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Two series expansions for the logarithm of the gamma function involving Stirling numbers and containing only rational coefficients for certain arguments related to π^{-1}

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ABSTRACT

In this paper, two new series for the logarithm of the Γ-function are presented and studied. Their polygamma analogs are also obtained and discussed. These series involve the Stirling numbers of the first kind and have the property to contain only rational coefficients for certain arguments related to π^{-1}. In particular, for any value of the form $\ln\Gamma(\frac{1}{2}n \pm \alpha\pi^{-1})$ and $\Psi_k(\frac{1}{2}n \pm \alpha\pi^{-1})$, where Ψ_k stands for the kth polygamma function, α is positive rational greater than $\frac{1}{6}\pi$, n is integer and k is non-negative integer, these series have rational terms only. In the specified zones of convergence, derived series converge uniformly at the same rate as $\sum(n\ln^m n)^{-2}$, where $m = 1, 2, 3, \ldots$, depending on the order of the polygamma function. Explicit expansions into the series with rational coefficients are given for the most attracting values, such as $\ln\Gamma(\pi^{-1})$, $\ln\Gamma(2\pi^{-1})$, $\ln\Gamma(\frac{1}{2} + \pi^{-1})$, $\Psi(\pi^{-1})$, $\Psi(\frac{1}{2} + \pi^{-1})$ and $\Psi_k(\pi^{-1})$. Besides, in this article, the reader will also find a number of other series involving Stirling numbers, Gregory's coefficients (logarithmic numbers, also known as Bernoulli numbers of the second kind), Cauchy numbers and generalized Bernoulli numbers. Finally, several estimations and full asymptotics for Gregory's coefficients, for Cauchy numbers, for certain generalized Bernoulli numbers and for certain sums with the Stirling numbers are obtained. In particular, these include sharp bounds for Gregory's coefficients and for the Cauchy numbers of the second kind.

1. Introduction

1.1. Motivation of the study

Numerous are expansions of the logarithm of the Γ-function and of polygamma functions into various series. For instance

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$$\ln\Gamma(z) = \left(z-\frac{1}{2}\right)\ln z - z + \frac{1}{2}\ln 2\pi + \sum_{n=1}^{N}\frac{B_{2n}}{2n(2n-1)z^{2n-1}} + O\big(z^{-2N-1}\big)\,, \qquad \begin{array}{l} N<\infty \\ |z|\to\infty \\ |\arg z| < \dfrac{\pi}{2} \end{array} \tag{1}$$

$$\ln\Gamma(z) = -\gamma z - \ln z + \sum_{n=1}^{\infty}\left[\frac{z}{n} - \ln\Big(1+\frac{z}{n}\Big)\right]\,, \qquad z\in\mathbb{C}, \quad z\neq 0,-1,-2,\ldots \tag{2}$$

$$\ln\Gamma(z) = \left(z-\frac{1}{2}\right)\ln z - z + \frac{1}{2}\ln 2\pi + \sum_{n=0}^{\infty}\left[\left(z+n+\frac{1}{2}\right)\ln\frac{z+n+1}{z+n} - 1\right]\,, \quad z\neq 0,-1,-2,\ldots \tag{3}$$

$$\ln\Gamma(z) = -z\,(\gamma+\ln 2\pi) - \frac{1}{2}\ln\frac{\sin\pi z}{\pi} + \frac{1}{2}(\gamma+\ln 2\pi) + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\sin 2\pi n z\cdot\ln n}{n}\,, \qquad 0<z<1 \tag{4}$$

$$\ln\Gamma(z) = -\gamma z - \ln z + \sum_{n=2}^{\infty}\frac{(-1)^n z^n}{n}\zeta(n)\,, \qquad |z|<1 \tag{5}$$

$$\ln\Gamma(z) = \left(z-\frac{1}{2}\right)\ln z - z + \frac{1}{2}\ln 2\pi + \frac{1}{2}\sum_{n=1}^{\infty}\frac{n\cdot\zeta(n+1,z+1)}{(n+1)(n+2)}\,, \qquad \operatorname{Re} z>0 \tag{6}$$

$$\ln\Gamma(z) = \left(z-\frac{1}{2}\right)\ln\left(z-\frac{1}{2}\right) - z + \frac{1}{2} + \frac{1}{2}\ln 2\pi - \sum_{n=1}^{\infty}\frac{\zeta(2n,z)}{2^{2n+1}n(2n+1)}\,, \qquad \operatorname{Re} z\geqslant\frac{1}{2} \tag{7}$$

which are respectively known as Stirling's series,[1] Weierstrass' series,[2] Guderman's series,[3] Malmsten–Kummer's series,[4] Legendre's series,[5] Binet's series[6] and Burnside's formula[7] for the logarithm of the Γ-function.[8] Usually, coefficients of such expansions are either highly transcendental, or seriously suspected to be so. Expansions into the series with rational coefficients are much less investigated, and especially for the gamma and polygamma functions of "exotic" arguments, such as, for example, π^{-1}, logarithms or complex values.

In one of our preceding works, in exercises nº 39–49 [13, Sect. 4], we have evaluated several curious integrals containing inverse circular and hyperbolic functions, which led to the gamma and polygamma functions at rational multiple of π^{-1}. It appears that some of these integrals are particularly suitable for power series expansions. In this paper, we derive two series expansions for the logarithm of the Γ-function, as well as their respective analogs for the polygamma functions, by making use of such a kind of integrals. These expansions are not simple and cannot be explicitly written in powers of z up to a given order, but they contain rational coefficients for any argument of the form $z = \frac{1}{2}n \pm \alpha\pi^{-1}$, where α is positive rational greater than $\frac{1}{6}\pi$ and n is integer, and therefore, may be of interest in certain situations. As examples, we

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1 This series expansion is one of the oldest and was known to Gauss [48, p. 33], Euler [44, part II, Chapter VI, § 159, p. 466], Stirling [145, p. 135] and De Moivre [42], who originated it. Note that this series should be used with care because for $N\to\infty$, as was remarked by De Moivre himself, it diverges. For more information, see [17, p. 329], [111, p. 111], [155, § 12-33], [72, § 15-05], [78, p. 530], [39, p. 1], [85], [114, § 4.1, pp. 293–294], [45], [50, pp. 286–288], [1, nº 6.1.40–6.1.41], [103], [23, pp. 43–50], [47,65,16,91,138], [37, p. 267], [104,101,9,115].

2 This series follows straightforwardly from the well-known Weierstrass' infinite product for the Γ-function [53, p. 10], [109, p. 12], [7, p. 14], [157, p. 236], [23, p. 20], [85], [90, pp. 21–22].

3 See e.g. [111, p. 111], [61, p. 76, nº 661].

4 This series is usually referred to as *Kummer's series* for the logarithm of the Γ-function, see, e.g., [8, vol. I, § 1.9.1], [157,139]. However, it was comparatively recently that we discovered that it was first obtained by Carl Malmsten and not by Ernst Kummer, see [13, Sect. 2.2].

5 See e.g. [8, vol. I, eq. 1.17(2)], [161, eq. (21)].

6 See e.g. [8, vol. I, p. 48, Eq. (10)]. To Binet are also due several other series which we discuss on p. 428.

7 See e.g. [20,160,103], [8, vol. I, p. 48, Eq. (11)]. Note that in the latter reference, there are two errors related to Burnside's formula, i.e. to our formula (7).

8 Some other series expansions for $\ln\Gamma(z)$ may be also found in [12, pp. 335–339], [134, p. 1076, Eq. (6)], [8, vol. I, § 1.17], [111, pp. 240–251], see also a remark on p. 428. For further information on the Γ-function, see [53,109], [157, Chapt. XII], [7,23], [8, vol. I, Chapt. I], [41].

provide explicit expansions into the series with rational coefficients for $\ln\Gamma(\pi^{-1})$, $\ln\Gamma(2\pi^{-1})$, $\ln\Gamma(\frac{1}{2}+\pi^{-1})$, $\Psi(\pi^{-1})$, $\Psi(\frac{1}{2}+\pi^{-1})$ and $\Psi_k(\pi^{-1})$. Coefficients of discovered expansions involve the Stirling numbers of the first kind, which often appear in combinatorics, as well as in various "exotic" series expansions, such as, for example

$$\begin{aligned} \ln\ln 2 &= -\frac{1}{2}+\frac{5}{24}-\frac{1}{8}+\frac{251}{2880}-\frac{19}{288}+\frac{19\,087}{362\,880}-\frac{751}{17\,280}+\frac{107\,001}{29\,030\,400}-\ldots \\ &= \sum_{n=1}^{\infty}\frac{(-1)^n}{n}\cdot\frac{1}{n!}\sum_{l=1}^{n}\frac{\left|S_1(n,l)\right|}{l+1} \end{aligned} \tag{8}$$

which is probably due to Arthur Cayley who gave it in 1859,[9] or a very similar expansion converging to Euler's constant

$$\begin{aligned} \gamma &= \frac{1}{2}+\frac{1}{24}+\frac{1}{72}+\frac{19}{2880}+\frac{3}{800}+\frac{863}{362\,880}+\frac{275}{169\,344}+\frac{33\,953}{29\,030\,400}+\ldots \\ &= \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\cdot\frac{1}{n!}\sum_{l=1}^{n}\frac{S_1(n,l)}{l+1} \end{aligned} \tag{9}$$

which was given by Lorenzo Mascheroni in 1790 [97, p. 23] and was subsequently rediscovered several times (in particular, by Ernst Schröder in 1879 [133, p. 115, Eq. (25a)], by Niels E. Nørlund in 1923 [111, p. 244], by Jan C. Kluyver in 1924 [77], by Charles Jordan in 1929 [73, p. 148], by Kenter in 1999 [76], by Victor Kowalenko in 2008 [82,81]).[10] At large n and moderate values of argument z, discovered series converge approximately at the same rate as $\sum(n\ln^m n)^{-2}$, where $m=1$ for $\ln\Gamma(z)$ and $\Psi(z)$, $m=2$ for $\Psi_1(z)$ and $\Psi_2(z)$, $m=3$ for $\Psi_3(z)$ and $\Psi_4(z)$, *etc.* At the same time, first partial sums of the series may behave quite irregularly, and the sign of its general term changes in a complex pattern. However, in all cases, including small and moderate values of n, the absolute value of the nth general term remains bounded by αn^{-2}, where α does not depend on n. Finally, in the manuscript, we also obtain a number of other series expansions containing Stirling numbers, Gregory's coefficients (logarithmic numbers, also known as Bernoulli numbers of the second kind), Cauchy numbers, ordinary and generalized Bernoulli numbers, binomial coefficients and harmonic numbers, as well as provide the convergence analysis for some of them.

