Unbiased Efficient Estimator of the Fourth-Order Cumulant for Random Zero-Mean Non-i.i.d. Signals: Particular Case of MA Stochastic Process

Iaroslav V. Blagouchine and Eric Moreau, *Senior Member, IEEE*

***Abstract*—Non-Gaussian processes may require not only the information provided by first two moments, but also that given by the higher-order statistics, in particular, by the third- and fourth-order moments or cumulants. This paper addresses a fourth-order cumulant estimation problem for real discrete-time random non-i.i.d. signal, that can be approximated as an MA stochastic process. An unbiased estimator is proposed, studied and compared to two other frequently used estimators of the fourth-order cumulant (natural estimator and fourth k-statistics). Statistical comparative studies are undertaken from both bias and MSE points of view, for different distribution laws and MA filters. Algorithms, aiming to reduce computational complexity of the proposed estimator, as well as that of the fourth k-statistics bias, are also provided.**

***Index Terms*—k-statistics, bias, consistency, cumulants, estimation, estimator, higher order statistics (HOS), mean square error (MSE), moments, non-i.i.d processes, random signals, semi-invariants, stochastic processes, unbiasedness.**

I. Introduction

IT IS well known that for the non-normal processes, their description given in terms of the first- and second-order statistics are often not sufficient, and the supplementary information, provided by the higher-order statistics, may be required. Among these statistics, the third- and fourth-order moments and cumulants have received some special interest. Over 20 past years the research community addressed many interesting aspects and applications of them: blind source separation (BSS) [1]–[6], identification of the finite impulse response (IR) systems [7]–[12], speech processing [13]–[16], cyclo- and almost-cyclo-stationarity [17]–[21], acoustics [22], biomedicine [23], etc. Thus, in practice, it is often necessary to accurately estimate these higher-order statistics.

Usually, *cumulants*, also known as *cumulative moment functions*, *semi-invariants* or *half-invariants* [24]–[28], are more often used in applications, and the moments have generally only an auxiliary function. In earlier paper [29], different estimators of the fourth-order cumulant for real discrete-time random i.i.d. zero-mean signal were presented, studied and compared in detail theoretically and empirically. This work *inter alia* showed that from many points of view, the unbiased estimator of the fourth-order cumulant, formulæ (11) and (20) in [29], behaves better or equally than other classically used estimators of the fourth-order cumulant. However, the work [29], as well as many others in domain, suffers from a strict requirement that the samples of the considered signal are supposed to be i.i.d., while in practice, this is not always the case. A very frequent case of the non-i.i.d. signals is that of the signals that can be modeled as the i.i.d. signals passed through an ARMA filter, i.e., the so-called ARMA processes. However, the general class of the ARMA processes is relatively huge (e.g., signals having Markov properties, i.e., AR processes, MA processes), and often, the MA processes' approximation may be sufficient, especially because the IR of the corresponding MA filter may be chosen long enough. The aim of this paper is therefore to address an extension of the work [29] to this non-i.i.d. case.

In a recent paper [30], it is shown that the unbiased estimator for the i.i.d. processes given in [29], formula (11), becomes biased for the dependent data; in particular, for the strongly-mixing and ρ-mixing processes (e.g., the ARMA processes). Moreover, it is also remarked in [30], that the strongly-mixing and ρ-mixing processes (the latter are actually contained in the former class) are quite rich classes of processes (see also [31]–[34] for more details). On the other hand, in the considered non-i.i.d. cases the proposed estimator [29], (11), remains consistent, and the estimation error is asymptotically normal [30]. Thus, the present paper aims not only to generalize the work [29], but also to respond to the "criticism" from [30] by proposing an unbiased estimator of the fourth-order cumulant for the dependent-data cases, which can be approximated as MA stochastic processes.

We briefly recall several basic statements related to the considered problem. First of all, we consider two real discrete-time random zero-mean signals $x_i \equiv x(i)$ and $u_i \equiv u(i)$, $i \in \mathbb{Z}$, where i is discrete time, i.e., number of current sample. Signal x_i is i.i.d. at least up to order 8, see [29]. Signal u_i is a non-i.i.d. process modeled as an i.i.d. stochastic signal x_i filtered by a real MA filter of order p, whose coefficients are denoted by $\{a_k\}_{k=0}^{p}$. In addition, for mathematical convenience, it is also supposed the coefficients a_k for $k \notin [0,p]$ exist and are all null. The IR of the considered MA filter, denoted via h_i, therefore is

$$h_i = \sum_{k=-\infty}^{+\infty} a_k \delta_{i,k} = \sum_{k=0}^{p} a_k \delta_{i,k}, \quad i \in \mathbb{Z}, \quad \forall\, a_k \in \mathbb{R}$$

Manuscript received January 14, 2010; revised April 01, 2010. Date of current version November 19, 2010.

The authors are with the Telecommunication Department, ISITV, University of Toulon, F-83162 Valette du Var, Toulon, France (e-mail: iaroslav.blagouchine@univ-tln.fr; moreau@univ-tln.fr).

Communicated by E. Serpedin, Associate Editor for Signal Processing.

Color versions of one or more of the figures in this paper are available online at http://ieeexplore.ieee.org.

Digital Object Identifier 10.1109/TIT.2010.2078270

where $\delta_{i,k}$ is the Kronecker symbol[1]. This MA filter is stable since it has a finite IR. Thus, the u_i may be written as

$$u_i = h_i * x_i = \sum_{k=-\infty}^{+\infty} a_k x_{i-k} = \sum_{k=0}^{p} a_k x_{i-k} \qquad (1)$$

where $*$ is the discrete convolution operator. Furthermore, it is also supposed that all raw moments up to order 8 of the signal x_i exist, and hence, so do those of u_i. The latter are denoted for simplicity by $\mathrm{E}[x^p] \equiv \mathrm{E}[x_i^p]$ and $\mathrm{E}[u^p] \equiv \mathrm{E}[u_i^p]$ respectively, $p \in \mathbb{N}^*$, where $\mathrm{E}[\cdot]$ is the operator of mathematical expectation. Also, for simplicity, we will write in further h instead of h_i for the IR of the above-mentioned MA filter. In addition, we will also use MA filters whose IRs, designated by h^n, are[2]

$$h^n \equiv h_i^n = \underbrace{\sum_k \cdots \sum_l}_{n \text{ times}} a_k \cdots a_l \delta_{i,k} \cdots \delta_{i,l} = \sum_{k=0}^{p} a_k^n \delta_{i,k}$$

where $n \in \mathbb{N}^*$, $i \in \mathbb{Z}$. That is to say, we again omit the discrete argument i by writing h^n instead of h_i^n.

