive cycle of length k, and $(q^k+1)\bar{e}_i = 0$ if $\bar{e}_i$ is associated with a negative k-cycle. The effect of the relation $(qw-1)\bar{q}_l = 0$ can best be seen by replacing $\bar{q}_l$ by another generator whose definition depends on w. Suppose we express w as a product of positive and negative cycles on the elements $e_1, e_2, \ldots, e_l$. Let $(e_{i_1}, \varepsilon_2 e_{i_2}, \varepsilon_3 e_{i_3}, \ldots, \varepsilon_r e_{i_r})$ be a typical cycle, where $w(\varepsilon_r e_{i_r}) = \pm e_{i_1}$. Let $x \in X$ be defined by

$$x = \tfrac{1}{2} \sum_{\text{cycles}} (e_{i_1} + \varepsilon_2 e_{i_2} + \varepsilon_3 e_{i_3} + \ldots + \varepsilon_r e_{i_r}),$$

with one summand for each cycle. (Note that the definition of x depends upon the choice of a first element e_{i_1} in each cycle.) X is generated by $e_1, e_2, \ldots, e_l$ and x. We shall replace $\bar{q}_l$ by $\bar{x}$ as the final generator in Γ. Now we have

$$w(\tfrac{1}{2}(e_{i_1} + \varepsilon_2 e_{i_2} + \ldots + \varepsilon_r e_{i_r})) = \begin{cases} \tfrac{1}{2}(e_i + \varepsilon_2 e_{i_2} + \ldots + \varepsilon_r e_{i_r}) \\ \qquad \text{for a positive cycle,} \\ \tfrac{1}{2}(e_{i_1} + \varepsilon_2 e_{i_2} + \ldots + \varepsilon_r e_{i_r}) - e_{i_1} \\ \qquad \text{for a negative cycle.} \end{cases}$$

Thus $w(x) = x - \sum e_{i_1}$ with one summand e_{i_1} for each negative cycle. Thus in X/P_1 we have

$$(qw-1)\bar{x} = (q-1)\bar{x} - q\sum \bar{e}_{i_1},$$

summed over negative cycles in X.

We consider next how the cycle $(e_{i_1}, \varepsilon_2 e_{i_2}, \ldots, \varepsilon_r e_{i_r})$ of w on the elements $\pm e_i$ gives rise to a cycle of w on the components of a given type in Φ_1. Suppose the e_i in the cycle correspond to components in Φ_1 of type $A_{\lambda_t - 1}$. Let k be the least positive integer such that $e_{i_{k+1}}$ corresponds to the same component in Φ_1 as e_{i_1}. Then k divides r. Let $r = ks$. Then in X/P_1, we see that w induces a corresponding cycle $(\bar{e}_{i_1}, \varepsilon_2\bar{e}_{i_2}, \ldots, \varepsilon_k\bar{e}_{i_k})$ of length k on the components of type A_{λ_i-1}. There will in general be several cycles $(e_{i_1}, \varepsilon_2 e_{i_2}, \ldots, \varepsilon_r e_{i_r})$ giving rise to a single cycle $(\bar{e}_{i_1}, \varepsilon_2\bar{e}_{i_2}, \ldots, \varepsilon_k\bar{e}_{i_k})$ in X/P_1. The number of such cycles of w in X mapping to a given cycle in X/P_1 is λ_i/s. We distinguish between three possibilities. If $\varepsilon_{k+1} = 1$ we have a positive r-cycle in X which induces a positive k-cycle in X/P_1. If $\varepsilon_{k+1} = -1$ and s is even we have a positive r-cycle in X which induces a negative k-cycle in X/P_1. If $\varepsilon_{k+1} = -1$ and s is odd we have a negative r-cycle in X which induces a negative k-cycle in X/P_1.

Now each r-cycle $(e_{i_1}, \varepsilon_2 e_{i_2}, \ldots, \varepsilon_r e_{i_r})$ of w in X contributes

$$\tfrac{1}{2}(e_{i_1} + \varepsilon_2 e_{i_2} + \ldots + \varepsilon_r e_{i_r})$$

toward the element x. It therefore contributes $\frac{1}{2}(\bar{e}_{i_1} + \varepsilon_2\bar{e}_{i_2} + \ldots + \varepsilon_r\bar{e}_{i_r})$ toward $\bar{x} \in X/P_1$. We consider the nature of this element in the three cases above. If we have a positive r-cycle in X inducing a positive k-cycle in X/P_1 then

$$\tfrac{1}{2}(\bar{e}_{i_1} + \varepsilon_2\bar{e}_{i_2} + \ldots + \varepsilon_r\bar{e}_{i_r}) = \tfrac{1}{2}s(\bar{e}_{i_1} + \varepsilon_2\bar{e}_{i_2} + \ldots + \varepsilon_k\bar{e}_{i_k}).$$

If we have a positive r-cycle in X inducing a negative k-cycle in X/P_1 then

$$\tfrac{1}{2}(\bar{e}_{i_1} + \varepsilon_2\bar{e}_{i_2} + \ldots + \varepsilon_r\bar{e}_{i_r}) = 0.$$

If we have a negative r-cycle in X then

$$\tfrac{1}{2}(\bar{e}_{i_1} + \varepsilon_2\bar{e}_{i_2} + \ldots + \varepsilon_r\bar{e}_{i_r}) = \tfrac{1}{2}(\bar{e}_{i_1} + \varepsilon_2\bar{e}_{i_2} + \ldots + \varepsilon_k\bar{e}_{i_k}).$$

We now take together the contributions to $\bar{x}$ from all r-cycles in X inducing a given k-cycle in X/P_1. Since there are λ_i/s such r-cycles we have a contribution to $\bar{x}$ given by

$$\tfrac{1}{2}\alpha(\bar{e}_{i_1} + \varepsilon_2\bar{e}_{i_2} + \ldots + \varepsilon_k\bar{e}_{i_k}),$$

where

$$\alpha = \begin{cases} \lambda_i & \text{for a positive } r\text{-cycle inducing a positive } k\text{-cycle,} \\ 0 & \text{for a positive } r\text{-cycle inducing a negative } k\text{-cycle,} \\ \lambda_i/s & \text{for a negative } r\text{-cycle.} \end{cases}$$

Moreover, the contribution to $\bar{x}$ from the component D_μ is $\frac{1}{2}\bar{e}_j$, since μ is odd. Thus we have

$$\bar{x} = \sum \tfrac{1}{2}\alpha(\bar{e}_{i_1} + \varepsilon_2\bar{e}_{i_2} + \ldots + \varepsilon_k\bar{e}_{i_k}) + \tfrac{1}{2}\bar{e}_j$$

summed over the w-orbits on components of type A in Φ_1. It follows that

$$\begin{aligned}(qw-1)\bar{x} &= (q-1)\bar{x} - q\sum \bar{e}_i \quad \text{(summed over negative cycles in } X\text{)} \\ &= \sum \tfrac{1}{2}(q-1)\alpha\,(\bar{e}_{i_1} + \varepsilon_2\bar{e}_{i_2} + \ldots + \varepsilon_k\bar{e}_{i_k}) - q\sum \bar{e}_{i_1} + \tfrac{1}{2}(q-1)\bar{e}_j,\end{aligned}$$

the latter sum being taken over all negative cycles of type A in X, since w induces the trivial graph automorphism on D_μ and therefore has an even number of negative cycles on the e_i in D_μ. Since $q \equiv -1 \bmod 4$ we have

$$(qw-1)\bar{x} = \sum \tfrac{1}{2}(q-1)\alpha(\bar{e}_{i_1}+\varepsilon_2\bar{e}_{i_2}+\ldots+\varepsilon_k\bar{e}_{i_k}) - q\sum(\lambda_i/s)\bar{e}_{i_1}+\bar{e}_j,$$

the latter sum being taken over all negative cycles of type A in X/P_1, since each negative cycle in X/P_1 comes from λ_i/s negative cycles in X. Thus the relation $(qw-1)\bar{x} = 0$ in Γ implies that in Γ we have

$$\bar{e}_j = \sum \tfrac{1}{2}(q-1)\alpha(\bar{e}_{i_1}+\varepsilon_2\bar{e}_{i_2}+\ldots+\varepsilon_k\bar{e}_{i_k}) - q\sum(\lambda_i/s)\bar{e}_{i_1},$$