1.2. Notations

Throughout the manuscript, following abbreviated notations are used: $\gamma=0.5772156649\ldots$ for Euler's constant, $\binom{k}{n}$ denotes the binomial coefficient C_k^n, B_n stands for the nth Bernoulli number,[11] $\lfloor x\rfloor$ for the integer part of x, $\operatorname{tg} z$ for the tangent of z, $\operatorname{ctg} z$ for the cotangent of z, $\operatorname{ch} z$ for the hyperbolic cosine of z, $\operatorname{sh} z$ for the hyperbolic sine of z, $\operatorname{th} z$ for the hyperbolic tangent of z, $\operatorname{cth} z$ for the hyperbolic cotangent of z. In

[9] Cayley [30] did not present the formula in the same form as we did, he only gave first six coefficients for formula (17) from p. 410. He noticed that the law for the formation of these coefficients "is a very complicated one" and that they are related in some way to Stirling numbers. The exact relationship between series (8) and Stirling's polynomials (and, hence, Stirling numbers) was established later by Niels Nielsen [109, p. 76], [110, p. 36, eq. (8)], see also [8, vol. III, p. 257, eqs. 19.7(58)–19.7(63)]. By the way, coefficients of this particular series are also strongly correlated with *Cauchy numbers of the second kind*, see (16), (18), footnote 23 and (51).

[10] The series itself was given by Gregorio Fontana, who, however, failed to find a constant to which it converges (he only proved that it should be lesser than 1). Mascheroni identified this *Fontana's constant* and showed that it equals Euler's constant [97, pp. 21–23]. Taking into account that both Fontana and Mascheroni did practically the equal work, series (9) is called throughout the paper *Fontana–Mascheroni's series*. Coefficients of this series are usually written in terms of *Gregory's coefficients*, see (15), (18) and footnote 22.

[11] In particular $B_0=+1$, $B_1=-\frac{1}{2}$, $B_2=+\frac{1}{6}$, $B_3=0$, $B_4=-\frac{1}{30}$, $B_5=0$, $B_6=+\frac{1}{42}$, $B_7=0$, $B_8=-\frac{1}{30}$, $B_9=0$, $B_{10}=+\frac{5}{66}$, $B_{11}=0$, $B_{12}=-\frac{691}{2730}$, *etc.*, see [1, Tab. 23.2, p. 810], [89, p. 5] or [50, p. 258] for further values. Note also that some authors may use slightly different definitions for the Bernoulli numbers, see e.g. [61, p. 19, n° 138] or [6, pp. 3–6].

order to avoid any confusion between compositional inverse and multiplicative inverse, inverse trigonometric and hyperbolic functions are denoted as arccos, arcsin, arctg, ... and not as $\cos^{-1}$, $\sin^{-1}$, tg^{-1}, ... Writings $\Gamma(z)$, $\Psi(z)$, $\Psi_1(z)$, $\Psi_2(z)$, $\Psi_3(z)$, $\Psi_4(z)$, $\zeta(z)$ and $\zeta(z,v)$ denote respectively the gamma, the digamma, the trigamma, the tetragamma, the pentagamma, the hexagamma, the Riemann zeta and the Hurwitz zeta functions of argument z. The Pochhammer symbol $(z)_n$, which is also known as the generalized factorial function, is defined as the rising factorial $(z)_n \equiv z(z+1)(z+2)\cdots(z+n-1) = \Gamma(z+n)/\Gamma(z)$.[12,13] For sufficiently large n, not necessarily integer, the latter can be given by this useful approximation

$$
\begin{aligned}
(z)_n &= \frac{n^{n+z-\frac{1}{2}}\sqrt{2\pi}}{\Gamma(z)\,e^n}\left\{1+\frac{6z^2-6z+1}{12n}+\frac{36z^4-120z^3+120z^2-36z+1}{288n^2}+O(n^{-3})\right\} \\
&= \frac{n^z\cdot\Gamma(n)}{\Gamma(z)}\left\{1+\frac{z(z-1)}{2n}+\frac{z(z-1)(z-2)(3z-1)}{24n^2}+O(n^{-3})\right\}
\end{aligned}
\tag{10}
$$

which follows from the Stirling formula for the Γ-function.[14] Writing $S_1(k,n)$ stands for the signed Stirling numbers of the first kind (see Sect. 2.1). Kronecker symbol (Kronecker delta) of arguments l and k is denoted by $\delta_{l,k}$: $\delta_{l,k}=1$ if $l=k$ and $\delta_{l,k}=0$ if $l\neq k$. $\operatorname{Re} z$ and $\operatorname{Im} z$ denote respectively real and imaginary parts of z. Natural numbers are defined in a traditional way as a set of positive integers, which is denoted by $\mathbb{N}$. Letter i is never used as index and is $\sqrt{-1}$. Finally, by the relative error between the quantity A and its approximated value B, we mean either $(A-B)/A$, or $|A-B|/|A|$, depending on the context. Other notations are standard.

2. Stirling numbers and their role in various expansions

2.1. General information in brief

Stirling numbers were introduced by the Scottish mathematician James Stirling in his famous treatise [145, pp. 1–11], and were subsequently rediscovered in various forms by numerous authors, including Christian Kramp, Pierre-Simon Laplace, Andreas von Ettingshausen, Ludwig Schläfli, Oskar Schlömilch, Paul Appel, Arthur Cayley, George Boole, James Glaisher, Leonard Carlitz and many others [67,84], [93, Book I, part I], [152,127–129], [130, pp. 186–187], [131, vol. II, pp. 23–31], [5,29–31,15,52], [25, p. 129], [109, pp. 67–78], [75, p. 1], [56,79].[15] Traditionally, Stirling numbers are divided in two different “kinds”: Stirling numbers of the first kind and those of the second kind, albeit there really is only one “kind” of Stirling numbers [see footnote 18].[16] The Stirling numbers of the first kind appear in numerous occasions in combinatorics, in calculus of finite differences, in numerical analysis, in number theory and even in calculus of variations. In combinatorics, Stirling numbers of the first kind, denoted $|S_1(n,l)|$, are defined as the number of ways to arrange n objects into l cycles or cyclic arrangements ($|S_1(n,l)|$ is often verbalized “n cycle l”). These numbers are also called *unsigned* (or *signless*) *Stirling numbers*, as opposed to $S_1(n,l)$, which are called *signed Stirling numbers* and which are related to the former as $S_1(n,l) = (-1)^{n\pm l}|S_1(n,l)|$. In the

[12] For nonpositive and complex n, only the latter definition $(z)_n = \Gamma(z+n)/\Gamma(z)$ holds.

[13] Note that some writers (mostly German-speaking) call such a function *faculté analytique* or *Facultät*, see e.g. [129], [130, p. 186], [131, vol. II, p. 12], [62, p. 119], [84]. Other names and notations for $(z)_n$ are briefly discussed in [75, pp. 45–47] and in [58, pp. 47–48].

[14] A simpler variant of the above formula may be found in [148,85].

[15] Although it is largely accepted that these numbers were introduced by James Stirling in his famous treatise [145] published in 1730, Donald E. Knuth [79, p. 416] rightly remarks that these numbers may be much older. In particular, the above-mentioned writer found them in an old unpublished manuscript of Thomas Harriot, dating about 1600.

[16] Within the framework of our study we are not concerned with the *Stirling numbers of the second kind*; we, therefore, will not treat them here. By the way, it is interesting that Stirling himself, first, introduced numbers of the second kind [145, p. 8], and then, those of the first kind [145, p. 11].