We now recall the moment-based definition (as opposed to the characteristic function based one) of the fourth-order cumulant κ_4 for an arbitrary real discrete-time random zero-mean signal y_i [24], [25], [27], [28], [35]–[37]

$$\kappa_4[y] = \mathrm{E}[y^4] - 3\mathrm{E}^2[y^2]. \qquad (2)$$

Basing on this definition, many people use the so-called "natural estimator" for the estimation of the fourth-order cumulant (2), in which the unknown moments are trivially replaced by the sample ones [29], [38]:

$$\hat{\kappa}_{4,\mathrm{nat}}[y] = \frac{1}{n}\sum_{i=1}^{n} y_i^4 - \frac{3}{n^2}\left(\sum_{i=1}^{n} y_i^2\right)^2. \qquad (3)$$

However, there is also another frequently used estimator of the fourth-order cumulant: the fourth k-statistics. This statistics is an unbiased estimator of the fourth-order cumulant for an i.i.d. process, and it does not make any assumption about the mean value of the stochastic process [24], [28], [29], [39]

$$\begin{aligned} k_4[y] &= \frac{n^3}{n^{[4]}}\left\{(n+1)m_4 - 3(n-1)m_2^2\right\} \\ &= \frac{1}{n^{[4]}}\left\{(n^3+n^2)s_4 - 4(n^2+n)s_3 s_1 \right. \\ &\qquad \left. - 3(n^2-n)s_2^2 + 12 n s_2 s_1^2 - 6 s_1^4\right\} \end{aligned} \qquad (4)$$

where

$$n^{[k]} \equiv \prod_{l=0}^{k-1}(n-l) = \frac{n!}{(n-k)!}, \quad n,\ k \in \mathbb{N}^*$$

m_r are the sample central moments

$$m_r \equiv \frac{1}{n}\sum_{i=1}^{n}(y_i - \widetilde{m}_1)^r,$$
$$\widetilde{m}_q \equiv \frac{1}{n}\sum_{i=1}^{n} y_i^q, \quad r,\ q \in \mathbb{N}^*$$

and s_r are the power sums $s_r \equiv \sum_{i=1}^{n} y_i^r$, $r \in \mathbb{N}^*$.

II. Constructionof the Estimator

A. Preliminaries

Since the samples of the signal x_i are i.i.d. up to order 8, the following fundamental property holds:

$$\mathrm{E}[x_i^p x_j^q] = \begin{cases} \mathrm{E}[x_i^p]\mathrm{E}[x_j^q], & i \neq j \\ \mathrm{E}[x_i^{p+q}], & i = j \end{cases} \qquad p,\ q \in \mathbb{N}^* \qquad (5)$$

provided that $p + q \leqslant 8$. For the signal u_i, the things are more complicated. From (1), it is straightforward that the samples of u_i are not i.i.d., because $\mathrm{E}[u_i u_j] \neq \mathrm{E}[u_i]\mathrm{E}[u_j]$ for $i \neq j$, where $i,\ j \in \mathbb{N}^*$. Indeed, from (1), by using (5) and the fact that x_i and u_i are zero-mean, we may verify the auto-correlation function of the latter:

$$\begin{aligned} \mathrm{E}[u_i u_j] &= \mathrm{E}\left[\sum_{k,l=0}^{p} a_k a_l x_{i-k} x_{j-l}\right] \\ &= \mathrm{E}[x^2]\sum_{k=0}^{p} a_k a_{k+i-j} + \mathrm{E}^2[x]\sum_{\substack{k,l=0 \\ k\neq l+i-j}}^{p} a_k a_l \\ &= \mathrm{E}[x^2]\sum_{k=0}^{p} a_k a_{k+i-j} \neq \mathrm{E}[u_i]\mathrm{E}[u_j] = 0 \end{aligned} \qquad (6)$$

where we used a somewhat similar method to the standard method employed for the i.i.d. processes to deal with multiple sums[3]. Thus, u_i is in general not i.i.d. However, note that since the coefficients a_k for $k \notin [0, p]$ are null, $\mathrm{E}[u_i u_j] \neq 0$ only for those i and j that satisfy $|i - j| \leqslant p$; otherwise $\mathrm{E}[u_i u_j] = 0$ and for these i and j the samples u_i and u_j are mutually independent. By the way, when $i = j$, the latter equation gives the well-known relationship between the powers (or second-order cumulants) of u_i and x_i:

$$\mathrm{E}[u^2] = \mathrm{E}[u_i^2] = \mathrm{E}[x^2]\sum_{k=0}^{p} a_k^2. \qquad (7)$$

Physically, the latter means that the power of the input signal x_i is not conserved through an MA filter, unless the filter is of unit energy (unit Euclidean norm).

[1]The IR may be also written in vector notation, e.g., $\boldsymbol{h} \equiv [a_0, a_1, \ldots, a_p]$, but we will not use this notation, by preferring to view the IR as a discrete function of i, which is given as $h_i = a_i$.

[2]By using the vector notation from the footnote 1, this IR may be regarded as $(n-1)$ Hadamard products of $\boldsymbol{h}$.

[3]Note that the method is not a straightforward generalization of the combinatorial method [24], [28], [39]. In particular, the calculation of the coefficients of the mathematical expectations, and that of the indexes of the IRs coefficients in multiple sums, require careful watching (in order to better understand the principle, one may want to study in detail (10), and Appendix).

B. Unbiasedness of the Estimator

We wish to build an unbiased frame-based estimator of the general form

$$\hat{\kappa}_{4,\text{unb. non i.i.d.}}[u] = \alpha \sum_{i=1}^{n} u_i^4 - \beta \left(\sum_{i=1}^{n} u_i^2 \right)^2. \quad (8)$$

where n is the size of the frame of u_i we have for the estimation, and the coefficients α and β are such that the bias b vanishes. The latter reads

$$\begin{aligned} b &= \mathrm{E}\left[\hat{\kappa}_{4,\text{unb. non i.i.d.}}\right] - \kappa_4[u] \\ &= (\alpha n - 1)\mathrm{E}\left[u^4\right] + 3\mathrm{E}^2\left[u^2\right] - \beta \sum_{i,j=1}^{n} \mathrm{E}\left[u_i^2 u_j^2\right]. \quad (9) \end{aligned}$$

First, we calculate the term $\mathrm{E}\left[u_i^2 u_j^2\right]$. Again, different cases may occur, depending on the current i and j. The calculation of $\mathrm{E}\left[u_i^2 u_j^2\right]$ yields

$$\begin{aligned} &\mathrm{E}\left[u_i^2 u_j^2\right] \\ &= \mathrm{E}\left[\sum_{k,l,m,q=0}^{p} a_k a_l a_m a_q x_{i-k} x_{i-l} x_{j-m} x_{j-q} \right] \\ &= \mathrm{E}\left[x^4\right] \sum_{k=0}^{p} a_k^2 a_{k+i-j}^2 + \mathrm{E}^2\left[x^2\right] \\ &\times \left(\sum_{\substack{k,l=0 \\ k \neq l+i-j}}^{p} a_k^2 a_l^2 + 2 \sum_{\substack{k,l=0 \\ k \neq l}}^{p} a_k a_{k+i-j} a_l a_{l+i-j} \right) \quad (10) \end{aligned}$$

where we took into account that x_i is zero-mean[4].