the latter sum being taken over all negative cycles of type A in X/P_1. But we also have $\bar{e}_{i_1} = q\varepsilon_2\bar{e}_{i_2} = q^2\varepsilon_3\bar{e}_{i_3} = \ldots$ in Γ. Hence, for the positive k-cycles, we obtain

$$\tfrac{1}{2}(q-1)\alpha(\bar{e}_{i_1}+\varepsilon_2\bar{e}_{i_2}+\ldots+\varepsilon_k\bar{e}_{i_k}) = \tfrac{1}{2}\lambda_i(q^k-1)\bar{e}_{i_1},$$

and, for the negative k-cycles induced by negative r-cycles, we have

$$\begin{aligned}
&\tfrac{1}{2}(q-1)\alpha(\bar{e}_{i_1}+\varepsilon_2\bar{e}_{i_2}+\ldots+\varepsilon_k\bar{e}_{i_k}) - q(\lambda_i/s)\bar{e}_{i_1}\\
&= \tfrac{1}{2}(q-1)(\lambda_i/s)(1-q-q^2-\ldots-q^{k-1})\bar{e}_{i_1} - q(\lambda_i/s)\bar{e}_{i_1}\\
&= -\tfrac{1}{2}(\lambda_i/s)(q^k-1)\bar{e}_{i_1} - (\lambda_i/s)\bar{e}_{i_1}\\
&= -\tfrac{1}{2}(\lambda_i/s)(q^k+1)\bar{e}_{i_1}.
\end{aligned}$$

Hence the relation $(qw-1)\bar{x} = 0$ in Γ is equivalent to

$$\bar{e}_j = \sum \tfrac{1}{2}\lambda_i(q^k-1)\bar{e}_{i_1} - \sum \tfrac{1}{2}(\lambda_i/s)(q^k+1)\bar{e}_{i_1},$$

where the first sum extends over all positive k-cycles on X/P_1 and the second over all negative k-cycles on X/P_1 induced by negative r-cycles on X. Thus Γ is generated by elements $\bar{e}_i$, one from each w-orbit on components of type A, by $\bar{e}_j$ and by $\bar{x}$ subject to the following relations:

$(q^k-1)\bar{e}_i = 0$ if $\bar{e}_i$ is in a positive k-cycle;

$(q^k+1)\bar{e}_i = 0$ if $\bar{e}_i$ is in a negative k-cycle;

$2\bar{e}_j = 0$;

$2\bar{x}$ is a linear combination of $\bar{e}_j$ and $\bar{e}_i$'s;

$\bar{e}_j = \sum \tfrac{1}{2}\lambda_i(q^k-1)\bar{e}_i - \sum \tfrac{1}{2}(\lambda_i/s)(q^k+1)\bar{e}_i$.

It follows that $\bar{e}_j = 0$ in Γ if and only if λ_i is even for each positive cycle on X/P_1 and λ_i/s is even for each negative cycle on X/P_1 induced by a negative cycle on X. The condition on negative cycles may be stated alternatively as follows. For each negative cycle on X/P_1 we have λ_i/s even whenever s is odd. This is equivalent to asserting that λ_i is even for all negative cycles of w on X/P_1. Thus $\bar{e}_j = 0$ in Γ if and only if all components A_{λ_i-1} of Φ_1 have λ_i even.

Thus, if w induces the trivial graph automorphism on D_μ, then $(G_1{}^\sigma)_\sigma$ is the connected centralizer of a semisimple element unless μ is odd, $q \equiv -1 \bmod 4$, and all λ_i are even.

Case 2. Suppose w induces the non-trivial graph automorphism on D_μ. This is equivalent to the condition that w has an odd number of negative cycles on the elements e_j appearing in D_μ. Γ is generated by elements $\bar{e}_i$ corresponding to w-orbits of type A, by $\bar{e}_j$ coming from the components D_μ, and by $\bar{q}_l$. Also we have $2\bar{q}_l = (\overline{\sum_{i=1}^l e_i})$ and $2\bar{e}_j = 0$. We have relations $(qw-1)\bar{e}_i = 0$ and $(qw-1)\bar{q}_l = 0$; and $(qw-1)\bar{e}_j = 0$ is a consequence of $2\bar{e}_j = 0$. Again we must decide whether $\bar{e}_j = 0$ in Γ. The only relation which could imply this is $(qw-1)\bar{q}_l = 0$. Now $q_l = \frac{1}{2}(e_1+e_2+\ldots+e_l)$ and so we have

$$w(\bar{q}_l)-\bar{q}_l = \bar{e}_j + \text{a linear combination of } \bar{e}_i\text{'s},$$

since w changes an odd number of signs for e_j's belonging to D_μ. Hence

$$(qw-1)\bar{q}_l = \bar{e}_j + \tfrac{1}{2}(q-1)(2\bar{q}_l) + \text{a linear combination of } \bar{e}_i\text{'s}.$$

Suppose μ is odd. Then $2\bar{q}_l = \bar{e}_j +$ a linear combination of $\bar{e}_i$'s. Suppose in addition that $q \equiv -1 \bmod 4$. Then $\frac{1}{2}(q-1)(2\bar{q}_l) = \bar{e}_j +$ a linear combination of $\bar{e}_i$'s. Hence in this case $(qw-1)\bar{q}_l = 0$ does not imply $\bar{e}_j = 0$. Thus $\bar{e}_j \neq 0$ in Γ.

We shall therefore assume that either μ is even or μ is odd and $q \equiv 1 \bmod 4$. The relations $(qw-1)\bar{e}_i = 0$ imply that $(q^k-1)\bar{e}_i = 0$ if $\bar{e}_i$ is associated with a positive k-cycle and $(q^k+1)\bar{e}_i = 0$ if $\bar{e}_i$ is associated with a negative k-cycle. We shall again replace the remaining generator $\bar{q}_l$ by the element $\bar{x}$ whose definition depends on w. As before we express w as a product of positive and negative cycles on the elements $e_1, e_2, \ldots, e_l$. Let $(e_{i_1}, \varepsilon_2 e_{i_2}, \varepsilon_3 e_{i_3}, \ldots, \varepsilon_r e_{i_r})$ be a typical cycle, where $w(\varepsilon_r e_{i_r}) = \pm e_{i_1}$. Then we define x by

$$x = \tfrac{1}{2}\sum_{\text{cycles}} (e_{i_1} + \varepsilon_2 e_{i_2} + \ldots + \varepsilon_r e_{i_r}),$$

with one summand for each cycle. We calculate $\bar{x}$ as before and obtain

$$\bar{x} = \sum \tfrac{1}{2}\alpha(\bar{e}_{i_1} + \varepsilon_2 \bar{e}_{i_2} + \ldots + \varepsilon_k \bar{e}_{i_k}) + \text{a multiple of } \bar{e}_j,$$

summed over w-orbits of type A_{λ_i-1}, where α is given by

$$\alpha = \begin{cases} \lambda_i & \text{for a positive } r\text{-cycle inducing a positive } k\text{-cycle,} \\ 0 & \text{for a positive } r\text{-cycle inducing a negative } k\text{-cycle,} \\ \lambda_i/s & \text{for a negative } r\text{-cycle.} \end{cases}$$

Consider the element $(qw-1)\bar{x}$. The contribution of $\bar{e}_j$ to $(qw-1)\bar{x}$ is $\frac{1}{2}r(q-1)\bar{e}_j$ from each positive r-cycle on the e_j's in D_μ and $(\frac{1}{2}r(q-1)-q)\bar{e}_j$ from each negative r-cycle. Since there are an odd number of negative cycles on the e_j's in D_μ the contribution of $\bar{e}_j$ to $(qw-1)\bar{x}$ is

$$(\tfrac{1}{2}\mu(q-1)-q)\bar{e}_j = \bar{e}_j,$$

since $2\bar{e}_j = 0$ and $\frac{1}{2}\mu(q-1)$ is even. The remaining contributions to $(qw-1)\bar{x}$ are as before and we have

$$(qw-1)\bar{x} = \sum \tfrac{1}{2}\lambda_i(q^k-1)\bar{e}_i - \sum \tfrac{1}{2}(\lambda_i/s)\,(q^k+1)\bar{e}_i + \bar{e}_j,$$

where the sums extend respectively over the positive k-cycles on X/P_1 and the negative k-cycles on X/P_1 induced by negative r-cycles on X. (Here $r = ks$.) Thus the relation $(qw-1)\bar{x} = 0$ is equivalent to

$$\bar{e}_j = \sum \tfrac{1}{2}\lambda_i(q^k-1)\bar{e}_i - \sum \tfrac{1}{2}(\lambda_i/s)\,(q^k+1)\bar{e}_i.$$

Hence $\bar{e}_j = 0$ if and only if λ_i is even for all positive cycles on X/P_1 and λ_i/s is even for all negative cycles on X/P_1 induced from negative cycles on X. As before, this is equivalent to the assertion that all λ_i are even. Thus if μ is even, or if μ is odd and $q \equiv 1 \bmod 4$, then $(G_1{}^g)_\sigma$ is the connected centralizer of a semisimple element unless all components A_{λ_i-1} have λ_i even. (We observe that the condition that λ_i should be even includes the fact that λ has no part equal to 1. This is equivalent to the condition that $\lambda_2+\lambda_3+\ldots+\mu+\nu = l$ and is a non-trivial assertion about Φ_1.)