analysis and related disciplines, the unsigned/signed Stirling numbers of the first kind are usually defined as the coefficients in the expansion of rising/falling factorial

$$\begin{cases} \prod_{k=0}^{n-1}(z+k) = (z)_n = \dfrac{\Gamma(z+n)}{\Gamma(z)} = \sum_{l=1}^{n}|S_1(n,l)|\cdot z^l & \text{(a)} \\ \prod_{k=0}^{n-1}(z-k) = (z-n+1)_n = \dfrac{\Gamma(z+1)}{\Gamma(z+1-n)} = \sum_{l=1}^{n} S_1(n,l)\cdot z^l & \text{(b)} \end{cases} \tag{11}$$

where $z \in \mathbb{C}$ and $n \geqslant 1$. Stirling numbers of the first kind are also often introduced via their generating functions

$$\begin{cases} \sum_{n=l}^{\infty}\dfrac{|S_1(n,l)|}{n!}z^n = (-1)^l\dfrac{\ln^l(1-z)}{l!}\,, & l=0,1,2,\ldots & \text{(a)} \\ \sum_{n=l}^{\infty}\dfrac{S_1(n,l)}{n!}z^n = \dfrac{\ln^l(1+z)}{l!}\,, & l=0,1,2,\ldots & \text{(b)} \end{cases} \tag{12}$$

both series on the left being uniformly and absolutely convergent on the disk $|z| < 1$.[17] Signed Stirling numbers of the first kind may be calculated explicitly via the following formula

$$S_1(n,l) = \begin{cases} \dfrac{(2n-l)!}{(l-1)!}\sum_{k=0}^{n-l}\dfrac{1}{(n+k)(n-l-k)!(n-l+k)!}\sum_{r=0}^{k}\dfrac{(-1)^r r^{n-l+k}}{r!(k-r)!}\,, & l\in[1,n] \\ 1\,, & n=0\,,\ l=0 \\ 0\,, & \text{otherwise} \end{cases} \tag{13}$$

where $S_1(0,0) = 1$ by convention.[18] From the above definitions, it is visible that numbers $S_1(n,l)$ are necessarily integers: $S_1(1,1) = +1$, $S_1(2,1) = -1$, $S_1(2,2) = +1$, $S_1(3,1) = +2$, $S_1(3,2) = -3$, $S_1(3,3) = +1$, ..., $S_1(8,5) = -1960$, ..., $S_1(9,3) = +118\,124$, *etc.*

Stirling numbers of the first kind were studied in a large number of works and have many various properties which we cannot describe in a small article. These numbers are of great utility, especially, for the summation of series, the fact which was noticed primarily by Stirling in his marvellous treatise [145] and which was later emphasized by numerous writers. In particular, Charles Jordan, who worked a lot on Stirling numbers, see e.g. [75,73,74], remarked that these numbers may be even more important than Bernoulli numbers. In what follows we give only a small amount of the information necessary for the understanding of the rest of our work. Readers interested in a more deep study of these numbers are kindly invited to refer to the above-cited historical references, as well as to the following specialized literature: [75, Chapt. IV], [73,74, 108], [109, pp. 67–78], [110,150], [58, Sect. 6.1], [79, pp. 410–422], [37, Chapt. V], [43], [116, Chapt. 4, § 3, n° 196–210], [62, p. 60 *et seq.*], [107], [121, p. 70 *et seq.*], [141, vol. 1], [11], [33, Chapt. 8], [1, n° 24.1.3, p. 824], [80, Sect. 21.5-1, p. 824], [8, vol. III, § 19.7], [111,144], [38, pp. 91–94], [156, pp. 2862–2865], [6, Chapt. 2], [100,55,57,56,154,25,27], [113, p. 642], [125,51,159,102,10,158,146,68,22,21,71,2,147,59,95,135, 136,126,122,123,64,83]. Note that many writers discovered these numbers independently, without realizing

[17] Remark that formally, in (12), the summation may be started not only from $n = l$, but from any n in the range $[0, l-1]$, because $S_1(n,l) = 0$ for such n.

[18] In the above, we always supposed that n and l are nonnegative, although, this, strictly speaking, is not necessary. In fact, for negative arguments n and l, Stirling numbers of the first kind reduce to those of the second kind and *vice versa*, see e.g. [79, p. 412], [57, p. 116], [62, p. 60 *et seq.*].

that they deal with the Stirling numbers. For this reason, in many sources, these numbers may appear under different names, different notations and even slightly different definitions. Actually, only in the beginning of the XXth century, the name “Stirling numbers” appeared in mathematical literature (mainly, thanks to Thorvald N. Thiele and Niels Nielsen [108,150], [79, p. 416]). Other names for these numbers include: *factorial coefficients*, *faculty's coefficients* (*Facultätencoefficienten*, *coefficients de la faculté analytique*), *differences of zero* and even *differential coefficients of nothing*. The Stirling numbers are also closely connected to the *generalized Bernoulli numbers* $B_n^{(s)}$, also known as *Bernoulli numbers of higher order*, see e.g. [25, p. 129], [55, p. 449], [57, p. 116], [8, vol. III, § 19.7], [162,19,18]; many of their properties may be, therefore, deduced from those of $B_n^{(s)}$. As concerns notations, there exist more than 50 notations for them, see e.g. [57], [75, pp. vii–viii, 142, 168], [79, pp. 410–422], [58, Sect. 6.1], and we do not insist on our particular notation, which may seem for certain not properly chosen. Lastly, we remark that there also are several slightly different definitions of the Stirling numbers of the first kind; our definitions (11)–(13) agree with those adopted by Jordan [75, Chapt. IV], [73,74], Riordan [121, p. 70 *et seq.*], Mitrinović [100], Abramowitz & Stegun [1, nº 24.1.3, p. 824] and many others.[19] A quick analysis of several alternative definitions may be found in [57,56], [75, pp. vii–viii and Chapt. IV], [79, pp. 410–422].

2.2. MacLaurin series expansions of certain composite functions and some other series with Stirling numbers

Let's now focus our attention on expansions (12). An appropriate use of these series provides numerous fascinating formulæ, and especially, the series expansions of the MacLaurin–Taylor type for the composite functions involving logarithms and inverse trigonometric and hyperbolic functions. The technique is based of the summation over l of (12), on the fact that $S_1(n,l)$ vanishes for $l \notin [1,n]$ and on the interchanging the order of summation.[20] For example, writing in (12b) $2l$ for l, and summing the result with respect to l from $l=1$ to $l=\infty$ yields immediately for the right-hand side of (12b) the MacLaurin expansion of $\operatorname{ch}\ln(1+z)$; equating both sides we obtain

$$\operatorname{ch}\ln(1+z) = 1+\sum_{n=1}^{\infty}\frac{z^n}{n!}\cdot\sum_{l=1}^{\lfloor\frac12 n\rfloor}S_1(n,2l) = 1+\frac12\sum_{n=2}^{\infty}(-1)^n z^n\,, \qquad |z|<1 \tag{14}$$

where the sum in the middle may be truncated at $l=\lfloor\frac12 n\rfloor$ thanks to (13), and where at the final stage we used a known property of the Stirling numbers.[21] By the same line of reasoning, if we divide the right-hand side of (12b) by $l+1$ and sum it over $l\in[1,\infty)$, then we get

$$\begin{aligned}\sum_{l=1}^{\infty}\frac{1}{l+1}\cdot\frac{\ln^l(1+z)}{l!} &= \frac{1}{\ln(1+z)}\sum_{l=1}^{\infty}\frac{\ln^{l+1}(1+z)}{(l+1)!} = \\ &= \frac{1}{\ln(1+z)}\Big[e^{\ln(1+z)}-\ln(1+z)-1\Big] = \frac{z}{\ln(1+z)}-1\end{aligned}$$

Applying the same operation to the left-hand side of (12b) and comparing both sides yields

[19] Modern CAS, such as *Maple* or *Mathematica*, also share these definitions; in particular `Stirling1(n,l)` in the former and `StirlingS1[n,l]` in the latter correspond to our $S_1(n,l)$.
[20] Series in question being absolutely convergent.
[21] This particular expansion may be obtained more easily if we remark that $\operatorname{ch}\ln(1+z)=\frac12\big(1+z+\frac{1}{1+z}\big)$. Similarly may be proved Eq. (21).

$$\frac{z}{\ln(1+z)} = 1 + \sum_{n=1}^{\infty} z^n \cdot \underbrace{\frac{1}{n!}\sum_{l=1}^{n} \frac{S_1(n,l)}{l+1}}_{G_n} = 1 + \sum_{n=1}^{\infty} G_n\, z^n\,, \qquad |z|<1\,, \tag{15}$$

the equality, which is more known as the generating equation for the *Gregory's coefficients* G_n (in particular, $G_1 = +\frac{1}{2}$, $G_2 = -\frac{1}{12}$, $G_3 = +\frac{1}{24}$, $G_4 = -\frac{19}{720}$, $G_5 = +\frac{3}{160}$, $G_6 = -\frac{863}{60\,480}$, ...).[22] Analogously, performance of same procedures with (12a), written for $-z$ instead of z, results in

$$\frac{z}{(1+z)\ln(1+z)} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n z^n}{n!} \cdot \underbrace{\sum_{l=1}^{n} \frac{|S_1(n,l)|}{l+1}}_{C_{2,n}} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n C_{2,n}}{n!}\, z^n\,, \qquad |z|<1\,, \tag{16}$$

which is also known as the generating series for the *Cauchy numbers of the second kind* $C_{2,n}$ (in particular, $C_{2,1} = \frac{1}{2}$, $C_{2,2} = \frac{5}{6}$, $C_{2,3} = \frac{9}{4}$, $C_{2,4} = \frac{251}{30}$, $C_{2,5} = \frac{475}{12}$, $C_{2,6} = \frac{19\,087}{84}$, ...).[23] Dividing by z, integrating and determining the constant of integration yields another interesting series

$$\ln\ln(1+z) = \ln z + \sum_{n=1}^{\infty} \frac{(-1)^n z^n}{n \cdot n!} \cdot \sum_{l=1}^{n} \frac{|S_1(n,l)|}{l+1} = \ln z + \sum_{n=1}^{\infty} \frac{(-1)^n C_{2,n}}{n \cdot n!}\, z^n\,, \qquad |z|<1\,, \tag{17}$$

which is an "almost MacLaurin series" for $\ln\ln(1+z)$. Asymptotic studies of general terms in series (15) and (17) reveal that for $n\to\infty$ both terms decrease logarithmically:

$$G_n = \frac{1}{n!}\sum_{l=1}^{n} \frac{S_1(n,l)}{l+1} \sim \frac{(-1)^{n-1}}{n\ln^2 n} \qquad \text{and} \qquad \frac{C_{2,n}}{n \cdot n!} = \frac{1}{n \cdot n!}\sum_{l=1}^{n} \frac{|S_1(n,l)|}{l+1} \sim \frac{1}{n\ln n} \tag{18}$$

respectively (see p. 414), and hence, series (15) and (17) converge not only in $|z|<1$, but also at $z=\pm 1$ at $z=1$ respectively. Thus, putting $z=1$ into (15), we have

$$\frac{1}{\ln 2} = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{l=1}^{n} \frac{S_1(n,l)}{l+1} = 1 + \sum_{n=1}^{\infty} G_n \tag{19}$$

while setting $z=1$ into (17) gives a series for $\ln\ln 2$, see (8). Moreover, application of Abel's theorem on power's series to (15) at $z\to -1^+$ yields *Fontana's series*[24]

$$1 = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!} \sum_{l=1}^{n} \frac{S_1(n,l)}{l+1} = \sum_{n=1}^{\infty} |G_n| \tag{20}$$