Then, we deal with the term $\mathrm{E}\left[u^4\right]$. The latter can be easily derived from (10) by putting $i = j$

$$\begin{aligned} \mathrm{E}\left[u^4\right] &= \mathrm{E}\left[u_i^4\right] \\ &= \mathrm{E}\left[x^4\right] \sum_{k=0}^{p} a_k^4 + 3\mathrm{E}^2\left[x^2\right] \sum_{\substack{k,l=0 \\ k \neq l}}^{p} a_k^2 a_l^2 \\ &= \underbrace{\left(\mathrm{E}\left[x^4\right] - 3\mathrm{E}^2\left[x^2\right] \right)}_{\kappa_4[x]} \sum_{k=0}^{p} a_k^4 \\ &\quad + 3\mathrm{E}^2\left[x^2\right] \left(\sum_{k=0}^{p} a_k^2 \right)^2. \quad (11) \end{aligned}$$

The last line is added in order to get rid of the double sum without crossed term, which is not very convenient. Note that from the last expression, by substituting it together with (7) into (2), we obtain the well-known relationship between the fourth-order cumulants of u_i and x_i

$$\kappa_4[u] = \kappa_4[x] \sum_{k=0}^{p} a_k^4. \quad (12)$$

[4]If x_i was not zero-mean, we would have several supplementary terms in the latter expression.

This relationship is very similar to the powers' one, defining the condition for the conservation of the fourth-order cumulant[5] through an MA filter.

Finally, the substitution of (7), (10) and (11) into (9) yields the bias of the estimator (8)

$$\begin{aligned} b = &\left\{ \alpha n f_1 - \beta f_3 - f_1 \right\} \mathrm{E}\left[x^4\right] \\ &+ 3\left\{ \alpha n (f_2 - f_1) - \beta f_4 + f_1 \right\} \mathrm{E}^2\left[x^2\right]. \quad (13) \end{aligned}$$

where the auxiliary functions, depending all on the coefficients $\{a_k\}_{k=0}^{p}$, are

$$\begin{aligned} &f_1 = \sum_{k=0}^{p} a_k^4, \quad f_2 = \left(\sum_{k=0}^{p} a_k^2 \right)^2, f_3 = \sum_{i,j=1}^{n} \sum_{k=0}^{p} a_k^2 a_{k+i-j}^2, \\ &f_4 = \frac{1}{3} \sum_{i,j=1}^{n} \left[\sum_{\substack{k,l=0 \\ k \neq l+i-j}}^{p} a_k^2 a_l^2 + 2 \sum_{\substack{k,l=0 \\ k \neq l}}^{p} a_k a_{k+i-j} a_l a_{l+i-j} \right]. \quad (14) \end{aligned}$$

By simultaneously equalizing the expressions between the braces behind the terms $\mathrm{E}\left[x^4\right]$ and $\mathrm{E}^2\left[x^2\right]$ in (13), we deduce that the coefficients α and β that make the bias vanishes, are

$$\alpha = \frac{1}{n} + \beta \frac{f_3}{n f_1}, \quad \beta = \frac{f_1 f_2}{f_1 f_4 + f_1 f_3 - f_2 f_3}. \quad (15)$$

The main computational complexity of the estimator is defined by f_4 in (14), and it is not negligible. Fortunately, it can be simplified. The expression in the brackets in f_4 may be rewritten as follows:

$$\begin{aligned} &\sum_{\substack{k,l=0 \\ k \neq l+i-j}}^{p} a_k^2 a_l^2 + 2 \sum_{\substack{k,l=0 \\ k \neq l}}^{p} a_k a_{k+i-j} a_l a_{l+i-j} \\ &= \left(\sum_{k=0}^{p} a_k^2 \right)^2 + 2 \left(\sum_{k=0}^{p} a_k a_{k+i-j} \right)^2 - 3 \sum_{k=0}^{p} a_k^2 a_{k+i-j}^2 \\ &= f_2 + 2R_{j-i}^2[h] - 3R_{j-i}\left[h^2\right] \quad (16) \end{aligned}$$

where $R_k[h^n]$ denotes discrete auto-correlation function of discrete time k and of IR h^n, $n \in \mathbb{N}^*$ [see also formula between (1) and (2) for h^n]. This function is defined as

$$\begin{aligned} R_k\left[h^n\right] &\equiv R_k\left[h_i^n\right] \\ &\equiv \sum_{l=-\infty}^{+\infty} a_l^n a_{l-k}^n = \sum_{l=0}^{p} a_l^n a_{l-k}^n, \quad k \in \mathbb{Z}. \end{aligned}$$

Whence

$$f_4 = \frac{n^2 f_2}{3} + \frac{1}{3} \sum_{i,j=1}^{n} \left(2R_{j-i}^2[h] - 3R_{j-i}\left[h^2\right] \right). \quad (17)$$

The resulting computation time is smaller, because the computational complexity of the auto-correlation function is lower than that of the direct computation of multiple sums, since it can be

[5]Moreover, it is well known that the latter property can be extended even to the rth-order cumulants: $\kappa_r[u] = \kappa_r[x] \sum_{k=0}^{p} a_k^r$, where $r \in \mathbb{N}^*$.

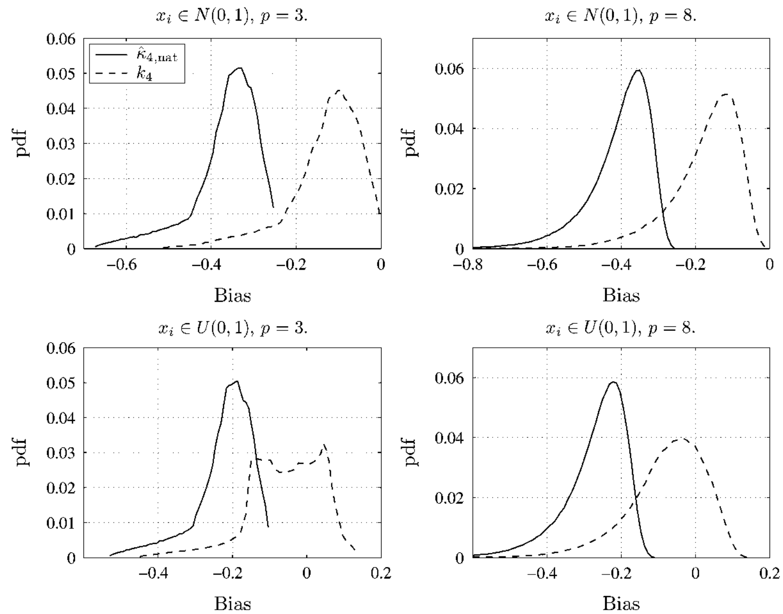

Fig. 1. Theoretical pdf of the biases of the natural estimator and the fourth k-statistics for a non-i.i.d. signal u_i obtained via specified MA κ_2-keeping filters.

calculated via FFT algorithm [40]. By the way, f_3 may be computed in the same manner[6]

$$f_3 = \sum_{i,j=1}^{n} \sum_{k=0}^{p} a_k^2 a_{k+i-j}^2 = \sum_{i,j=1}^{n} R_{j-i}\left[h^2\right]. \tag{18}$$

Finally, note that according to the theorem of uniqueness of symmetric polynomial unbiased estimators [28], [29] the derived estimator (8) is the unique unbiased estimator of the fourth-order cumulant for the considered MA process (1).