We have therefore justified the entry in Table 2 for simply-connected groups of type B_l.

Groups of type B_l in characteristic 2

We conclude this section with a discussion of the case when G is a group of type B_l over an algebraically closed field K of characteristic 2. (Then G is abstractly isomorphic to a group of type C_l over K). Let G_1 be a reductive subgroup of G. By Proposition 12 the root system Φ_1 of G_1 has form

$$\Phi_1 = \bigcup_\alpha \{e_i - e_j : i \neq j,\, j \in I_\alpha\} \bigcup_\alpha \{\pm e_i \pm e_j : i \neq j,\, i,j \in J_\alpha\}$$
$$\bigcup_\alpha \{\pm e_i \pm e_j,\ \pm e_i : i \neq j,\, i,j \in K_\alpha\},$$

where $I_1, I_2, \ldots, J_1, J_2, \ldots, K_1, K_2, \ldots$, are disjoint subsets of $\{1, 2, \ldots, l\}$ whose union is $\{1, 2, \ldots, l\}$. Let $|I_\alpha| = \lambda_\alpha$, $|J_\alpha| = \mu_\alpha$, $|K_\alpha| = \nu_\alpha$, and let

$$\lambda = (\lambda_1, \lambda_2, \ldots), \quad \mu = (\mu_1, \mu_2, \ldots), \quad \nu = (\nu_1, \nu_2, \ldots).$$

Then λ, μ, ν are partitions such that $|\lambda|+|\mu|+|\nu| = l$. Φ_1 is a subsystem of type

$$A_{\lambda_1-1} \times A_{\lambda_2-1} \times \ldots \times D_{\mu_1} \times D_{\mu_2} \times \ldots \times B_{\nu_1} \times B_{\nu_2} \times \ldots.$$

Since K has characteristic 2 we may assume G is adjoint. Then we have

$$p_1 = e_1 - e_2, \quad p_2 = e_2 - e_3, \quad \ldots, \quad p_{l-1} = e_{l-1} - e_l, \quad p_l = e_l.$$

Also we have $X = \{\sum_{i=1}^{l} a_i e_i : a_i \in Z\}$. X/P_1 is generated by elements $\bar{e}_i$, one for each I_α, and by elements $\bar{e}_j$ satisfying $2\bar{e}_j = 0$, one for each J_α. Hence $X/P_1 \cong Z \oplus Z \oplus \ldots \oplus Z_2 \oplus Z_2 \oplus \ldots$, with one component Z for each part of λ and one component Z_2 for each part of μ. The torsion subgroup $\bar{P}_1/P_1$ of X/P_1 is isomorphic to $Z_2 \oplus Z_2 \oplus \ldots$ with one component for each part of μ. It is therefore an elementary abelian 2-group and its only character of order prime to 2 is the unit character. This fails to be regular unless $\Phi_1 = \bar{\Phi}_1$. This justifies the entries in Tables 1 and 2 for groups of type B_l in characteristic 2.

7. Groups of type D_l

Suppose G is a group of type D_l over an algebraically closed field K. Let $\Phi = \{\pm e_i \pm e_j : i,j \in \{1,2,\ldots,l\}\}$ be the root system of G, and let G_1 be a reductive subgroup with root system Φ_1 given by

$$\Phi_1 = \bigcup_\alpha \{e_i - e_j : i \neq j,\ i,j \in I_\alpha\} \bigcup_\alpha \{\pm e_i \pm e_j : i \neq j,\ i,j \in J_\alpha\},$$

where $I_1, I_2, \ldots, J_1, J_2, \ldots$ are disjoint subsets of $\{1,2,\ldots,l\}$ whose union is $\{1,2,\ldots,l\}$. Let $|I_\alpha| = \lambda_\alpha$, $|J_\alpha| = \mu_\alpha$, and $\lambda = (\lambda_1, \lambda_2, \ldots)$, $\mu = (\mu_1, \mu_2, \ldots)$. Then $|\lambda| + |\mu| = l$. Φ_1 is a root system of type

$$A_{\lambda_1 - 1} \times A_{\lambda_2 - 1} \times \ldots \times D_{\mu_1} \times D_{\mu_2} \times \ldots.$$

The roots in $\bar{\Phi}_1 - \Phi_1$ have form $\pm e_i \pm e_j$ where i, j are in distinct components J_α, J_β. Their images in $\bar{P}_1/P_1$ have form $\bar{e}_i - \bar{e}_j$. Let ψ be a character of $\bar{P}_1/P_1$. Then $\psi(e_i - e_j) = \pm 1$. If there are at least three components of type D, giving rise to $\bar{e}_i, \bar{e}_j, \bar{e}_k$ in X/P_1, we have $\psi(\bar{e}_i - \bar{e}_j) = \pm 1$, $\psi(\bar{e}_j - \bar{e}_k) = \pm 1$, $\psi(\bar{e}_i - \bar{e}_k) = \pm 1$. So ψ annihilates some element of $\bar{\Phi}_1 - \Phi_1$.

If there is at most one component of type D we have $\Phi_1 = \bar{\Phi}_1$. If there are exactly two components of type D then $\bar{\Phi}_1 - \Phi_1$ consists of a single element $\bar{e}_i - \bar{e}_j$ in $\bar{P}_1/P_1$. This element has order 2. There is a character ψ of order 2 which does not annihilate it. Thus if p is odd G_1 is a connected centralizer, but if $p = 2$ it is not. This justifies the entry in Table 1 for groups of type D_l.

Turning to the finite group $(G_1{}^g)_\sigma$, we see that if G_1 has at most one component of type D then G_1 is a Levi subgroup of some parabolic subgroup of G and so by [**5**, Theorem 21], $(G_1{}^g)_\sigma$ is the connected centralizer in G_σ of some semisimple element when q is sufficiently large. If there are at least three components of type D, or if K has characteristic 2 and there are at least two components of type D, Table 1 shows that $(G_1{}^g)_\sigma$ cannot be the connected centralizer in G_σ of a semisimple element. We shall therefore assume subsequently that K has characteristic $p \neq 2$ and that Φ_1 has exactly two components of type D.

We must now distinguish between the different isogeny types for G. The fundamental roots $p_1, p_2, \ldots, p_l$ and fundamental weights $q_1, q_2, \ldots, q_l$ may be taken as

$$p_1 = e_1 - e_2, \quad p_2 = e_2 - e_3, \quad \ldots, \quad p_{l-1} = e_{l-1} - e_l, \quad p_l = e_{l-1} + e_l,$$
$$q_1 = e_1, \quad q_2 = e_1 + e_2, \quad \ldots, \quad q_{l-2} = e_1 + e_2 + \ldots + e_{l-2},$$
$$q_{l-1} = \tfrac{1}{2}(e_1 + e_2 + \ldots + e_{l-1} - e_l), \quad q_l = \tfrac{1}{2}(e_1 + e_2 + \ldots + e_{l-1} + e_l).$$

The different isogeny types to be considered are the following.

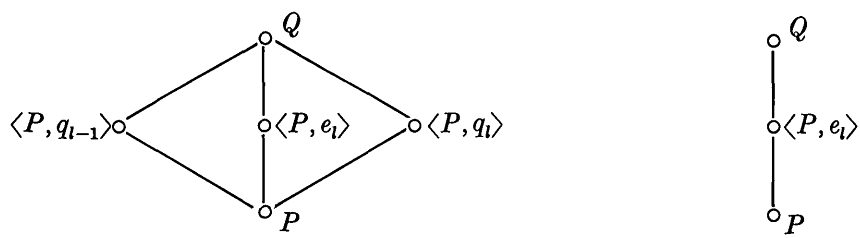
(i) $X = \{\sum_{i=1}^{l} a_i e_i : a_i \in Z\}$. In this case G is the special orthogonal group $SO_{2l}(K)$.