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[22] Coefficients $G_n = \frac{1}{n!}\sum\limits_{l=1}^{n}\frac{S_1(n,l)}{l+1} = \frac{(-1)^n}{n!}\int\limits_0^1 (-x)_n\, dx = -\frac{B_n^{(n-1)}}{(n-1)\,n!} = \frac{C_{1,n}}{n!}$ are also called *(reciprocal) logarithmic numbers*, *Bernoulli numbers of the second kind*, normalized *generalized Bernoulli numbers* $B_n^{(n-1)}$, *Cauchy numbers* and normalized *Cauchy numbers of the first kind* $C_{1,n}$. They were introduced by James Gregory in 1670 in the context of area's interpolation formula (which is known nowadays as *Gregory's interpolation formula*) and were subsequently rediscovered in various contexts by many famous mathematicians, including Gregorio Fontana, Lorenzo Mascheroni, Pierre-Simon Laplace, Augustin-Louis Cauchy, Jacques Binet, Ernst Schröder, Oskar Schlömilch, Charles Hermite, Jan C. Kluyver and Joseph Ser [120, vol. II, pp. 208–209], [149, vol. 1, p. 46, letter written on November 23, 1670 to John Collins], [72, pp. 266–267, 284], [54, pp. 75–78], [32, pp. 395–396], [97, pp. 21–23], [92, T. IV, pp. 205–207], [15, pp. 53–55], [151], [54, pp. 192–194], [94,154,133,132], [66, pp. 65, 69], [77,134]. For more information about these important coefficients, see [111, pp. 240–251], [112], [73, p. 132, Eq. (6), p. 138], [74, p. 258, Eq. (14)], [75, pp. 266–267, 277–280], [109,110,142,143], [144, pp. 106–107], [40], [156, p. 190], [61, p. 45, n° 370], [8, vol. III, pp. 257–259], [140], [89, p. 229], [117, n° 600, p. 87], [78, p. 216, n° 75-a] [37, pp. 293–294, n° 13], [26,70,163,3,165,24], [99, Eq. (3)], [98,105], [4, pp. 128–129], [6, Chapt. 4], [83].

[23] These numbers $C_{2,n} = \sum\limits_{l=1}^{n}\frac{|S_1(n,l)|}{l+1} = \int\limits_0^1 (x)_n\, dx = |B_n^{(n)}|$, called by some authors signless *generalized Bernoulli numbers* $|B_n^{(n)}|$ and signless *Nørlund numbers*, are much less famous than Gregory's coefficients G_n, but their study is also very interesting, see e.g. [111, pp. 150–151], [112], [8, vol. III, pp. 257–259], [37, pp. 293–294, n° 13], [69,3,165,119].

[24] This series appears in a letter of Gregorio Fontana to which Lorenzo Mascheroni refers in [97, pp. 21–23].

converging at the same rate as $\sum n^{-1}\ln^{-2}n$, see (18).

The use of the same and of similar techniques allows to readily obtain power series for even more complicated functions. Further examples demonstrate better than words the powerfulness of the method:

$$\operatorname{sh}\ln(1+z) = \sum_{n=1}^{\infty}\frac{z^n}{n!}\cdot\sum_{l=0}^{\lfloor\frac{1}{2}n\rfloor}S_1(n,2l+1) = z-\frac{1}{2}\sum_{n=2}^{\infty}(-1)^n z^n\,, \qquad |z|<1 \tag{21}$$

$$\cos\ln(1+z) = 1+\sum_{n=1}^{\infty}\frac{z^n}{n!}\cdot\sum_{l=1}^{\lfloor\frac{1}{2}n\rfloor}(-1)^l S_1(n,2l)\,, \qquad |z|<1 \tag{22}$$

$$\sin\ln(1+z) = \sum_{n=1}^{\infty}\frac{z^n}{n!}\cdot\sum_{l=0}^{\lfloor\frac{1}{2}n\rfloor}(-1)^l S_1(n,2l+1)\,, \qquad |z|<1 \tag{23}$$

$$\ln\bigl[1+\ln(1+z)\bigr] = \sum_{n=1}^{\infty}\frac{z^n}{n!}\cdot\sum_{l=0}^{n-1}(-1)^l l!\cdot S_1(n,l+1)\,, \qquad |z|<1-e^{-1}\approx 0.63 \tag{24}$$

$$\frac{1}{\ln^2(1+z)} = \frac{1}{z^2}+\frac{1}{z}+\sum_{n=0}^{\infty}\frac{z^n}{(n+2)!}\cdot\sum_{l=1}^{n+1}\frac{1-n(l+1)}{(l+1)(l+2)}\cdot S_1(n+1,l)\,, \qquad |z|<1 \tag{25}$$

$$\begin{aligned}\frac{1}{\ln^m(1+z)} &= \frac{1}{z}\cdot\sum_{k=1}^{m-1}\frac{1}{k!\cdot\ln^{m-k}(1+z)}+\\ &\quad+\frac{1}{m!\cdot z}+\sum_{n=1}^{\infty}\frac{z^{n-1}}{n!}\cdot\sum_{l=1}^{n}\frac{S_1(n,l)}{(l+1)_m}\,, \qquad \begin{array}{l} m=2,3,4,\ldots\\ |z|<1\end{array}\end{aligned} \tag{26}$$

$$\frac{\ln^m(1+z)}{1+z} = (-1)^m m!\cdot\sum_{n=0}^{\infty}\frac{(-1)^n\bigl|S_1(n+1,m+1)\bigr|}{n!}\,z^n\,, \qquad \begin{array}{l} m=0,1,2,\ldots\\ |z|<1\end{array} \tag{27}$$

$$\operatorname{arctg}\ln(1+z) = \sum_{n=1}^{\infty}\frac{z^n}{n!}\cdot\sum_{l=0}^{\lfloor\frac{1}{2}n\rfloor}(-1)^l(2l)!\cdot S_1(n,2l+1)\,, \qquad |z|<2\sin\frac{1}{2}\approx 0.96 \tag{28}$$

$$\operatorname{arcth}\ln(1+z) = \sum_{n=1}^{\infty}\frac{z^n}{n!}\cdot\sum_{l=0}^{\lfloor\frac{1}{2}n\rfloor}(2l)!\cdot S_1(n,2l+1)\,, \qquad |z|<1-e^{-1}\approx 0.63 \tag{29}$$

$$\frac{\operatorname{arcth}^m z}{m!} = \sum_{n=m}^{\infty}z^n\cdot\sum_{l=m}^{n}\binom{n-1}{l-1}\cdot\frac{2^{l-m}\cdot S_1(n,l)}{l!}\,, \qquad m=1,2,3,\ldots\,, \qquad |z|<1 \tag{30}$$

$$\operatorname{tg}\ln(1+z) = \sum_{n=1}^{\infty}\frac{z^n}{n!}\cdot\sum_{l=0}^{\lfloor\frac{1}{2}n\rfloor}\frac{2^{2l+1}(2^{2l+2}-1)\cdot|B_{2l+2}|\cdot S_1(n,2l+1)}{l+1}\,, \qquad |z|<1-e^{-\pi/2}\approx 0.79 \tag{31}$$

$$\begin{aligned}\operatorname{th}\ln(1+z) &= \sum_{n=1}^{\infty}\frac{z^n}{n!}\cdot\sum_{l=0}^{\lfloor\frac{1}{2}n\rfloor}(-1)^l\frac{2^{2l+1}(2^{2l+2}-1)\cdot|B_{2l+2}|\cdot S_1(n,2l+1)}{l+1}=\\ &=\sum_{n=0}^{\infty}(-1)^n\frac{z^{4n+1}}{2^{2n}}-\sum_{n=0}^{\infty}(-1)^n\frac{z^{4n+2}}{2^{2n+1}}+\sum_{n=0}^{\infty}(-1)^n\frac{z^{4n+4}}{2^{2n+2}}\,, \qquad |z|<\sqrt{2}\approx 1.41\end{aligned} \tag{32}$$

Derived expansions coincide with the corresponding MacLaurin/Laurent series, converge everywhere where expanded functions are analytic[25] and contain rational coefficients only. The main advantage of this tech-

[25] The radius of convergence of such series r is conditioned by the singularities of expanded functions. For instance, $\ln\bigl[1+\ln(1+z)\bigr]$ is analytic on the entire complex z-plane except points at which $1+z=0$ and $1+\ln(1+z)=0$, which are both branch points.

nique is that we do not need to "mechanically" compute the nth derivative of the composite function, which often may be a very laborious task.[26,27]