C. Mean Square Error of the Estimator

Besides the bias, which is a measure of the estimator's performances based on its first-order moment, the mean square error (MSE), based on both its first- and second-order moments, is another important characteristic of an estimator. Unfortunately, the analytical calculation of the latter criterion, whose complexity is mainly defined by the term $\mathrm{E}\left[\hat{\kappa}_4^2\right]$, is a long and laborious procedure, especially because it implies a lot of calculations related of the numerous multiple sums (up to 12-fold sums). Thus, the MSE studies will be performed empirically in Section III-B, using numerical simulation approach.

III. Statistical Study of the Estimator Performances

A. Bias Comparisons

In order to find out the bias performances of the proposed estimator, we undertake a statistical comparative study between the proposed estimator, and two other estimation techniques of the fourth-order cumulant: the natural estimator (3) and the fourth k-statistics (4) for the considered non-i.i.d. signal u_i. The bias of the natural estimator for u_i is obtained from (13) by setting $\alpha = 1/n$ and $\beta = 3/n^2$

$$b_{\text{nat}} = -\frac{3f_3}{n^2}\mathrm{E}\left[x^4\right] + 3\left(f_2 - \frac{3f_4}{n^2}\right)\mathrm{E}^2\left[x^2\right]. \tag{19}$$

That of the fourth k-statistics, whose calculation is put into Appendix, is given by (25).

The auxiliary functions $f_1, \ldots, f_{10}$ depend on the given numeric values of the IR of the MA filter, and so do the biases. We therefore carried out a statistical study of the bias for the randomly generated IRs, given by $\{a_k\}_{k=0}^{p}$. Two different filter's orders are considered: $p = 3$ and $p = 8$. In both cases, the coefficients of filter $\{a_k\}_{k=0}^{p}$ are generated randomly according to the normal law $N(0,1)$. Since in many applications, it is important to conserve either the power (i.e., the second-order cumulant κ_2) or the fourth-order cumulant of the signal passing through an MA filter, the IR is respectively normalized either by its Euclidean norm, see (7), or by its 4-norm, see (12). Finally, in all experiments, the initial i.i.d. signal x_i is of the unit power and distributed according to the normal and uniform laws. The fourth-order cumulant of such a signal is 0 for the normal law and -1.2 for the uniform one. As to the non-i.i.d. signal u_i, its power in the case of the κ_2-keeping filter is 1 in both normal and uniform scenarios, its fourth-order cumulant in the case of κ_4-keeping filter is 0 for the normal law and -1.2 for the uniform one.

The obtained results are reported in Figs. 1 and 2. The number of performed simulations used to draw each pdf from the corresponding histograms[7] is set to one million (in other words, we generated 8 million different filter's realizations in all, i.e., one million different IRs per panel). The length of the estimated frames n is set to 24.

[6]For example, for the numerical simulations that we come to show, $p = 8$, $n = 24$, this gain is about 16 times for the formula (17) and 1.5 times for the formula (18) with respect to f_4 and f_3 respectively computed directly from (14) [empirical estimation from 1000 Monte-Carlo runs].

[7]All pdfs are obtained via normalized histogram envelopes for both theoretical bias (Figs. 1 and 2), and empirical MSE (Figs. 3 and 4). The normalization is carried out in such a way that the 1-norm of the obtained discrete pdf is equated to 1. Thus, depending on the number of bins of the histogram, the absolute amplitude of pdfs may vary, while the relative one remains invariant.

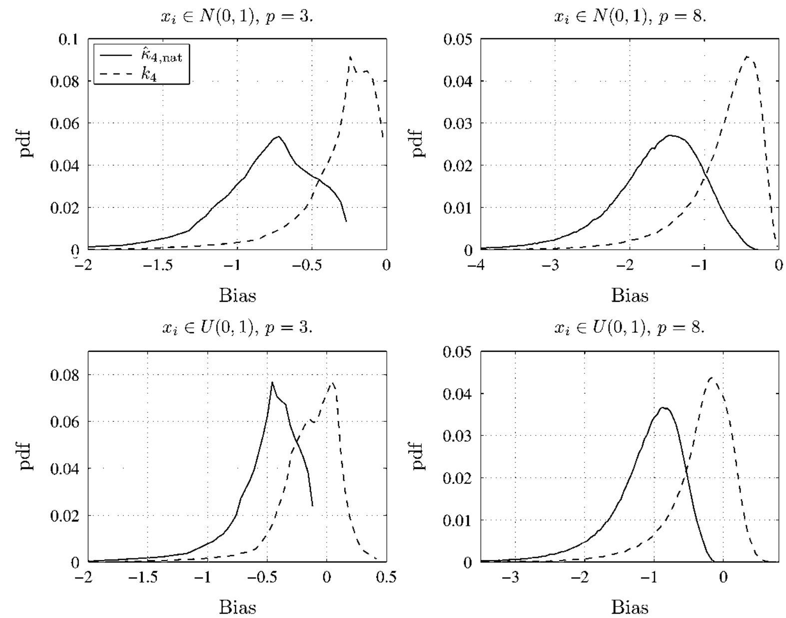

Fig. 2. Theoretical pdf of the biases of the natural estimator and the fourth k-statistics for a non-i.i.d. signal u_i obtained via specified MA κ_4-keeping filters.

For the natural estimator, from the power-keeping case, Fig. 1, we may mainly deduce three results. Firstly, the bias in the normal case is greater than that in the uniform one for the same order p (the same result for the i.i.d. case was previously obtained in [29]). Secondly, the more the order of the MA filter p, the greater the bias. Thirdly, the bias of the non-i.i.d. case is always greater than that in the i.i.d. case: $6/n = 0.25$ for $N(0,1)$ and $2.4/n = 0.1$ for $U(0,1)$ [29]. In the κ_4-keeping case, Fig. 2, we observe the same tendencies as above, except that the absolute values of biases are greater. Since in this case the true (or real) value of cumulant is conserved through an MA filter [i.e., it is still 0 for the $N(0,1)$ law and -1.2 for the $U(0,1)$ one], such great values of bias might be not acceptable for many applications, while the proposed estimator remains always unbiased.

As to the fourth k-statistics' bias, globally, its behavior has the same features as that of the natural estimator, but its values are much smaller (we recall that it is unbiased for the i.i.d. case). Also, there are two small differences: fourth k-statistics' bias can be positive, and in the case of the power-keeping filter for the uniformly distributed samples of x_i with filter's order $p = 3$, bias' distribution is multimodal. However, the fourth k-statistics has also a drawback with respect to the proposed unbiased and natural estimators: greater computational complexity.