(ii) $X = \{\sum_{i=1}^{l} a_i e_i : a_i \in Z, \sum_{i=1}^{l} a_i \text{ even}\}$. In this case G is the adjoint group of type D, viz. $G = PSO_{2l}(K)$.

(iii) X is the group generated by $e_1, e_2, \ldots, e_l, q_l$. Here G is the simply-connected group of type D_l, viz. the group $\mathrm{Spin}_{2l}(K)$.

(iv) If l is even there is a fourth possibility. Here X is the group generated by $\sum_{i=1}^{l} a_i e_i$, where $a_i \in Z$ with $\sum_{i=1}^{l} a_i$ even, and by q_l. Here G is the half-spin group.

Let Q be the group generated by $q_1, q_2, \ldots, q_l$, and let P be the group generated by $p_1, p_2, \ldots, p_l$. Then $P \subset Q, |Q/P| = 4$, and the subgroups between P and Q are as follows:



l even $\qquad\qquad$ l odd

In the case when l is even the two groups $\langle P, q_{l-1}\rangle$ and $\langle P, q_l\rangle$ are interchanged by the graph automorphism of G. Thus only one of the cases $X = \langle P, q_{l-1}\rangle$ and $X = \langle P, q_l\rangle$ needs to be considered.

We make some general comments on the σ-action before looking in detail at the individual isogeny types. We have $\sigma = q\sigma_0$, where σ_0 is an isometry which has order 1 or 2. σ_0 acts on the components of G_1. At most two of these are assumed to have type D. Suppose σ_0 has order 2. Then, at most one of these components has trivial σ_0-action and at most one has non-trivial σ_0-action.

(i) Suppose first that G_1 has exactly one component of type D. The action of $\tau \in \mathrm{Aut}_W(\Delta_1)$ on this can be either trivial or non-trivial. The action of σ_0 can also be either trivial or non-trivial. Combining these possibilities we see that there are four cases which can arise.

(ii) Suppose G_1 has two components of type D not interchanged by τ. σ_0 acts on both of these and its action is trivial on one, say D_{μ_1}, and non-trivial on the other D_{μ_2}. The τ-action can be trivial or non-trivial on D_{μ_1} and D_{μ_2}. Thus there are again four cases which can arise.

(iii) Suppose G_1 has two components of type D which are interchanged by τ. σ_0 acts trivially on one of these and non-trivially on the other. τ^2 can be either trivial or non-trivial, giving rise to two possible cases.

Consider the components of $(M^g)_\sigma$ to which these various cases give rise. We obtain one component for each orbit of $\sigma_0\tau$ on the components of type D. For each $\sigma_0\tau$-orbit of length 1 we obtain $D_l(q)$ if $\sigma_0\tau = 1$ and ${}^2D_l(q^2)$ if $\sigma_0\tau \neq 1$. For each $\sigma_0\tau$-orbit of length 2 we obtain $D_l(q^2)$ if $(\sigma_0\tau)^2 = 1$ and ${}^2D_l(q^4)$ if $(\sigma_0\tau)^2 \neq 1$.

We summarize the possible components of type D which can arise when σ_0 has order 2 in the diagrams on p. 32.

The case where $G = \mathrm{SO}_{2l}(K)$

Suppose $X = \{\sum_{i=1}^{l} a_i e_i : a_i \in Z\}$. Then X/P_1 is generated by elements $\bar{e}_i$, one for each I_α, and by $\bar{e}_{j_1}, \bar{e}_{j_2}$ satisfying $2\bar{e}_{j_1} = 2\bar{e}_{j_2} = 0$ corresponding to the two components of type D. Hence

$$X/P_1 \cong Z \oplus Z \oplus \ldots \oplus Z_2 \oplus Z_2,$$

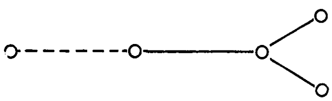
with one component Z for each part of λ. Thus the torsion subgroup $\bar{P}_1/P_1$ is isomorphic to $Z_2 \oplus Z_2$.

Let $w \in \mathcal{N}_W(W_1)$ and let w map to $\tau \in \mathrm{Aut}_W(\Delta_1)$. By Proposition 10 we have

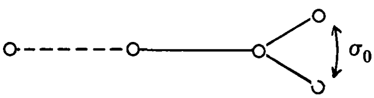
$$\mathrm{Aut}_W(\Delta_1) \cong S_{m_2} \times (Z_2 \wr S_{m_3}) \times (Z_2 \wr S_{m_4}) \times \ldots \times (Z_2 \wr S_{n_2}) \times (Z_2 \wr S_{n_3}) \times \ldots,$$

if $m_1 \neq 0$, where m_i, n_i are the number of parts of λ, μ respectively equal to i, and $\mathrm{Aut}_W(\Delta_1)$ is a subgroup of index 2 in the above if $m_1 = 0$. If $m_1 = 0$ the induced symmetries are those for which the total number of negative cycles on components $A_{\lambda_i - 1}$, where λ_i is even, and components of type D is even. Now $\sigma = q\sigma_0$ where the order of σ_0 is 1 or 2. When $\sigma_0 = 1$, G_σ is the split group $\mathrm{SO}_{2l}(q)$, and when σ_0 has order 2, G_σ is the quasi-split group $\mathrm{SO}^-_{2l}(q)$. Suppose $\sigma_0\tau$ gives rise to a pair of partitions $\xi^{(i)}, \eta^{(i)}$ with $|\xi^{(i)}| + |\eta^{(i)}| = m_i$, where the parts of the partitions give the lengths of the positive and negative cycles on the components of type A_{i-1}.

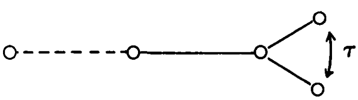
In order to pass from X/P_1 to Γ we impose additional relations $(q\sigma_0 w - 1)x = 0$ for all $x \in X$. Γ is generated by elements $\bar{e}_i$, one for each



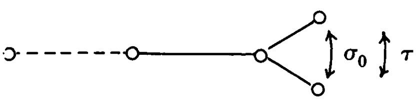
$\sigma_0 = 1,\ \tau = 1,\ \sigma_0\tau = 1;\ D_l(q).$



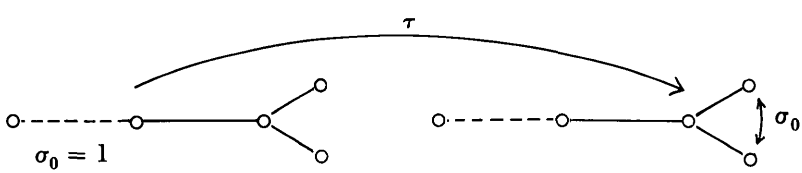
$\tau = 1, \qquad \sigma_0\tau \neq 1;\ {}^2D_l(q^2).$



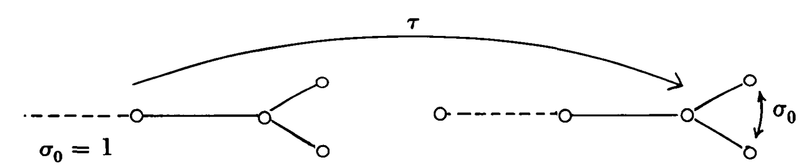
$\sigma_0 = 1, \qquad \sigma_0\tau \neq 1;\ {}^2D_l(q^2).$



$\sigma_0\tau = 1;\ D_l(q).$



$\tau^2 = 1,\ (\sigma_0\tau)^2 \neq 1;\ {}^2D_l(q^4).$



$\tau^2 \neq 1,\ (\sigma_0\tau)^2 = 1;\ D_l(q^2).$

$\sigma_0 w$-orbit of type A, subject to relations

$$(q^k - 1)\bar{e}_i = 0 \quad \text{for positive } k\text{-cycles},$$

$$(q^k + 1)\bar{e}_i = 0 \quad \text{for negative } k\text{-cycles},$$

and by $\bar{e}_j$'s corresponding to the $\sigma_0 w$-orbits of type D. Since there are just two components of type D there are either two $\sigma_0 w$-orbits of type D of length 1 or one $\sigma_0 w$-orbit of type D of length 2. If both components of type D are fixed by $\sigma_0 w$, Γ has two additional generators $\bar{e}_{j_1}, \bar{e}_{j_2}$ satisfying $2\bar{e}_{j_1} = 2\bar{e}_{j_2} = 0$. The image in Γ of $\bar{\Phi}_1 - \Phi_1$ consists of the single element

$\bar{e}_{j_1}+\bar{e}_{j_2}$. This element is non-zero in Γ and there is a character of Γ of order 2 which does not annihilate it. If, on the other hand, the two components of type D are interchanged by $\sigma_0 w$ there is a single generator $\bar{e}_j$ of Γ coming from an orbit of type D satisfying $2\bar{e}_j = 0$. The elements of $\bar{\Phi}_1 - \Phi_1$ then have image 0 in Γ. This justifies the entry in Table 2 for the group SO_{2l}.