Generating equations for Stirling numbers of the first kind may be also successfully used for the derivation of more complicated and quite unexpected results. For instance, it is known that

$$\zeta(k+1) = \sum_{n=k}^{\infty} \frac{|S_1(n,k)|}{n\cdot n!}\,, \qquad k=1,2,3,\ldots \tag{33}$$

see e.g. Jordan's book [75, pp. 166, 194–195].[28] This result was recently rediscovered by several modern writers, e.g. by Shen [135] and Sato [126]; however, their proofs are exceedingly long. Using (12) the whole procedure takes only a few lines:

$$\begin{aligned}\sum_{n=k}^{\infty} \frac{|S_1(n,k)|}{n\cdot n!} &= \sum_{n=k}^{\infty} \frac{|S_1(n,k)|}{n!}\underbrace{\int_0^1 x^{n-1}dx}_{1/n} = \int_0^1 \underbrace{\sum_{n=k}^{\infty} \frac{|S_1(n,k)|}{n!}\,x^n}_{\text{see (12)}}\,\frac{dx}{x} = \frac{(-1)^k}{k!}\int_0^1 \frac{\ln^k(1-x)}{x}\,dx = \\ &= \frac{(-1)^k}{k!}\int_0^\infty \frac{(-t)^k}{e^t\big(1-e^{-t}\big)}\,dt = \frac{1}{\Gamma(k+1)}\int_0^\infty \frac{t^k}{e^t-1}\,dt = \zeta(k+1)\end{aligned} \tag{34}$$

where in last integrals we made a change of variable $x = 1-e^{-t}$. The above formula may be readily generalized to

$$\zeta(k+1,v) = \sum_{n=k}^{\infty} \frac{|S_1(n,k)|}{n\cdot (v)_n}\,, \qquad k=1,2,3,\ldots\,, \qquad \operatorname{Re} v>0\,, \tag{35}$$

where at large n

$$\frac{|S_1(n,k)|}{n\cdot (v)_n} \sim \frac{\Gamma(v)}{(k-1)!}\cdot\frac{\ln^{k-1}n}{n^{v+1}}\,, \qquad n\to\infty\,, \tag{36}$$

in virtue of (10) and a known asymptotics for the Stirling numbers [74, p. 261], [75, p. 161], [1, n° 24.1.3, p. 824], [158, p. 348, Eq. (8)]. Moreover, by a slight modification of the above technique, we may also obtain the following results:

$$\begin{aligned}&\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}\cdot\frac{1}{n!}\sum_{l=1}^{n} f(l)\,S_1(n,l) = \sum_{l=1}^{\infty}(-1)^{l+1}f(l)\zeta(l+1)\,,\\ &\sum_{n=1}^{\infty} \frac{(-1)^{n-1}H_n}{n}\cdot\frac{1}{n!}\sum_{l=1}^{n} f(l)\,S_1(n,l) = \sum_{l=1}^{\infty}(-1)^{l+1}(l+1)f(l)\zeta(l+2)\,,\end{aligned} \tag{37}$$

From the former we conclude that the radius of convergence cannot be greater than 1, and from the latter, it follows that it cannot exceed $1-e^{-1}$ which is even lesser than 1. Hence $r = 1-e^{-1} \approx 0.63$.

[26] Some other power series expansions involving Stirling numbers are also given in works of Wilf [159], Kruchinin [86,88,87] and Rządkowski [124]. Moreover, series expansions of certain composite functions, not necessarily containing Stirling numbers, may be found in [78, Chapt. VI], [61, p. 20 & 63], [63] and [118, vol. I] (in the third reference, the author also provides a list of related references).

[27] Since these expansions are not particularly difficult to obtain and also may be derived by other techniques, it is possible that some of them could appear in earlier works. The same remark also concerns formulæ (37)–(46). For instance, formula (41) may be found in other sources as well, see e.g. [82, p. 431, Eq. (76)], [4, p. 128, Eq. (7.3.11)], [164, p. 4006] (the same series also appears in [35, p. 14, Eq. (2.39)], but the result is incorrect). Series (44) is also known, see e.g. [163, p. 2952, Eq. (1.3)], [35, p. 20, Eq. (3.6)], [24, p. 307, Eq. for $F_0(2)$].

[28] Jordan derives this formula and remarks that particular cases of it were certainly known to Stirling [145] (see also [108, p. 302, Eq. (36bis)], [150, p. 10], and compare it to formulæ from [145, p. 11]).

where $f(l)$ is an arbitrary function ensuring the convergence and H_n is the nth harmonic number;

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\cdot\frac{1}{n!}\sum_{l=1}^{n}\frac{S_1(n,l)}{l+k} &= \sum_{l=2}^{\infty}\frac{(-1)^l\cdot\zeta(l)}{l+k-1} = \frac{1}{k-1}-\frac{\ln 2\pi}{k}+\frac{\gamma}{2}+\\ &+\sum_{l=1}^{\lfloor\frac12(k-1)\rfloor}(-1)^l\binom{k-1}{2l-1}\frac{(2l)!\cdot\zeta'(2l)}{l\cdot(2\pi)^{2l}}+\sum_{l=1}^{\lfloor\frac12 k\rfloor-1}(-1)^l\binom{k-1}{2l}\frac{(2l)!\cdot\zeta(2l+1)}{2\cdot(2\pi)^{2l}}\end{aligned} \tag{38}$$

where $k = 2, 3, 4, \ldots$ and where the series on the left converges as $\sum n^{-2}\ln^{-k-1}n$; for $k = 1, 2, 3, \ldots$

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n+k}\cdot\frac{1}{n!}\sum_{l=1}^{n}\frac{S_1(n,l)}{l+1} &= \sum_{n=1}^{\infty}\frac{|G_n|}{n+k} = \frac{1}{k}+\sum_{m=1}^{k}(-1)^m\binom{k}{m}\ln(m+1) =\\ &= \frac{1}{k}+\Delta^k\ln(x)\Big|_{x=1}\end{aligned} \tag{39}$$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\cdot\frac{1}{n!}\sum_{l=1}^{n}\frac{S_1(n,l)}{l+1} = \sum_{n=1}^{\infty}\frac{|G_n|}{n} = \gamma \tag{40}$$

$$\sum_{n=2}^{\infty}\frac{(-1)^{n-1}}{n-1}\cdot\frac{1}{n!}\sum_{l=1}^{n}\frac{S_1(n,l)}{l+1} = \sum_{n=2}^{\infty}\frac{|G_n|}{n-1} = -\frac{1}{2}+\frac{\ln 2\pi}{2}-\frac{\gamma}{2} \tag{41}$$

$$\sum_{n=3}^{\infty}\frac{(-1)^{n-1}}{n-2}\cdot\frac{1}{n!}\sum_{l=1}^{n}\frac{S_1(n,l)}{l+1} = \sum_{n=3}^{\infty}\frac{|G_n|}{n-2} = -\frac{1}{8}+\frac{\ln 2\pi}{12}-\frac{\zeta'(2)}{2\pi^2} \tag{42}$$

$$\sum_{n=4}^{\infty}\frac{(-1)^{n-1}}{n-3}\cdot\frac{1}{n!}\sum_{l=1}^{n}\frac{S_1(n,l)}{l+1} = \sum_{n=4}^{\infty}\frac{|G_n|}{n-3} = -\frac{1}{16}+\frac{\ln 2\pi}{24}-\frac{\zeta'(2)}{4\pi^2}+\frac{\zeta(3)}{8\pi^2} \tag{43}$$

where Δ^k is the kth finite difference, see e.g. [153, p. 270, Eq. (14.17)], and where all series on the left converges as $\sum(n\ln n)^{-2}$;

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}H_n}{n}\cdot\frac{1}{n!}\sum_{l=1}^{n}\frac{S_1(n,l)}{l+1} = \sum_{n=1}^{\infty}\frac{|G_n|\cdot H_n}{n} = \frac{\pi^2}{6}-1 \tag{44}$$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}H_n}{n}\cdot\frac{1}{n!}\sum_{l=1}^{n}\frac{S_1(n,l)}{(l+1)(l+2)} = \frac{\pi^2}{12}-\gamma \tag{45}$$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}H_n}{n}\cdot\frac{1}{n!}\sum_{l=1}^{n}\frac{S_1(n,l)}{(l+1)(l+3)} = \frac{\pi^2}{18}+\frac{1}{2}\ln 2\pi-\frac{\gamma}{2}-1 \tag{46}$$

which all converge as $\sum n^{-2}\ln^{-1}n$, and even this beautiful alternating series

$$\sum_{n=1}^{\infty}\frac{1}{n}\cdot\frac{1}{n!}\sum_{l=1}^{n}\frac{S_1(n,l)}{l+1} = \sum_{n=1}^{\infty}\frac{G_n}{n} = \operatorname{Ei}(\ln 2)-\gamma = \operatorname{li}(2)-\gamma\,, \tag{47}$$

where $\operatorname{Ei}(\cdot)$ and $\operatorname{li}(\cdot)$ denote exponential integral and logarithmic integral functions respectively.[29] Finally, Stirling numbers of the first kind may also appear in the evaluation of certain integrals, which, at first sight, have nothing to do with Stirling numbers. For instance, if k is positive integer and $\operatorname{Re} s > k-1$, then

[29] Numbers G_n are strictly alternating: $G_n = (-1)^{n-1}|G_n|$. The left side of (47) is, therefore, the alternating variant of Fontana–Mascheroni's series (9), (40), and from various points of view the constant $\operatorname{li}(2)-\gamma = 0.4679481152\ldots$ may be regarded as the *alternating Euler's constant*, by analogy to $\ln\frac{4}{\pi}$, which was earlier proposed as such by Jonathan Sondow in [137].