Lastly, in order to complete the estimator performances studies, we also have to perform the MSE measures.

B. MSE Comparisons

The MSE $\equiv \mathrm{E}\left[(\hat{\kappa}_4 - \kappa_4)^2\right]$ of all studied estimators are obtained empirically. Since this procedure is highly time-consuming, the number of the performed simulations used to draw each pdf is set to 50 000 instead of one million (in other words, we generated $8 \times 50\,000$ different filter's realizations in all, i.e., 50 000 random normally distributed IRs per panel). Then, for one given filter's configuration (i.e., for each IR), the MSE of the studied estimators is calculated via the sample mean of 5000 Monte-Carlo runs (i.e., we generated $8 \times 50\,000 \times 5000 = 20$ millions simulations in all). Other simulation conditions are the same as in bias studies.

The obtained results are reported in Figs. 3 and 4. For the κ_2-keeping case, Fig. 3, we observe that in the normal case, the MSE of the proposed estimator is worse with respect to other estimators, while in the uniform case it is practically the same. As to the κ_4-keeping case, Fig. 4, all estimators showed approximately same performances for the low filter's order $p = 3$, and the unbiased estimator is slightly worse for the high filter's order $p = 8$. For both κ_2- and κ_4-keeping cases, we ascertain that the more the order of the filter, the greater the MSEs of the estimators (note that we also had a similar tendency in bias studies).

In conclusion, the MSE studies showed that the MSE performances of the proposed estimator are variable and depend, as it is in many cases, on the distribution law. On the other hand, the proposed estimator was designed according to the principle of unbiasedness, that also explains why the MSE performances are not always optimum.

IV. Numerical Simulation Test

In order to achieve estimator performances studies, we finally propose to test it in conditions close to the real ones. For this aim, we will test it in dynamic conditions in which power and distribution law of x_i are both variable. The results of this test are shown in Fig. 5. First 80 frames of x_i are normally distributed according to $N(0, \sigma_{x,k})$ law, while the second ones are distributed according to the law $U(0, \sigma_{x,k})$. Standard deviation $\sigma_{x,k}$ (k is the frame's number), of each frame of the input signal x_i is time-depending according to the amplitude modulation law of the form

$$\sigma_{x,k} = \sigma_{x,0}\big(1 + a \sin \omega_0 k\big), \quad \omega_0 = \frac{2\pi}{k_{\max}}$$

with $\sigma_{x,0} = 1$, depth of modulation $a = 0.5$, $k_{\max} = 80$ and $k = 1, 2, \ldots, k_{\max}$, for both normal and uniform laws. Other

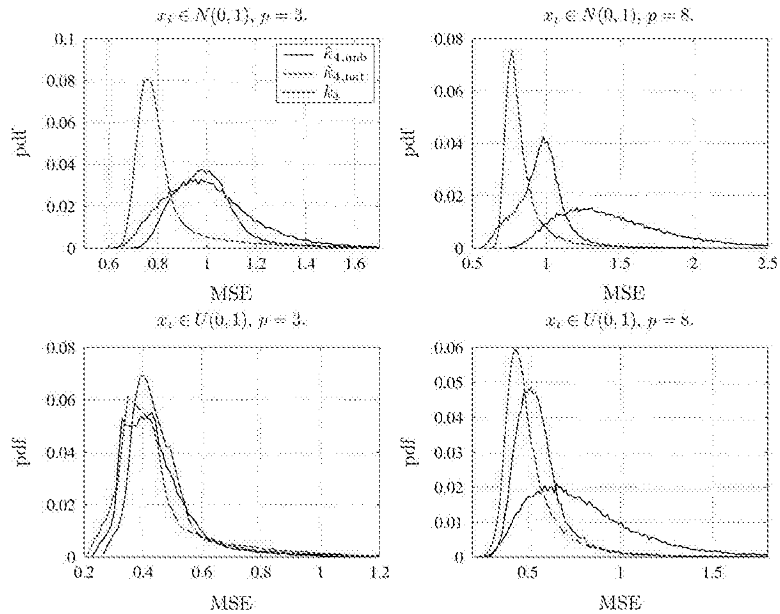

Fig. 3. Empirical pdf of the MSE of the unbiased and natural estimators, as well as the fourth k-statistics, for a non-i.i.d. signal u_i obtained via specified MA κ_2-keeping filters.

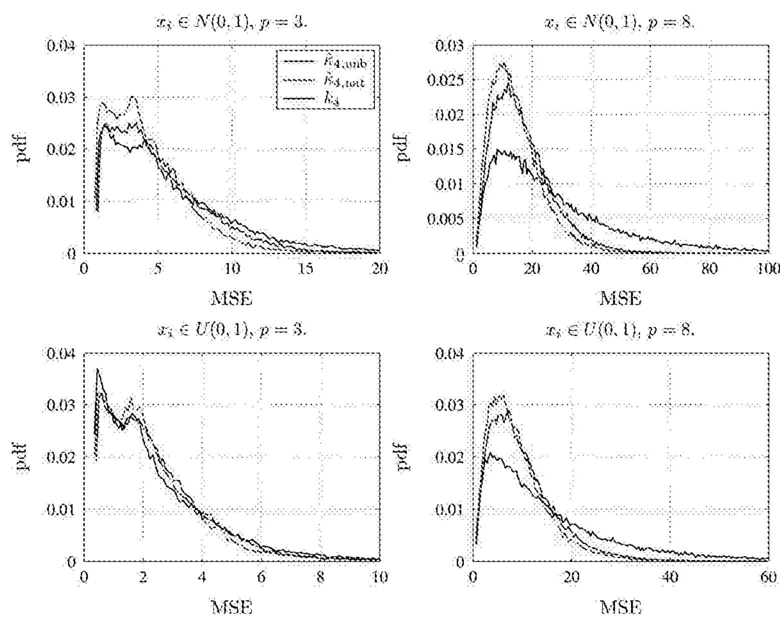

Fig. 4. Empirical pdf of the MSE of the unbiased and natural estimators, as well as the fourth k-statistics, for a non-i.i.d. signal u_i obtained via specified MA κ_4-keeping filters.

parameters are: $p = 8$, $n = 24$, number of Monte-Carlo runs is set to 5000 (for each k).

The simulation shows that the proposed statistics $\hat{\kappa}_{4,\mathrm{unb}}$ estimates very well the true (or real) value of cumulant, while keeping its MSE at the same level as other estimators.