The simple components of $(G_1{}^\sigma)_\sigma$ in this case are of type $A_{i-1}(q^{\xi^{(i)}_j})$, ${}^2A_{i-1}(q^{2\eta^{(i)}_j})$, $D_{\mu_1}(q)$ or ${}^2D_{\mu_1}(q^2)$, and $D_{\mu_2}(q)$ or ${}^2D_{\mu_2}(q^2)$. Moreover, the total number of twisted components of type ${}^2D$ and type ${}^2A_{i-1}$ with even i is even if $\sigma_0 = 1$ and odd if $\sigma_0 \neq 1$.

The adjoint case

Now suppose G is an adjoint group of type D_l. Then

$$X = \left\{\sum_{i=1}^{l} a_i e_i \colon a_i \in Z,\ \sum_{i=1}^{l} a_i \text{ even}\right\}.$$

X/P_1 is the set of elements $\sum a_i\bar{e}_i + a_{j_1}\bar{e}_{j_1} + a_{j_2}\bar{e}_{j_2}$ with one $\bar{e}_i$ for each component of type A in Φ_1 subject to the conditions $\sum a_i + a_{j_1} + a_{j_2}$ is even and $a_{j_1}, a_{j_2} \in \{0, 1\}$. The torsion subgroup $\bar{P}_1/P_1$ of X/P_1 is the subgroup of order 2 generated by $\bar{e}_{j_1}+\bar{e}_{j_2}$.

We consider the additional relations needed to pass from X/P_1 to its quotient group Γ. These relations have the form $(q\sigma_0 w - 1)x = 0$ for all $x \in X$. We have, for each i, $w(\bar{e}_i) = \varepsilon_i \bar{e}_{i'}$ for some i', where $\varepsilon_i = \pm 1$. As in the adjoint case in type C_l we choose a new system of generators for X/P_1 as follows. X/P_1 is generated by $\bar{e}_i + \varepsilon_i\bar{e}_{i'}$, $\bar{e}_i - \varepsilon_i\bar{e}_{i'}$ for all $\bar{e}_i$ corresponding to components of type A, by $\bar{e}_{j_1}+\bar{e}_{j_2}$, and by $\bar{e}_i - \bar{e}_j$ where $\bar{e}_i, \bar{e}_j$ are in distinct w-orbits of any type. Thus Γ is generated by elements $\bar{e}_i + \varepsilon_i\bar{e}_{i'}$, $\bar{e}_i - \varepsilon_i\bar{e}_{i'}$, $\bar{e}_{j_1}+\bar{e}_{j_2}$, $\bar{e}_i - \bar{e}_j$ subject to relations

$$(q\sigma_0 w - 1)(\bar{e}_i + \varepsilon_i\bar{e}_{i'}) = 0,$$
$$(q\sigma_0 w - 1)(\bar{e}_i - \varepsilon_i\bar{e}_{i'}) = 0,$$
$$(q\sigma_0 w - 1)(\bar{e}_i - \bar{e}_j) = 0,$$
$$2(\bar{e}_{j_1}+\bar{e}_{j_2}) = 0,$$
$$(\bar{e}_i - \bar{e}_j) + (\bar{e}_j - \bar{e}_k) = (\bar{e}_i - \bar{e}_k),$$
$$2(\bar{e}_i - \bar{e}_j) = (\bar{e}_i + \varepsilon_i\bar{e}_{i'}) + (\bar{e}_i - \varepsilon_i\bar{e}_{i'}) - (\bar{e}_j + \varepsilon_j\bar{e}_{j'}) - (\bar{e}_j - \varepsilon_j\bar{e}_{j'}).$$

Consider the image of $\bar{\Phi}_1 - \Phi_1$ in Γ. This consists of a single element $\bar{e}_{j_1}+\bar{e}_{j_2}$. $\bar{e}_{j_1}, \bar{e}_{j_2}$ may or may not be in the same $\sigma_0 w$-orbit. Suppose that $\bar{e}_{j_1}, \bar{e}_{j_2}$ are not in the same $\sigma_0 w$-orbit. Then suppose we impose on Γ the additional relations $\bar{e}_i + \varepsilon_i\bar{e}_{i'} = 0$, $\bar{e}_i - \varepsilon_i\bar{e}_{i'} = 0$ for each orbit of type A. We then obtain a quotient group of Γ generated by $\bar{e}_{j_1}+\bar{e}_{j_2}$, $\bar{e}_i - \bar{e}_j$ (one term for each pair of distinct $\sigma_0 w$-orbits) subject to relations

$$2(\bar{e}_{j_1}+\bar{e}_{j_2}) = 0, \quad (\bar{e}_i-\bar{e}_j)+(\bar{e}_j-\bar{e}_k) = (\bar{e}_i-\bar{e}_k), \quad 2(\bar{e}_i-\bar{e}_j) = 0.$$

These relations do not imply $\bar{e}_{j_1}+\bar{e}_{j_2} = 0$. Thus $\bar{e}_{j_1}+\bar{e}_{j_2} \neq 0$ in Γ.

We therefore assume that $\bar{e}_{j_1}, \bar{e}_{j_2}$ are in the same w-orbit. We simplify the given system of generators and relations by considering the orbits of $\sigma_0 w$ on the components of Φ_1 of type A. These orbits correspond to parts $\xi^{(i)}{}_j, \eta^{(i)}{}_j$ of the partitions $\xi^{(i)}, \eta^{(i)}$, where $\xi^{(i)}{}_j$ is a positive cycle on components of type A_{i-1} and $\eta^{(i)}{}_j$ is a negative cycle on components of type A_{i-1}. In addition there is one $\sigma_0 w$-orbit on the two components of type D. We put $c_i = \bar{e}_i - q\varepsilon_i \bar{e}_{i'}, d_i = 2\varepsilon_i \bar{e}_{i'}$, one pair c_i, d_i for each w-orbit of type A. Then, just as in the case of an adjoint group of type C_l (Lemma 13), we obtain the following result.

LEMMA 15. *Suppose Φ_1 has exactly two components of type D. Then Γ is isomorphic to the abelian group with generators c_i, d_i (one pair for each $\sigma_0 w$-orbit of type A) and $\bar{e}_i - \bar{e}_j$ (one term for each pair of distinct $\sigma_0 w$-orbits) subject to the following relations:*

(i) *when* $k = \xi^{(i)}{}_j$,

$$\tfrac{1}{2}(q^k-1)d_i = \begin{cases} 0 & \text{if } k \text{ is even,} \\ c_i & \text{if } k \text{ is odd;} \end{cases}$$

(ii) *when* $k = \eta^{(i)}{}_j$,

$$\tfrac{1}{2}(q^k+1)d_i = \begin{cases} 0 & \text{if } k \text{ is even,} \\ c_i & \text{if } k \text{ is odd;} \end{cases}$$

(iii) $c_i = c_j$ *for all pairs of w-orbits;*
(iv) $2c_i = 0$ *for all orbits;*
(v) $(\bar{e}_i-\bar{e}_j)+(\bar{e}_j-\bar{e}_k) = (\bar{e}_i-\bar{e}_k)$;
(vi) $2(\bar{e}_i-\bar{e}_j) = \varepsilon_i d_i - \varepsilon_j d_j$.