$$\int_0^1 \frac{\ln^s(1-x)}{x^k}\,dx = \frac{(-1)^s\cdot\Gamma(s+1)}{(k-1)!}\cdot\begin{cases}\zeta(s+1)\,, & k=1\\ \zeta(s)\,, & k=2\\ \zeta(s-1)+\zeta(s)\,, & k=3\\ \zeta(s-2)+3\zeta(s-1)+2\zeta(s)\,, & k=4\\ \displaystyle\sum_{r=1}^{k-1} S_1(k-1,r)\sum_{m=0}^{r}\binom{r}{m}(k-2)^{r-m}\zeta(s+1-m)\,, & k\geqslant 3\end{cases} \tag{48}$$

The proofs of some of these results being quite long, we, accordingly to the remarks of the reviewer and of the associated editor, placed them in Appendix A of the arXiv version of the paper arXiv:1408.3902.[30] In Appendix B of the same arXiv version, the reader will also find several full asymptotics for the sums involving Stirling numbers of the first kind, such as, for example,

$$\begin{aligned}\frac{1}{n!}\sum_{l=1}^{n}\frac{|S_1(n,l)|}{l+k} &\sim \frac{1}{\ln n}+\sum_{l=1}^{\infty}\frac{(-1)^l}{\ln^{l+1} n}\cdot\left[\frac{x^{k-1}}{\Gamma(x)}\right]^{(l)}_{x=1} = \frac{1}{\ln n}-\frac{k-1+\gamma}{\ln^2 n}+\\ &+\frac{6\gamma^2+12\gamma(k-1)-\pi^2+6(k^2-3k+2)}{6\ln^3 n}+O\!\left(\frac{1}{\ln^4 n}\right), \qquad \begin{array}{l}k=1,2,3,\ldots\\ n\to\infty\end{array}\end{aligned} \tag{49}$$

or

$$\begin{aligned}\frac{1}{n!}\sum_{l=1}^{n}\frac{S_1(n,l)}{l+k} &\sim \frac{(-1)^{n-1}}{n}\sum_{m=0}^{k}\binom{k}{m}(-1)^m\left\{\frac{1}{\ln n}+\sum_{l=1}^{\infty}\frac{(-1)^l}{\ln^{l+1} n}\cdot\left[\frac{x^m}{\Gamma(x)}\right]^{(l)}_{x=1}\right\} =\\ &= \frac{(-1)^{n-1}}{n}\left\{\frac{k!}{\ln^{k+1} n}-\frac{\gamma\,(k+1)!}{\ln^{k+2} n}+O\!\left(\frac{1}{\ln^{k+3} n}\right)\right\}, \qquad \begin{array}{l}k=1,2,3,\ldots\\ n\to\infty\,,\end{array}\end{aligned} \tag{50}$$

or the full asymptotics for the Cauchy numbers of the second kind

$$\begin{aligned}\frac{C_{2,n}}{n!} = \frac{\left|B_n^{(n)}\right|}{n!} = \frac{1}{n!}\sum_{l=1}^{n}\frac{|S_1(n,l)|}{l+1} &= \frac{1}{\ln n}+\sum_{l=1}^{\infty}\frac{(-1)^l}{\ln^{l+1} n}\cdot\left[\frac{1}{\Gamma(x)}\right]^{(l)}_{x=1}+O\!\left(\frac{1}{n\ln^2 n}\right) =\\ &= \frac{1}{\ln n}-\frac{\gamma}{\ln^2 n}-\frac{\pi^2-6\gamma^2}{6\ln^3 n}+O\!\left(\frac{1}{\ln^4 n}\right), \qquad n\to\infty\,,\end{aligned} \tag{51}$$

as well as that of Gregory's coefficients G_n

$$\begin{aligned}G_n = \frac{1}{n!}\sum_{l=1}^{n}\frac{S_1(n,l)}{l+1} &= \frac{(-1)^{n-1}}{n}\cdot\sum_{l=1}^{\infty}\frac{(-1)^l}{\ln^{l+1} n}\cdot\left[\frac{1-x}{\Gamma(x)}\right]^{(l)}_{x=1}+O\!\left(\frac{1}{n^2\ln n}\right) =\\ &= \frac{(-1)^{n-1}}{n}\cdot\left\{\frac{1}{\ln^2 n}-\frac{2\gamma}{\ln^3 n}-\frac{\pi^2-6\gamma^2}{2\ln^4 n}+O\!\left(\frac{1}{\ln^5 n}\right)\right\}, \qquad n\to\infty\,.\end{aligned} \tag{52}$$
[31]

[30] Mostly, these formulæ are derived in the manner analogous to (34), except for (48). The latter may be derived as follows. By differentiating k times the well-known expansion $(1-y)^{-1}=\sum y^n$ and by using (11), we get $(1-y)^{-k-1}k! = \sum\limits_{n=k}^{\infty}\sum\limits_{r=1}^{k} S_1(k,r)\,n^r y^{n-k}$ in $|y|<1$. Putting $k-1$ instead of k and $x=1-y$ yields, after some algebra similar to (34), equality (48).

In Appendix B, we also obtained accurate upper and lower bounds for the Cauchy numbers of the second kind

$$\begin{aligned}\frac{1}{\ln(n+1)}-\frac{1}{\ln^2(n+1)}+\frac{1}{(n+1)\ln^2(n+1)}\leqslant\frac{C_{2,n}}{n!}\leqslant\frac{1}{\ln n}-\frac{\gamma}{\ln^2 n}-\\ -\frac{2(1-\gamma)}{\ln^3 n}+\frac{2-\gamma}{n\ln^2 n}+\frac{2(1-\gamma)}{n\ln^3 n}\,,\qquad n\geqslant 2\,,\end{aligned}\tag{53}$$

which also imply that

$$\frac{1}{\ln n}-\frac{1}{\ln^2 n}\leqslant\frac{C_{2,n}}{n!}\leqslant\frac{1}{\ln n}-\frac{\gamma}{\ln^2 n}\,,\qquad n\geqslant 3\,,\tag{54}$$

as well as those for Gregory's coefficients

$$\begin{aligned}\frac{1}{n\ln^2 n}-\frac{2}{n\ln^3 n}+\frac{1}{n^2\ln^2 n}+\frac{2}{n^2\ln^3 n}\leqslant\big|G_n\big|\leqslant\frac{1}{(n-1)\ln^2(n-1)}-\frac{2\gamma}{(n-1)\ln^3(n-1)}-\\ -\frac{6\,(1-\gamma)}{(n-1)\ln^4(n-1)}+\frac{1-\gamma}{n(n-1)\ln^2(n-1)}+\frac{2\,(3-\gamma)}{n(n-1)\ln^3(n-1)}+\frac{12\,(1-\gamma)}{n(n-1)\ln^4(n-1)}\end{aligned}\tag{55}$$

where $n\geqslant 3$. The latter also imply a weaker relation

$$\frac{1}{n\ln^2 n}-\frac{2}{n\ln^3 n}\leqslant\big|G_n\big|\leqslant\frac{1}{n\ln^2 n}-\frac{2\gamma}{n\ln^3 n}\,,\qquad n\geqslant 5\,,\tag{56}$$

which may be sufficient in many applications and which is also very accurate. Corresponding figures illustrate the quality of the bounds.[32]

Historical remark. The first-order asymptotics for $C_{2,n}$, the second formulæ in (18), was probably known to Binet as early as 1839 (see the final remark on p. 428) and may be also found in a later work of Comtet [37, p. 294]. As regards the higher-order terms given in (51), as well as upper and lower bounds given in (53), we have not found them in previously published literature. The first-order approximation for Gregory's coefficients G_n at $n\to\infty$, the first formulæ in (18), was found by Ernst Schröder in 1879 [133, p. 115, Eq. (25a)]. It was rediscovered by Johan Steffensen in 1924 [143, pp. 2–4], [144, pp. 106–107], and was slightly bettered in 1957 by Davis [40, p. 14, Eq. (14)]. Higher-order terms of this asymptotics were obtained by S.C. Van Veen in 1950 [151, p. 336], [112, p. 29], Gergő Nemes in 2011 [105] and the author in 2014. S.C. Van Veen and Gergő Nemes used different methods to derive their results and obtained

[31] All these asymptotics are derived in a similar manner. For instance, formula (49) is obtained as follows. First, proceeding analogously to (34) and using Stirling approximation (10), we have $\sum\limits_{l=1}^{n}\frac{|S_1(n,l)|}{l+k}=\int\limits_0^1 x^{k-1}(x)_n\,dx\sim(n-1)!\int\limits_0^1\frac{n^x x^{k-1}}{\Gamma(x)}\,dx$ when $n\to\infty$. Using the MacLaurin series (70) and performing the term-by-term integration yields (49). Similarly, $\sum\limits_{l=1}^{n}\frac{S_1(n,l)}{l+k}=(-1)^n\int\limits_0^1 x^{k-1}(-x)_n\,dx\sim(-1)^{n-1}(n-1)!\int\limits_0^1\frac{n^{z-1}(1-z)^k}{\Gamma(z)}\,dz$ at $n\to\infty$. Using binomial expansion for $(1-z)^k$ and proceeding analogously to the previous case yields (50). Setting $k=1$, we obtain asymptotics (51) and (52). Readers interested in a more deep study of asymptotical methods might also wish to consult the following literature: [46, Chapt. I, § 4], [45,114,43].