V. Conclusions

In this paper we designed and studied an unbiased and quite efficient estimator of the fourth-order cumulant for real random zero-mean non-i.i.d. signal that can be approximated by an MA stochastic process. We undertook statistical comparative studies of the proposed estimator with two other classically used estimators of the fourth-order cumulant: the natural estimator and the fourth k-statistics. These studies were devoted to two main estimator's criteria: bias and MSE. The bias studies showed that in contrast to the proposed estimator, the natural one is strongly biased, while the fourth k-statistics turned out to be moderately biased. As to the MSE performances, as expected the proposed unbiased estimator showed variable MSE performances: they depend on the distribution law, on the order of the MA filter and on the concrete choice of the IR coefficients. Hence, the proposed unbiased estimator may be particularly useful in the applications where the performances depend on the mean of the cumulant (e.g., [6], [16], something similar to [13], etc.) and not on the mean square. In order the estimators to be appreciated in versatile short-term and real-time applications, we also reduced, wherever possible, computational complexity of the algorithms. Finally, an adaptive version of the proposed estimator

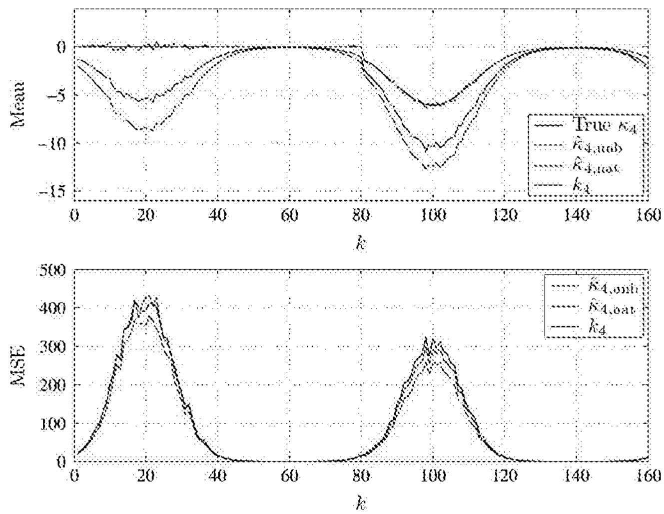

Fig. 5. Numerical simulation test for three compared estimators for the dynamic power stochastic signal u_i.

may be also designed by using the method described in [29], but the derivation procedure is more complicated, because the coefficients α and β of the estimator (8) do not depend on n explicitly, but implicitly, via the IR. In a very general way, the recursive version of such an estimator for arbitrary values of α and β [see (8)] is given as

$$\hat{\kappa}_{4,n+1} = \frac{\alpha_{n+1}}{\alpha_n}\hat{\kappa}_{4,n} + n^2\left(\frac{\alpha_{n+1}}{\alpha_n}\beta_n - \beta_{n+1}\right)\hat{\sigma}_n^4 \\ -2n\beta_{n+1}\hat{\sigma}_n^2 u_{n+1}^2 + \left(\alpha_{n+1} - \beta_{n+1}\right)u_{n+1}^4$$

where indices n and $(n+1)$ denote the size of the signal's frames from which estimations are performed (i.e., α_n and β_n are the coefficients from (8) corresponding to an estimate made from n samples; α_{n+1} and β_{n+1}, are those corresponding to an estimate made from $(n+1)$ samples), and $\hat{\sigma}_n \equiv \sqrt{s_2/n}$. The usual performance analysis of such an adaptive estimator (i.e., convergence in mean and in mean square) may be performed by using the recursive method that was described in [29] (see especially appendices B and C). Also, higher-order cumulants can be always estimated without bias by using the same method (e.g., see the discussion in [29] for $\kappa_3,\ \kappa_5,\ \kappa_6,\ldots$).

Appendix
Bias of the Fourth k-Statistics for the MA Process

The bias of the fourth k-statistics for non-i.i.d. signal u_i can be calculated from (4). By separately evaluating the mathematical expectation of each term, we first obtain

$$\mathrm{E}\left[s_1^4\right] = \sum_{i,j,k,l=1}^{n}\mathrm{E}\left[u_i u_j u_k u_l\right] = \sum_{i,j,k,l=1}^{n}\mathrm{E}\left[\sum_{m,q,o,h=0}^{p} a_m a_q a_o a_h \, x_{i-m}x_{j-q}x_{k-o}x_{l-h}\right] = f_5\mathrm{E}\left[x^4\right] + f_6\mathrm{E}^2\left[x^2\right] \tag{20}$$

where we designated

$$\begin{aligned} f_5 &= \sum_{i,j,k,l=1}^{n}\left[\sum_{m=0}^{p} a_m a_{m+j-i} a_{m+k-i} a_{m+l-i}\right] \\ &= \sum_{i=1}^{n}\sum_{m=0}^{p} a_m\left(\sum_{j=1}^{n} a_{m+j-i}\right)^3, \\ f_6 &= 3\sum_{i,j,k,l=1}^{n}\left[\sum_{\substack{m,q=0\\ m\neq q+i-j}}^{p} a_m a_q a_{q+k-j} a_{m+l-i}\right] \\ &= 3\sum_{i,j,k,l=1}^{n} R_{j-k}[h]R_{i-l}[h] - 3f_5 \\ &= 3\left(\sum_{j,k=1}^{n} R_{j-k}[h]\right)^2 - 3f_5. \end{aligned}$$

Note that we try everywhere where is possible to reduce computational complexity.

Then, by using the same procedure as above, we obtain

$$\mathrm{E}\left[s_2 s_1^2\right] = \sum_{i,j,k=1}^{n}\mathrm{E}\left[u_i^2 u_j u_k\right] = \sum_{i,j,k=1}^{n}\mathrm{E}\left[\sum_{m,q,o,h=0}^{p} a_m a_q a_o a_h \, x_{i-m}x_{i-q}x_{j-o}x_{k-h}\right] = f_7\mathrm{E}\left[x^4\right] + f_8\mathrm{E}^2\left[x^2\right]. \tag{21}$$

where we denoted

$$\begin{aligned} f_7 &= \sum_{i,j,k=1}^{n}\left[\sum_{m=0}^{p} a_m^2 a_{m+j-i} a_{m+k-i}\right] \\ &= \sum_{i=1}^{n}\sum_{m=0}^{p} a_m^2\left(\sum_{j=1}^{n} a_{m+j-i}\right)^2, \\ f_8 &= \sum_{i,j,k=1}^{n}\left[\sum_{\substack{m,o=0\\ m\neq o+i-j}}^{p} a_m^2 a_o a_{o+k-j} + 2\sum_{\substack{m,q=0\\ m\neq q}}^{p} a_m a_q a_{m+j-i} a_{q+k-i}\right] \\ &= \sum_{i,j,k=1}^{n}\left(R_{j-k}[h]\sqrt{f_2} + 2R_{i-j}[h]R_{i-k}[h]\right) - 3f_7 \\ &= n\sqrt{f_2}\sum_{j,k=1}^{n} R_{j-k}[h] + 2\sum_{i=1}^{n}\left(\sum_{j=1}^{n} R_{i-j}[h]\right)^2 - 3f_7. \end{aligned}$$