Lemma 15 asserts in particular that $c_i = c_j$ for all pairs of w-orbits. We write $c = c_i$ for all i. Since $\bar{e}_{j_1}, \bar{e}_{j_2}$ are in the same w-orbit the element $\bar{e}_{j_1}+\bar{e}_{j_2}$ is equal to c in Γ. Thus we wish to decide whether the relations of Lemma 15 imply that $c = 0$. By applying Lemma 14 and considering the two generator subgroups $\langle c_i, d_i\rangle$ we deduce that $c \neq 0$ in Γ. This justifies the entry in Table 2 for adjoint groups of type D_l.

We note that if $p \neq 2$ and there are two components D_{μ_1}, D_{μ_2} of type D then the simple components of $(G_1{}^0)_\sigma$ of type A are of form $A_{i-1}(q^{\xi^{(i)}{}_j})$, ${}^2A_{i-1}(q^{2\eta^{(i)}{}_j})$, and the simple components of type D have form $D_{\mu_1}(q)$ or ${}^2D_{\mu_1}(q^2)$, $D_{\mu_2}(q)$ or ${}^2D_{\mu_2}(q^2)$ if $\sigma_0 w$ does not interchange the two components D_{μ_1}, D_{μ_2}, and have form $D_\mu(q^2)$ or ${}^2D_\mu(q^4)$ if $\sigma_0 w$ does interchange D_{μ_1}, D_{μ_2}.

The simply-connected case

Now suppose G is a simply-connected group of type D_l, i.e. $G \cong \mathrm{Spin}_{2l}(K)$. Then X is the group generated by $e_1, e_2, \ldots, e_l, q_l$, where

$$q_l = \tfrac{1}{2}(e_1 + e_2 + \ldots + e_l).$$

X/P_1 is generated by elements $\bar{e}_i$, one for each component of type A in Φ_1, by $\bar{e}_{j_1}, \bar{e}_{j_2}$ corresponding to the two components of type D, and by $\bar{q}_l$. In order to pass from X/P_1 to Γ we impose the additional relations $(q\sigma_0 w - 1)x = 0$ for all $x \in X$.

Suppose first that $\sigma_0 w$ interchanges the two components of type D. Then the relation $(q\sigma_0 w - 1)\bar{e}_{j_1} = 0$ implies that $\bar{e}_{j_1} + \bar{e}_{j_2} = 0$ in Γ. However, the image of $\bar{\Phi}_1 - \Phi_1$ in Γ consists of the single element $\bar{e}_{j_1} + \bar{e}_{j_2}$. Thus Γ has no regular character in this case and so we cannot have the connected centralizer of a semisimple element.

Thus we assume that $\sigma_0 w$ fixes both components D_{μ_1}, D_{μ_2} of type D. Then Γ is generated by elements $\bar{e}_i$ corresponding to components of type A, by $\bar{e}_{j_1}, \bar{e}_{j_2}$, and by $\bar{q}_l$ subject to relations

$$(q\sigma_0 w - 1)\bar{e}_i = 0,$$
$$2\bar{e}_{j_1} = 2\bar{e}_{j_2} = 0,$$
$$(q\sigma_0 w - 1)\bar{q}_l = 0,$$
$$2\bar{q}_l = (\overline{\textstyle\sum_{i=1}^{l} e_i}).$$

The image of $\bar{\Phi}_1 - \Phi_1$ in Γ consists of the single element $\bar{e}_{j_1} + \bar{e}_{j_2}$ and the only relation which could possibly imply $\bar{e}_{j_1} + \bar{e}_{j_2} = 0$ is the relation $(q\sigma_0 w - 1)\bar{q}_l = 0$. We therefore consider this relation in detail.

Since $2q_l = e_1 + e_2 + \ldots + e_l$, we have

$$2\bar{q}_l = \zeta_1 \bar{e}_{j_1} + \zeta_2 \bar{e}_{j_2} + \text{a linear combination of } \bar{e}_i\text{'s},$$

where $\zeta_1 = 0$ if μ_1 is even and $\zeta_1 = 1$ if μ_1 is odd, and ζ_2 is defined similarly for μ_2. Also we have

$$\bar{q}_l - \sigma_0 w(\bar{q}_l) = \eta_1 \bar{e}_{j_1} + \eta_2 \bar{e}_{j_2} + \text{a linear combination of } \bar{e}_i\text{'s},$$

where $\eta_1 = 0$ if $\sigma_0 w$ does not twist D_{μ_1} and $\eta_1 = 1$ if $\sigma_0 w$ does twist D_{μ_1}, and η_2 is defined similarly for D_{μ_2}. Hence the relation $(q\sigma_0 w - 1)\bar{q}_l = 0$ has the form $\tfrac{1}{2}(q-1)(2\bar{q}_l) = q(\bar{q}_l - \sigma_0 w(\bar{q}_l))$, which gives

$$\tfrac{1}{2}(q-1)(\zeta_1 \bar{e}_{j_1} + \zeta_2 \bar{e}_{j_2}) = q(\eta_1 \bar{e}_{j_1} + \eta_2 \bar{e}_{j_2}) + \text{a linear combination of } \bar{e}_i\text{'s}.$$

Thus we have

$$(\tfrac{1}{2}(q-1)\zeta_1 - \eta_1)\bar{e}_{j_1} + (\tfrac{1}{2}(q-1)\zeta_2 - \eta_2)\bar{e}_{j_2} = \text{a linear combination of } \bar{e}_i\text{'s}.$$

We consider when this equation has the form

$$\bar{e}_{j_1} + \bar{e}_{j_2} = \text{a linear combination of } \bar{e}_i\text{'s}.$$

If $q \equiv 1 \bmod 4$ this is so if and only if $\eta_1 = 1, \eta_2 = 1$, i.e. $\sigma_0 w$ twists both D_{μ_1}, D_{μ_2}. If $q \equiv -1 \bmod 4$ this is so if and only if $\zeta_1 \not\equiv \eta_1 \bmod 2$ and $\zeta_2 \not\equiv \eta_2 \bmod 2$. This means the components have the form $D_{\mu_i}(q)$, where μ_i is odd, or ${}^2D_{\mu_i}(q)$, where μ_i is even. If these conditions are not satisfied we cannot deduce $\bar{e}_{j_1} + \bar{e}_{j_2} = 0$ in Γ and so Γ has a regular character.

Thus we suppose henceforward that either $q \equiv 1 \bmod 4$ and the D-components have the form ${}^2D_{\mu_1}(q^2)$, ${}^2D_{\mu_2}(q^2)$ or that $q \equiv -1 \bmod 4$ and the D-components are $D_{\mu_i}(q)$, for μ_i odd, or ${}^2D_{\mu_i}(q^2)$, for μ_i even. Then the contributions to $(q\sigma_0 w - 1)\bar{q}_l$ from the components of type A are calculated just as in the simply-connected groups of type B_l. The relation

$$(q\sigma_0 w - 1)\bar{q}_l = 0$$

becomes

$$\bar{e}_{j_1} + \bar{e}_{j_2} = \sum \tfrac{1}{2}\lambda_i(q^k - 1)\bar{e}_i - \sum \tfrac{1}{2}(\lambda_i/s)\,(q^k + 1)\bar{e}_i,$$

where the first sum extends over all the positive k-cycles of $\sigma_0 w$ on X/P_1 and the second over all negative k-cycles on X/P_1 induced by negative r-cycles on X. (Here $s = r/k$.) Now $\bar{e}_i$ has order $q^k - 1$ for a positive k-cycle, and order $q^k + 1$ for a negative k-cycle. Thus we obtain $\bar{e}_{j_1} + \bar{e}_{j_2} = 0$ if and only if λ_i is even for each positive cycle on X/P_1 and λ_i/s is even for each negative cycle on X/P_1 induced by a negative cycle on X. This is equivalent to the assertion that λ_i is even for all cycles of $\sigma_0 w$ on X/P_1. This completes the discussion in the simply-connected case and we have justified the entry in Table 2 in this case.

We note that the simple components of $(G_1{}^g)_\sigma$ in this case are of type $A_{i-1}(q^{\xi^{(i)}_j})$, ${}^2A_{i-1}(q^{2\eta^{(i)}_j})$, $D_{\mu_1}(q)$ or ${}^2D_{\mu_1}(q^2)$, and $D_{\mu_2}(q)$ or ${}^2D_{\mu_2}(q^2)$. Also the total number of twisted components of type ${}^2D$ and of type ${}^2A_{i-1}$, with even i, is even if $\sigma_0 = 1$ and odd if $\sigma_0 \neq 1$.