[32] In brief, bounds (53)–(56) are obtained as follows. First, we use the integral formulæ from footnotes 23 and 22 to show that $C_{2,n}=\int\limits_0^1\frac{\Gamma(x+n)}{\Gamma(x)}\,dx$ and $G_n\,n!=(-1)^{n-1}\int\limits_0^1\frac{(1-z)\,\Gamma(n-1+z)}{\Gamma(z)}\,dz$, both integrands being strictly nonnegative. Then, for each of the Γ-functions, we use two inequalities: first $(n+1)^{x-1}n!\leqslant\Gamma(x+n)\leqslant n^{x-1}n!$ where $0\leqslant x\leqslant 1$, $n=1,2,3,\ldots$, see [49, Eqs. (6)–(7), Fig. 2], and second, $x\leqslant\frac{1}{\Gamma(x)}\leqslant(\gamma-1)x^2+(2-\gamma)x$ for $0\leqslant x\leqslant 1$, see Appendix B for more information on the latter bounds. This yields bounds (53) and (55), and, after a little algebra, (54) and (56).

different expressions; however, one can show that their formulæ are equivalent. The former employed an elegant contour integration method, while the latter used Watson's lemma. Our formula (52) differs from both Van Veen's result and Nemes' result, but it is also equivalent to them. Note also that many researchers (e.g. Nørlund [112, p. 29], Davis [40, p. 14, Eq. (14)], Nemes [105]) incorrectly attribute the first-order asymptotics of G_n to Steffensen, who only rediscovered it. The same first-order asymptotics also appears in the well-known monograph [37, p. 294], but Comtet did not specify the source of the formula. As regards the bounds for G_n, in 1922 Steffensen found that $\frac{1}{6n\,(n-1)} < |G_n| < \frac{1}{6n}$ for $n > 2$, see [142, p. 198, Eq. (27)], [143, p. 2, Eq. (3)], [144, p. 106, Eq. (9)]. A stronger result (at large n) was stated by Kluyver in 1924 [77, p. 144]: $|G_n| < \frac{1}{n\ln n}$ with $n \geqslant 2$. In 2010, Rubinstein [122, p. 30, Theorem 1.1] found a bound for more general numbers, from which it *inter alia* follows that[33] $|G_n| \leqslant \frac{4\,(1+\ln(1+n))}{1+n}$ for $n \geqslant 1$; this bound is nevertheless much weaker than both preceding bounds. In another recent paper [36, p. 473], Coffey remarked that numerical simulations suggest that $|G_n|$ should be lesser than $\frac{1}{n\ln^2 n}$ for all $n \geqslant 2$, but that the proof of this result was missing.[34] Inequalities (56) include the missing proof. By the way, as far as we know, our bounds (55) are currently the best bounds for Gregory's coefficients.

2.3. An inspiring example for the derivation of the series for $\ln\Gamma(z)$

Let's now consider the example which was originally our inspiration for this work. In exercise n° 39-b in [13, Sect. 4] we established that

$$\int\limits_0^1 \frac{\operatorname{arctg}\operatorname{arcth} x}{x}\,dx = \pi\left\{\ln\Gamma\left(\frac{1}{\pi}\right) - \ln\Gamma\left(\frac{1}{2}+\frac{1}{\pi}\right) - \frac{1}{2}\ln\pi\right\} = 1.025760510\ldots \tag{57}$$

The arctangent of the hyperbolic arctangent is analytic in the whole disk $|x| < 1$, and therefore, can be expanded into the MacLaurin series.[35] The coefficients of such an expansion require a careful watching, the law for their formation being difficult to derive by inductive or semi-inductive methods. So we resort again to the method employing Stirling numbers:

$$\begin{aligned}
\operatorname{arctg}\operatorname{arcth} x &= \sum_{l=0}^{\infty}(-1)^l(2l)!\cdot\frac{\operatorname{arcth}^{2l+1} x}{(2l+1)!} = \sum_{n=1}^{\infty} x^n\cdot\sum_{k=1}^{n}\binom{n-1}{k-1}\frac{2^k}{k!}\cdot\sum_{l=0}^{\lfloor\frac{1}{2}n\rfloor}(-1)^l\cdot\frac{(2l)!\cdot S_1(k,2l+1)}{2^{2l+1}} = \\
&= \sum_{n=0}^{\infty} x^{2n+1}\cdot\underbrace{\sum_{k=1}^{2n+1}\binom{2n}{k-1}\frac{2^k}{k!}\cdot\sum_{l=0}^{n}(-1)^l\cdot\frac{(2l)!\cdot S_1(k,2l+1)}{2^{2l+1}}}_{A_n} = x + \frac{1}{15}x^5 + \frac{1}{45}x^7 + \\
&\qquad + \frac{64}{2835}x^9 + \frac{71}{4725}x^{11} + \frac{5209}{405\,405}x^{13} + \frac{2\,203\,328}{212\,837\,625}x^{15} + \ldots, \qquad |x| < 1,
\end{aligned}$$

where we used result (30), as well as the oddness of the expanded function.[36] Inserting this expansion into (57) and performing the term-by-term integration, we obtain the following series for the difference of first two terms in curly brackets in (57)

---

[33] Rubinstein's $\alpha_n(s)$ at $s = 0$ are our $-|G_n|$.

[34] Note that Coffey's p_{n+1} are our $|G_n|$ (Coffey's notation are probably borrowed from Ser's paper [134]).

[35] Function $\operatorname{arctg}\operatorname{arcth} x$ has branch points at $x = \pm 1$ and $x = \pm i\operatorname{tg} 1 \approx \pm 1.56i$.

[36] Note that although the MacLaurin series for the arctangent is valid only in the unit circle, i.e. formally only for such x that $|\operatorname{arcth} x| < 1$, the above expansion holds uniformly in the whole disk $|x| < 1$ (in virtue of the Cauchy's theorem on the representation of analytic functions by power series, as well as of the principle of analytic continuation). Moreover, an advanced study of this series, analogous to that performed in the next section, shows that it also converges for $x = 1$.

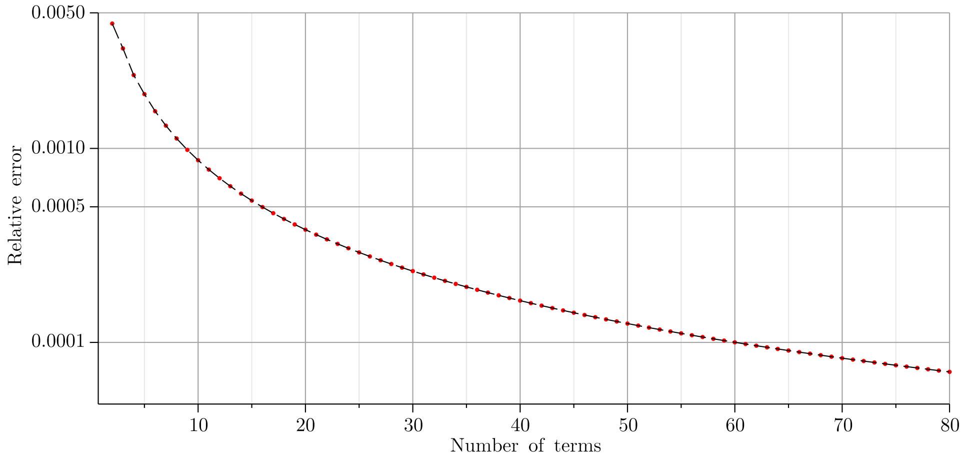

Fig. 1. Relative error of series expansion (58), logarithmic scale.

$$\ln\Gamma\left(\frac{1}{\pi}\right)-\ln\Gamma\left(\frac{1}{2}+\frac{1}{\pi}\right)=\frac{1}{2}\ln\pi+\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{A_n}{2n+1}=\frac{1}{2}\ln\pi+\frac{1}{\pi}\left\{1+\frac{1}{75}+\frac{1}{315}+\frac{64}{25\,515}+\right.$$
$$\left.+\frac{71}{51\,975}+\frac{5209}{5\,270\,265}+\frac{2\,203\,328}{3\,192\,564\,375}+\frac{132\,313}{253\,127\,875}+\ldots\right\}=0.8988746544\ldots \tag{58}$$

with A_n defined in the preceding equation. The derived series does not converge rapidly, see Fig. 1, but the most remarkable is that it contains rational coefficients only, which is quite unusual, especially for the arguments related to π^{-1}. This suggests that there might be some more general series similar in nature to (58), which allows to expand the logarithm of the Γ-function at certain points related to π^{-1} into the series with rational coefficients only. Such series expansions are the subject of our study in the next section.

3. Series expansions for the logarithm of the Γ-function and polygamma functions

3.1. *First series expansion for the logarithm of the* Γ*-function*

3.1.1. *Derivation of the series expansion*

Consider the general form of the second Binet's integral formula for the logarithm of the Γ-function

$$\int_0^{\infty}\frac{\operatorname{arctg} ax}{e^{bx}-1}\,dx=\frac{\pi}{b}\ln\Gamma\left(\frac{b}{2\pi a}\right)+\frac{1}{2a}\left(1-\ln\frac{b}{2\pi a}\right)+\frac{\pi}{2b}\ln\frac{b}{4\pi^2 a} \tag{59}$$

$a>0$ and $\operatorname{Re} b>0$, see e.g. [118, vol. I, nº 2.7.5-6], [12, pp. 335–336], [157, pp. 250–251], [8, vol. I, p. 22, Eq. 1.9(9)] or [13, Sect. 4, exercise nº 40]. The general idea of the method consists in finding such a change of variable that reduces the integrand in the left-hand side of (59) to a function (probably, a composite function) which may be "easily" expanded into the MacLaurin series. In our case, this change of variable may be easily found by requiring, for example, that

$$\int\frac{dx}{e^{bx}-1}=\alpha\int\frac{du}{u}$$

where u is the new variable and α is some normalizing coefficient, which can be chosen later at our convenience. Other changes of variables, of course, are possible as well (see, e.g., numerous examples in exercises 39 & 45 [13, Sect. 4]), but this one is particularly successful, especially if we set $\alpha=1/b$. Thus, putting