For the term $\mathrm{E}[s_3 s_1]$, the same procedure yields

$$\begin{aligned}\mathrm{E}[s_3 s_1] &= \sum_{i,j=1}^{n} \mathrm{E}\left[u_i^3 u_j\right] \\ &= \sum_{i,j=1}^{n} \mathrm{E}\left[\sum_{m,q,o,h=0}^{p} a_m a_q a_o a_h x_{i-m} x_{i-q} x_{i-o} x_{j-h}\right] \\ &= f_9 \mathrm{E}\left[x^4\right] + f_{10} \mathrm{E}^2\left[x^2\right] \qquad (22)\end{aligned}$$

where we designated

$$f_9 = \sum_{i,j=1}^{n} \sum_{m=0}^{p} a_m^3 a_{m+j-i} = \sum_{i,j=1}^{n} R_{i-j}\left[h^3, h\right],$$

and

$$\begin{aligned}f_{10} &= 3 \sum_{i,j=1}^{n} \sum_{\substack{m,q=0\\ m\neq q}}^{p} a_m^2 a_q a_{q+j-i} \\ &= 3\sqrt{f_2} \sum_{i,j=1}^{n} R_{i-j}[h] - 3 f_9\end{aligned}$$

and $R_k[\cdot,\cdot]$ denotes discrete cross-correlation function. For the IRs h^n [given in the formula between (1) and (2)], and g^m, $m \in \mathbb{N}^*$, defined analogously

$$g^m \equiv g_i^m = \sum_{k=0}^{p} b_k^m \delta_{i,k}, \quad i \in \mathbb{Z}, \quad \forall\, b_k \in \mathbb{R}$$

this function is given as

$$R_k\left[h^n, g^m\right] \equiv \sum_{l=-\infty}^{+\infty} a_l^n b_{l-k}^m = \sum_{l=0}^{p} a_l^n b_{l-k}^m, \quad k \in \mathbb{Z}$$

where k is discrete time. By the way, note that as usual we have $R_k[h^n, h^n] = R_k[h^n]$, i.e., cross-correlation function becomes auto-correlation one when its both arguments coincide.

Finally, we deal with

$$\mathrm{E}\left[s_2^2\right] = \sum_{i,j=1}^{n} \mathrm{E}\left[u_i^2 u_j^2\right] = f_3 \mathrm{E}\left[x^4\right] + 3 f_4 \mathrm{E}^2\left[x^2\right] \qquad (23)$$

that follows from (10) and (14); as well as with $\mathrm{E}[s_4]$, that may be easily derived from (11) and (14):

$$\mathrm{E}[s_4] = n f_1 \mathrm{E}\left[x^4\right] + 3n\left(f_2 - f_1\right) \mathrm{E}^2\left[x^2\right]. \qquad (24)$$

By substituting (20), (21), (22), (23) and (24) into (4), we finally arrive to the mathematical expectation of the fourth k-statistics

$$\begin{aligned}\mathrm{E}[k_4] = \frac{1}{n(n-1)(n-2)(n-3)} \times \Big\{ &\Big(n^3(n+1) f_1 - 4n(n+1) f_9 \\ &- 3n(n-1) f_3 + 12 n f_7 - 6 f_5\Big) \mathrm{E}\left[x^4\right] \\ &+ \Big(3n^3(n+1)(f_2 - f_1) - 4n(n+1) f_{10} \\ &- 9n(n-1) f_4 + 12 n f_8 - 6 f_6\Big) \mathrm{E}^2\left[x^2\right] \Big\}\end{aligned}$$

and consequently, to its bias

$$b_{\mathrm{kst}} = \mathrm{E}[k_4] - f_1\left(\mathrm{E}\left[x^4\right] - 3\mathrm{E}^2\left[x^2\right]\right) \qquad (25)$$

both for the considered MA process u_i defined in (1).

Lastly, we note that the overall reduction of the computational complexity for the bias b_{kst} (25) is about 480 times[8] (comparisons between bias calculated with auxiliary functions directly computed via multiple sums and between bias calculated with auxiliary functions after computational reduction, e.g., first line for f_5 *versus* the second one, or first line for f_6 *versus* the third one, or first line for f_8 *versus* the last one, etc.).

References

[1] J. K. Tugnait, "Identification and deconvolution of multichannel linear non-Gaussian processes using higher order statistics and inverse filter criteria," *IEEE Trans. Signal Process.*, vol. 45, pp. 658–672, Mar. 1997.
[2] C. Simon, Ph. Loubaton, C. Vignat, C. Jutten, and G. D'Urso, "Blind source separation of convolutive mixtures by maximization of fourth-order cumulants: the non i.i.d. case," in *Conf. Rec. 32nd Asilomar Conf. Signals, Systems & Computers*, Asilomar, Nov. 1998, vol. 2, pp. 1584–1588.
[3] C. Simon, Ph. Loubaton, and C. Jutten, "Separation of a class of convolutive mixtures: A contrast function approach," *Signal Process.*, vol. 81, no. 4, pp. 883–887, Apr. 2001.
[4] P. Comon, "Separation of sources using high-order cumulants," in *SPIE Conf. Advanced Algorithms and Architectures for Signal Processing, vol. Real-time Signal Processing XII*, San Diego, CA, Aug. 8–10, 1989, pp. 170–181.
[5] J.-C. Pesquet and E. Moreau, "Cumulant-based independence measures for linear mixtures," *IEEE Trans. Inf. Theory*, vol. 47, pp. 1947–1956, Jul. 2001.
[6] M. Castella, S. Rhioui, E. Moreau, and J.-C. Pesquet, "Quadratic higher order criteria for iterative blind separation of a MIMO convolutive mixture of sources," *IEEE Trans. Signal Process.*, vol. 55, pp. 218–232, 2007.
[7] P. Comon, "MA identification using fourth order cumulants," *Signal Process.*, vol. 26, no. 3, pp. 381–388, Mar. 1992.
[8] D. Dembélé and G. Favier, "Recursive estimation of the fourth-order cumulants with application to identification," *Signal Process.*, vol. 68, no. 2, pp. 127–139, 1998.
[9] A. Swami and J. M. Mendel, "Closed-form recursive estimation of MA coefficients using autocorrelations and third-order cumulants," *Proc. IEEE*, vol. 37, pp. 1794–1795, 1989.
[10] T. W. Chow and H.-Z. Tan, "Semiblind identification of nonminimum-phase ARMA models via order recursion with higher order cumulants," *IEEE Trans. Ind. Electron.*, vol. 45, pp. 663–671, 1998.
[11] H.-Z. Tan and T. W. Chow, "Blind and total identification of ARMA models in higher order cumulants domain," *IEEE Trans. Ind. Electron.*, vol. 46, pp. 1233–1240, 1999.
[12] J. Liang and Z. Ding, "FIR channel estimation through generalized cumulant slice weighting," *IEEE Trans. Signal Process.*, vol. 52, pp. 657–667, 2004.
[13] K. Forrest, G. Weismer, P. Milencovic, and R. N. Dougall, "Statistical analysis of word-initial voiceless obstruents: Preliminary data," *J. Acoust. Soc. Amer.*, vol. 84, pp. 115–123, 1988.

[8]Empirical estimation from 1000 Monte-Carlo runs for $p = 8$, $n = 24$.