The half-spin group

There is one further type of group D_l to be considered when l is even. This is the half-spin group. In this group one must have $\sigma_0 = 1$. The situation here is more complicated than in the other isogenous groups of type D_l, and we shall state the results without proof.

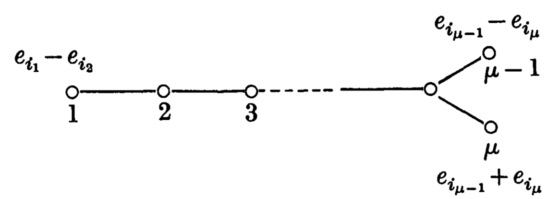
PROPOSITION 16. *Let G be the half-spin group of type D_l over an algebraically closed field of characteristic $p \neq 2$, and let G_1 be a reductive subgroup of maximal rank in G of type $A_{\lambda_1-1} \times A_{\lambda_2-1} \times \ldots \times D_{\mu_1} \times D_{\mu_2}$. Let $w \in \mathcal{N}_W(W_1)$, and suppose w fixes the two components D_{μ_1}, D_{μ_2}. Let $G_1{}^g$ be a σ-stable subgroup of G obtained by twisting G_1 by w, that is, $\pi(g^\sigma g^{-1}) = w$. Then $(G_1{}^g)_\sigma$ is the connected centralizer in G_σ of a semisimple element for q sufficiently large unless all λ_i are even and the D-components have the form ${}^2D_{\mu_1}(q^2)$, ${}^2D_{\mu_2}(q^2)$ if $q \equiv 1 \bmod 4$, and have the form $D_{\mu_i}(q)$, for μ_i odd, or ${}^2D_{\mu_i}(q^2)$, for μ_i even, if $q \equiv -1 \bmod 4$.*

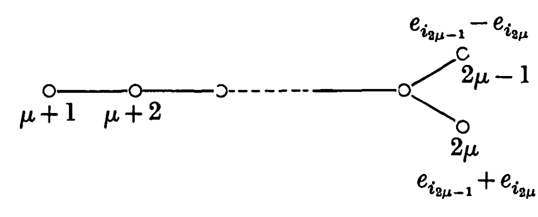
PROPOSITION 17. *Let G be as in Proposition* 16, *and let G_1 be a reductive subgroup of maximal rank in G of type $A_{\lambda_1-1} \times A_{\lambda_2-1} \times \ldots \times D_\mu \times D_\mu$. Let $w \in \mathcal{N}_W(W_1)$ interchange and twist the two components of type D_μ, thus giving rise to a component ${}^2D_\mu(q^4)$ in $(G_1{}^g)_\sigma$. Let $G_1{}^g$ be a σ-stable subgroup of G obtained by twisting G_1 by w. Then $(G_1{}^g)_\sigma$ is the connected centralizer in G_σ of a semisimple element for q sufficiently large.*

PROPOSITION 18. *Let G be as in Proposition* 16, *and let G_1 be a reductive subgroup of maximal rank in G of type $A_{\lambda_1-1} \times A_{\lambda_2-1} \times \ldots \times D_\mu \times D_\mu$. Let $w \in \mathcal{N}_W(W_1)$ interchange the components D_μ without twisting them, thus giving rise to a component $D_\mu(q^2)$ in $(G_1{}^g)_\sigma$. Let $G_1{}^g$ be a σ-stable subgroup of G obtained by twisting G_1 by w. Then $(G_1{}^g)_\sigma$ is the connected centralizer in G_σ of a semisimple element for q sufficiently large unless all λ_i are even. If all λ_i are even $(G_1{}^g)_\sigma$ is a connected centralizer for q sufficiently large if and only if*

$$\tfrac{1}{2}(q-1)\mu + M + \nu \equiv 0 \mod 2,$$

where M, ν are defined as follows. M is the number of simple components of form $A_{\lambda_i-1}(q^e)$ or ${}^2A_{\lambda_i-1}(q^{2e})$, where $\lambda_i \equiv 2 \bmod 4$ and e is odd. ν is 0 if w induces the positive graph automorphism on $D_\mu + D_\mu$, and ν is 1 if w induces the negative graph automorphism. The positive and negative graph automorphisms are defined as follows. Suppose the two components D_{μ_1}, D_{μ_2} of Φ_1 have fundamental roots as shown.





We are assuming that w interchanges these two components but does not twist them. There are two possible graph automorphisms of the Dynkin diagram which could be induced by w. If the nodes are numbered as above these automorphisms are τ_1, τ_2 defined by

$$\tau_1 = (1, \mu+1)(2, \mu+2)\ldots(\mu-1, 2\mu-1)(\mu, 2\mu),$$
$$\tau_2 = (1, \mu+1)(2, \mu+2)\ldots(\mu-1, 2\mu)(\mu, 2\mu-1).$$

τ_1, τ_2 are called the positive and negative graph automorphisms respectively.

NOTE. It may appear odd that the criterion for being a connected centralizer should depend upon whether w induces the positive or negative graph automorphism on $D_\mu + D_\mu$. However, these two graph automorphisms do not play a symmetric rôle for a half-spin group G with given character group X. We observed earlier that there are two possible lattices X between the root lattice and the weight lattice which are character groups for half-spin groups. If the other possible lattice X were chosen instead as the character group the rôles of the positive and negative graph automorphisms of $D_\mu + D_\mu$ would be interchanged in the criterion for being a connected centralizer.

The results for the half-spin group are summarized in Table 2 of § 4.

8. The character degrees

Finally, we shall apply the results obtained for the centralizers of semisimple elements in G_σ when G is simply-connected to give the degrees of the irreducible semisimple representations in the dual group $\tilde{G}_\sigma$, which is adjoint. In order to give these degrees the following notation will be useful, and we assume q is odd:

$$\begin{aligned} \alpha_l(q) &= |A_l(q)|_{q'}(q-1);\\ {}^2\alpha_l(q^2) &= |{}^2A_l(q^2)|_{q'}(q+1);\\ \beta_l(q) &= |B_l(q)|_{q'};\\ \gamma_l(q) &= |C_l(q)|_{q'};\\ \delta_l(q) &= |D_l(q)|_{q'};\\ {}^2\delta_l(q^2) &= |{}^2D_l(q^2)|_{q'}. \end{aligned}$$

Here $|A_l(q)|_{q'}$ denotes the part of $|A_l(q)|$ prime to q, etc. Moreover, we define, for small values of l,

$$\begin{aligned} \alpha_0(q) &= q-1,\\ {}^2\alpha_0(q^2) &= q+1,\\ \alpha_1(q) &= (q^2-1)(q-1),\\ {}^2\alpha_1(q^2) &= (q^2-1)(q+1),\\ \delta_2(q) &= (q^2-1)^2,\\ {}^2\delta_2(q^2) &= q^4-1. \end{aligned}$$

The group $(A_l)_{\mathrm{ad}}(q) = \mathrm{PGL}_{l+1}(q)$

The dual group is $(A_l)_{\mathrm{sc}}(q) = \mathrm{SL}_{l+1}(q)$. There is one genus of semisimple classes in $(A_l)_{\mathrm{sc}}(q)$ for each partition $\lambda = (1^{n_1} 2^{n_2} 3^{n_3} \ldots)$ of $l+1$ and each partition $\mu^{(i)} = (\mu^{(i)}{}_1 \mu^{(i)}{}_2 \ldots)$ of n_i for $i = 1, 2, \ldots$. The degree of the corresponding family of irreducible representations of $(A_l)_{\mathrm{ad}}(q)$ is

$$\alpha_l(q)/\prod_{i,j} \alpha_{i-1}(q^{\mu^{(i)}_j}).$$

The group $({}^2A_l)_{\mathrm{ad}}(q^2) = \mathrm{PU}_{l+1}(q)$

The dual group is $({}^2A_l)_{\mathrm{sc}}(q^2) = \mathrm{SU}_{l+1}(q)$. There is one genus of semisimple classes in $({}^2A_l)_{\mathrm{sc}}(q^2)$ for each partition $\lambda = (1^{n_1}2^{n_2}3^{n_3}\dots)$ of $l+1$ and each partition $\mu^{(i)} = (\mu^{(i)}{}_1\,\mu^{(i)}{}_2\dots)$ of n_i for $i = 1, 2, \dots$. The degree of the corresponding family of irreducible representations of $({}^2A_l)_{\mathrm{ad}}(q^2)$ is

$${}^2\alpha_l(q^2)/\prod_{\substack{i,j\\ \mu^{(i)}_j\text{even}}} \alpha_{i-1}(q^{\mu^{(i)}_j}) \prod_{\substack{i,j\\ \mu^{(i)}_j\text{odd}}} {}^2\alpha_{i-1}(q^{2\mu^{(i)}_j}).$$