[14] M. Rangoussi, A. Delopoulos, and M. Tsatsanis, "On the use of the higher-order statistics for robust endpoint detection of speech," in *IEEE Signal Processing Workshop on Higher-Order Statistics*, 1993, pp. 56–60.
[15] J. W. Fackrell and S. McLaughlin, "The higher-order statistics of speech signals," *IEEE Colloq. Techniques for Speech Processing and their Application*, pp. 7/1–7/6, 1994.
[16] L.-R. Shen, Li X.-Y, W. Wei, R.-B. Zhang, and H.-Q. Wang, "Speech stream detection based on higher-order statistics," in *IEEE Int. Conf. Machine Learning and Cybernetics*, 2003, vol. 5, pp. 3086–3089.
[17] A. Napolitano, "Cyclic higher-order statistics: Input/output relations for discrete- and continuous-time MIMO linear almost-periodically time-variant systems," *Signal Process.*, vol. 42, pp. 147–166, 1995.
[18] L. Izzo and A. Napolitano, "Higher-order cyclostationarity properties of sampled time-series," *Signal Process.*, vol. 54, pp. 303–307, 1996.
[19] L. Izzo and A. Napolitano, "Higher-order statistics for rice's representation of cyclostationarity signals," *Signal Process.*, vol. 56, pp. 279–292, 1997.
[20] L. Izzo and A. Napolitano, "The higher order theory of generalized almost-cyclostationarity time series," *IEEE Trans. Signal Process.*, vol. 46, pp. 2975–2989, Nov. 1998.
[21] W. A. Gardner, A. Napolitano, and L. Paura, "Cyclostationarity: Half a century of research," *Signal Process.*, vol. 86, pp. 639–697, 2006.
[22] J. J. Gonzalez de la Rosa, R. Piotrkowski, and J. E. Ruzzante, "Higher order statistics and independent component analysis for spectral characterization of acoustic emission signals in steel pipes," *IEEE Trans. Instrum. Meas.*, vol. 56, pp. 2312–2321, Dec. 2007.
[23] J. Jakubowski, K. Kwiatos, A. Chwaleba, and S. Osowski, "Higher order statistics and neural network for tremor recognition," *IEEE Trans. Biomed. Eng.*, vol. 49, pp. 152–159, Feb. 2002.
[24] R. A. Fisher, "Moments and product moments of sampling distributions," *Proc. Lond. Math. Soc. Ser. 2*, vol. 30, pp. 199–238, 1929.
[25] E. A. Cornish and R. A. Fisher, "Moments and cumulants in the specification of distributions," *Rev. Inst. Int. Statist.*, vol. 5, pp. 307–320, 1937.
[26] H. Jeffreys, *Theory of Probability. International Series of Monographs on Physics*, R. H. Flower and P. Kapitza, Eds. Oxford, U.K.: Clarendon, 1939.
[27] A. N. Shiryaev, *Probability*, Second ed. Berlin: Springer-Verlag, 1995.
[28] M. G. Kendall, *The Advanced Theory of Statistics*, Second ed. London, U.K.: Charles Griffin, 1945, vol. I.
[29] I. V. Blagouchine and E. Moreau, "Unbiased adaptive estimations of the fourth-order cumulant for real random zero-mean signal," *IEEE Trans. Signal Process.*, vol. 57, pp. 3330–3346, Sep. 2009.
[30] E. Masry, "The estimation of the fourth-order cumulant for dependent data: Consistency and asymptotic normality," *IEEE Trans. Signal Process.*, vol. 58, pp. 1991–1998, Apr. 2010.
[31] A. N. Kolmogorov and Y. A. Rozanov, "On strong mixing conditions for stationary Gaussian processes," *Theory Prob. Applicat.*, vol. 5, no. 2, pp. 204–208, Jan. 1960.
[32] M. Rosenblatt, "A central limit theorem and a strong mixing condition," *Proc. Nat. Acad. Sci. USA*, vol. 42, no. 7, pp. 412–413, Jul. 1956.
[33] V. V. Gorodetskii, "On the strong mixing property for linear sequences," *Theory Prob. Applicat.*, vol. 22, pp. 411–413, 1977.
[34] R. C. Bradley, "Basic properties of strong mixing conditions. A survey and some open questions," *Prob. Surv.*, vol. 2, pp. 107–144, 2005.
[35] J. F. Kenney and E. S. Keeping, *Mathematics of Statistics. Part I*, Third ed. New-York: Van Nostrand, 1961.
[36] J. M. Mendel, "Tutorial on higher-order statistics (spectra) in signal processing and system theory: Theoretical results and some applications," *Proc. IEEE*, vol. 79, pp. 278–305, 1991.
[37] D. R. Brillinger, *Time Series. Data Analysis and Theory*. New York: Holt, Rinehart and Winston, 1975.
[38] P. Amblard and J.-M. Brossier, "Adaptive estimation of the fourth-order cumulant of a white stochastic process," *Signal Process.*, vol. 42, no. 1, pp. 37–43, 1995.
[39] J. F. Kenney and E. S. Keeping, *Mathematics of Statistics. Part II*, 2nd ed. New York: Van Nostrand , 1959.
[40] S. J. Orfanidis, *Optimum Signal Processing. An Introduction*, 2nd ed. Englewood Cliffs, NJ: Prentice-Hall, 1996.

Iaroslav V. Blagouchine was born in Saint-Petersburg, Russia, on December 22, 1979. He received the B.S. degree in physics from the Saint-Petersburg State University in 2000, the M.S. degree in electronic engineering from the Grenoble Institute of Technology, France, and the Ph.D. degree in signal processing and applied mathematics from the École Centrale, France, in 2001 and 2010, respectively.

From 2001 to 2002, he was with the Department Señales, Sistemas y Radiocomunicaciones, Universidad Politécnica de Madrid, Spain. During 2003, he was a Research Engineer with the Grenoble Institute of Technology, where he was also a Teacher Assistant from 2004 to 2007. From 2007 to 2009, he was a Postdoctoral Researcher and Teacher Assistant with the Telecommunication Department of the University of Toulon, France. Since September 2009, he has been a Research Engineer with the Mobile Communication Department of Eurécom, France. His main research interests are in estimation problems, statistical signal processing, higher-order statistics, entropy-based applications as well as the constraint optimization problems.

Eric Moreau (M'96–SM'08) was born in Lille, France. He graduated from the École Nationale Supérieure des Arts et Métiers, Paris, France, in 1989. He received the Agrégation de Physique degree from the Ecole Normale Supérieure de Cachan, France, in 1990 and the DEA and Ph.D. degrees in signal processing from the Université Paris-Sud, France, in 1991 and 1995, respectively.

From 1995 to 2001, he was an Assistant Professor with the Telecommunications Department, Institut des Sciences de l'Ingénieur de Toulon et du Var, La Valette, France. He is currently a Professor with the University of Toulon, France. His main research interests are in constraint optimization, neural networks applications and in statistical signal processing.