The group $(B_l)_{\mathrm{ad}}(q) = \mathrm{SO}_{2l+1}(q)$

The dual group is $(C_l)_{\mathrm{sc}}(q) = \mathrm{Sp}_{2l}(q)$. We take a pair of partitions $\lambda = (1^{m_1}2^{m_2}3^{m_3}\dots), \mu = (1^{n_1}2^{n_2}3^{n_3}\dots)$, with $|\lambda|+|\mu| = l$, and assume that μ has at most two parts. We then take sets of partitions

$$\xi^{(i)} = (\xi^{(i)}{}_1\,\xi^{(i)}{}_2\dots), \quad \eta^{(i)} = (\eta^{(i)}{}_1\,\eta^{(i)}{}_2\dots),$$

with $|\xi^{(i)}|+|\eta^{(i)}| = m_i$, and partitions

$$\zeta^{(i)} = (\zeta^{(i)}{}_1\,\zeta^{(i)}{}_2\dots),$$

with $|\zeta^{(i)}| = n_i$. There is one genus of semisimple classes in $(C_l)_{\mathrm{sc}}(q)$ for each such set of partitions $\lambda, \mu, \xi^{(i)}, \eta^{(i)}, \zeta^{(i)}$ except that if $\mu = (i\,i)$ then $\zeta^{(i)} \neq (2)$. The degree of the corresponding family of irreducible representations of $(B_l)_{\mathrm{ad}}(q)$ is

$$\gamma_l(q)/\prod_{i,j} \alpha_{i-1}(q^{\xi^{(i)}_j}) \prod_{i,j} {}^2\alpha_{i-1}(q^{2\eta^{(i)}_j}) \prod_{i,j} \gamma_i(q^{\zeta^{(i)}_j}).$$

The group $(C_l)_{\mathrm{ad}}(q) = \mathrm{PG\,Sp}_{2l}(q)$

The dual group is $(B_l)_{\mathrm{sc}}(q) = \mathrm{Spin}_{2l+1}(q)$. We take a pair of partitions $\lambda = (1^{m_1}2^{m_2}3^{m_3}\dots), \mu = (2^{n_2}3^{n_3}\dots)$, and a number ν with $|\lambda|+|\mu|+\nu = l$, and assume that μ has at most one part. We then take sets of partitions

$$\xi^{(i)} = (\xi^{(i)}{}_1\,\xi^{(i)}{}_2\dots), \quad \eta^{(i)} = (\eta^{(i)}{}_1\,\eta^{(i)}{}_2\dots),$$

with $|\xi^{(i)}|+|\eta^{(i)}| = m_i$, and

$$\zeta^{(i)} = (\zeta^{(i)}{}_1\,\zeta^{(i)}{}_2\dots), \quad \omega^{(i)} = (\omega^{(i)}{}_1\,\omega^{(i)}{}_2\dots),$$

with $|\zeta^{(i)}|+|\omega^{(i)}| = n_i$. (Of course $n_i = 0$ or 1 so $\zeta^{(i)}, \omega^{(i)}$ have an extremely simple form.) There is one genus of semisimple classes in $(B_l)_{\mathrm{sc}}(q)$ for each such set of partitions $\lambda, \mu, \xi^{(i)}, \eta^{(i)}, \zeta^{(i)}, \omega^{(i)}$, except that we must exclude those giving rise to a critical subgroup as defined in § 4, Table 2. The degree of the corresponding family of irreducible representations of $(C_l)_{\mathrm{ad}}(q)$ is

$$\beta_l(q)/\prod_{i,j} \alpha_{i-1}(q^{\xi^{(i)}_j}) \prod_{i,j} {}^2\alpha_{i-1}(q^{2\eta^{(i)}_j}) \prod_{i,j} \delta_i(q^{\zeta^{(i)}_j}) \prod_{i,j} {}^2\delta_i(q^{2\omega^{(i)}_j}).$$

The group $(D_l)_{\mathrm{ad}}(q) = \mathrm{PO}_{2l}(q)$

The dual group is $(D_l)_{\mathrm{sc}}(q) = \mathrm{Spin}_{2l}(q)$. We take a pair of partitions $\lambda = (1^{m_1} 2^{m_2} 3^{m_3} \ldots), \mu = (2^{n_2} 3^{n_3} \ldots)$, with $|\lambda| + |\mu| = l$, and assume that μ has at most two parts. We then take sets of partitions

$$\xi^{(i)} = (\xi^{(i)}{}_1\, \xi^{(i)}{}_2 \ldots), \quad \eta^{(i)} = (\eta^{(i)}{}_1\, \eta^{(i)}{}_2 \ldots),$$

with $|\xi^{(i)}| + |\eta^{(i)}| = m_i$, and

$$\zeta^{(i)} = (\zeta^{(i)}{}_1\, \zeta^{(i)}{}_2 \ldots), \quad \omega^{(i)} = (\omega^{(i)}{}_1\, \omega^{(i)}{}_2 \ldots),$$

with $|\zeta^{(i)}| + |\omega^{(i)}| = n_i$. There is a family of semisimple classes in $(D_l)_{\mathrm{sc}}(q)$ for each such set of partitions $\lambda, \mu, \xi^{(i)}, \eta^{(i)}, \zeta^{(i)}, \omega^{(i)}$ except that if μ has two parts then we must exclude those giving rise to critical subgroups as described in § 4, Table 2, and if $\mu = (i\,i)$ then we cannot have $\zeta^{(i)} = (2)$ or $\omega^{(i)} = (2)$. We note also in this case that a family may contain more than one genus of semisimple classes. The degree of the corresponding family of irreducible representations of $(D_l)_{\mathrm{ad}}(q)$ is

$$\delta_l(q) / \prod_{i,j} \alpha_{i-1}(q^{\xi^{(i)}{}_j}) \prod_{i,j} {}^2\alpha_{i-1}(q^{2\eta^{(i)}{}_j}) \prod_{i,j} \delta_i(q^{\zeta^{(i)}{}_j}) \prod_{i,j} {}^2\delta_i(q^{2\omega^{(i)}{}_j}).$$

The group $({}^2D_l)_{\mathrm{ad}}(q^2) = \mathrm{PO}^-_{2l}(q)$

The dual group is $({}^2D_l)_{\mathrm{sc}}(q) = \mathrm{Spin}^-_{2l}(q)$. We take sets of partitions $\lambda, \mu, \xi^{(i)}, \eta^{(i)}, \zeta^{(i)}, \omega^{(i)}$ as for $(D_l)_{\mathrm{ad}}(q)$. There is a family of semisimple classes in $({}^2D_l)_{\mathrm{sc}}(q^2)$ for each such set of partitions, except that if μ has two parts then we must exclude those giving rise to critical subgroups as described in § 4, Table 2, and if $\mu = (i\,i)$ then we cannot have $\zeta^{(i)} = (2)$ or $\omega^{(i)} = 2$. The degree of the corresponding family of irreducible representations of $({}^2D_l)_{\mathrm{ad}}(q^2)$ is

$${}^2\delta_l(q^2) / \prod_{i,j} \alpha_{i-1}(q^{\xi^{(i)}{}_j}) \prod_{i,j} {}^2\alpha_{i-1}(q^{2\eta^{(i)}{}_j}) \prod_{i,j} \delta_i(q^{\zeta^{(i)}{}_j}) \prod_{i,j} {}^2\delta_i(q^{2\omega^{(i)}{}_j}).$$

Finally, if K has characteristic 2 the degrees of the irreducible semisimple characters of the adjoint group G when q is sufficiently large are given by

$$|\tilde{G}_\sigma|_{q'} / |\tilde{L}_\sigma|_{q'},$$

where $\tilde{L}$ is a σ-stable reductive part of some parabolic subgroup of the dual group $\tilde{G}$.

Analogous results for the exceptional groups have been obtained by Deriziotis **[10]**.

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