$x = -\frac{1}{b}\ln(1-u)$ and rewriting the result for $z = \frac{b}{2\pi a}$, Binet's formula takes the form

$$-\int_0^1 \operatorname{arctg}\Big[\frac{1}{2\pi z}\ln(1-u)\Big]\frac{du}{u} \;=\; \pi\ln\Gamma(z) + \pi z(1-\ln z) + \frac{\pi}{2}\ln\frac{z}{2\pi} \tag{60}$$

where $\operatorname{Re} z > 0$. The integrand on the left may be expanded into the MacLaurin series in powers of u accordingly to the method described in Section 2. This yields

$$\begin{aligned}\operatorname{arctg}\Big[\frac{1}{2\pi z}\ln(1-u)\Big] \;&=\; \sum_{l=0}^{\infty}(-1)^l(2l)!\cdot\frac{\big[\frac{1}{2\pi z}\ln(1-u)\big]^{2l+1}}{(2l+1)!} = \\ &= -\sum_{l=0}^{\infty}(-1)^l\frac{(2l)!}{(2\pi z)^{2l+1}}\cdot\sum_{n=2l+1}^{\infty}\frac{|S_1(n,2l+1)|}{n!}\,u^n = \\ &= -\sum_{n=1}^{\infty}\frac{u^n}{n!}\cdot\sum_{l=0}^{\lfloor\frac{1}{2}n\rfloor}(-1)^l\frac{(2l)!\cdot|S_1(n,2l+1)|}{(2\pi z)^{2l+1}}\end{aligned} \tag{61}$$

This expansion converges in the disk $|u| < r$ in which $\operatorname{arctg}\big[\frac{1}{2\pi z}\ln(1-u)\big]$ is analytic. The radius of this disk r depends on the parameter z and is conditioned by the singularities of the arctangent, which occur at $u = 1 - \exp(\pm 2\pi i z)$ [branch points], and by that of the logarithm, which is located at $u = 1$ [branch point as well]. The latter restricts the value of r to 1, and the unit radius of convergence corresponds to such z that $2\cos(2\pi\operatorname{Re} z) = \exp(\pm 2\pi\operatorname{Im} z)$. The zone of convergence of series (61) for $|u| < 1$ consists, therefore, in the intersection of two zones, each of which lying to the right of curves

$$\operatorname{Im} z \;=\; \frac{1}{2\pi}\cdot\begin{cases} +\ln 2 + \ln\cos\big(2\pi\operatorname{Re} z\big) \\ -\ln 2 - \ln\cos\big(2\pi\operatorname{Re} z\big)\end{cases} \tag{62}$$

respectively, see Fig. 2. Now, a close study of the general term of series (61) reveals that it also converges for $u = 1$. Indeed, from (75), it follows that one can always find such a constant $C > 1$ that for sufficiently large n_0, inequality

$$\frac{1}{n!}\sum_{l=0}^{\lfloor\frac{1}{2}n\rfloor}(-1)^l\frac{(2l)!\cdot|S_1(n,2l+1)|}{(2\pi z)^{2l+1}} \;\leqslant\; \frac{2\pi z\,C}{n\ln^2 n}\,, \qquad n > n_0\,,$$

holds. Hence, since series $\sum n^{-1}\ln^{-\alpha}n$ converges for $\alpha > 1$, so does series (61) at $u = 1$. An interesting consequence of the latter statement is this curious identity

$$\frac{\pi}{2} \;=\; \sum_{n=1}^{\infty}\frac{1}{n!}\cdot\sum_{l=0}^{\lfloor\frac{1}{2}n\rfloor}(-1)^l\frac{(2l)!\cdot|S_1(n,2l+1)|}{(2\pi z)^{2l+1}}\,, \tag{63}$$

which holds in the region of convergence of z. Thus, expansion (61) converges uniformly in each point of the disk $|u| < 1$ and can be integrated term-by-term.[37] Substituting series (61) into (60) and performing

[37] Another way to show that (61) is uniformly convergent is to directly verify that

$$\int_0^1\left[\sum_{n=N}^{\infty}\frac{u^n}{n!}\sum_{l=0}^{\lfloor\frac{1}{2}n\rfloor}(-1)^l\frac{(2l)!\cdot|S_1(n,2l+1)|}{(2\pi z)^{2l+1}}\right]du \;\to\; 0 \qquad \text{as } N\to\infty$$

see e.g. [60, pp. 161–162].

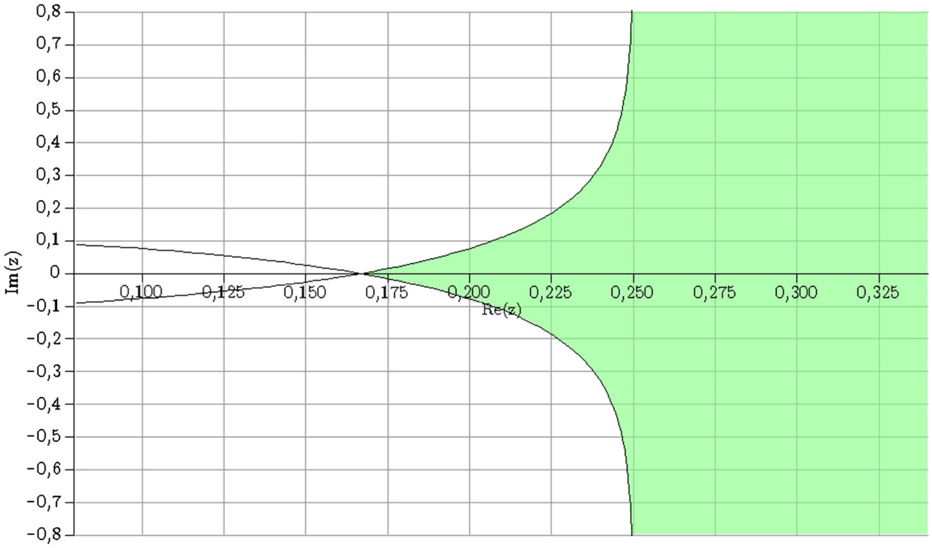

Fig. 2. The region of convergence of series (61) and (64) in the complex z-plane for $|u| < 1$ is the common part of two zones, each of which lying to the right of curves (62) [green zone]. Both curves start from the point $z = \frac{1}{6}$ and have vertical asymptotes in the complex z-plane at the line $\operatorname{Re} z = \frac{1}{4}$. The convergence of the series in the vertical strip $\frac{1}{6} < \operatorname{Re} z < \frac{1}{4}$ depends, therefore, on the imaginary part of z. On the contrary, in the half-plane $\operatorname{Re} z \geqslant \frac{1}{4}$ both series converge everywhere independently of the imaginary part of z. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

the indicated term-by-term integration from $u = 0$ to $u = 1$, we obtain the following series expansion for the logarithm of the Γ-function

$$\begin{aligned}\ln\Gamma(z) &= \left(z-\frac{1}{2}\right)\ln z - z + \frac{1}{2}\ln 2\pi + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n\cdot n!}\sum_{l=0}^{\lfloor\frac{1}{2}n\rfloor}(-1)^l\,\frac{(2l)!\cdot|S_1(n,2l+1)|}{(2\pi z)^{2l+1}} = \\ &= \left(z-\frac{1}{2}\right)\ln z - z + \frac{1}{2}\ln 2\pi + \frac{1}{\pi}\left\{\frac{1}{2\pi z}+\frac{1}{8\pi z}+\frac{1}{18}\left(\frac{1}{\pi z}-\frac{1}{4\pi^3 z^3}\right)+\frac{1}{32}\left(\frac{1}{\pi z}-\frac{1}{2\pi^3 z^3}\right)+\right. \\ &\qquad \left. +\frac{1}{600}\left(\frac{12}{\pi z}-\frac{35}{4\pi^3 z^3}+\frac{3}{4\pi^5 z^5}\right)+\frac{1}{4320}\left(\frac{60}{\pi z}-\frac{225}{4\pi^3 z^3}+\frac{45}{4\pi^5 z^5}\right)+\ldots\right\}\end{aligned} \tag{64}$$

converging in the same region as series (61), see (62) and Fig. 2. In particular, if z is real, it converges for $z > \frac{1}{6}$; on the contrary, if z is complex, then, independently of its imaginary part, it converges everywhere in the right half-plane $\operatorname{Re} z \geqslant \frac{1}{4}$. A quick analysis of the above series shows that for z rational multiple of π^{-1}, it contains rational coefficients only. Another important observation is that this series, unlike the classic Stirling series (1), cannot be explicitly written in powers of z. To illustrate this point, we write down its first 2, 3 and 4 terms respectively:

$$\begin{aligned}&\sum_{n=1}^{N}\frac{1}{n\cdot n!}\sum_{l=0}^{\lfloor\frac{1}{2}n\rfloor}(-1)^l\,\frac{(2l)!\cdot|S_1(n,2l+1)|}{(2\pi z)^{2l+1}} = \\ &= \begin{cases}\dfrac{1}{2\pi z}+\dfrac{1}{8\pi z} = \dfrac{5}{8\pi z}\,, & N=2\\[2mm] \dfrac{1}{2\pi z}+\dfrac{1}{8\pi z}+\dfrac{1}{18}\left(\dfrac{1}{\pi z}-\dfrac{1}{4\pi^3 z^3}\right) = \dfrac{49}{72\pi z}-\dfrac{1}{72\pi^3 z^3}\,, & N=3\\[2mm] \dfrac{1}{2\pi z}+\dfrac{1}{8\pi z}+\dfrac{1}{18}\left(\dfrac{1}{\pi z}-\dfrac{1}{4\pi^3 z^3}\right)+\dfrac{1}{32}\left(\dfrac{1}{\pi z}-\dfrac{1}{2\pi^3 z^3}\right) = \dfrac{205}{288\pi z}-\dfrac{17}{576\pi^3 z^3}\,, & N=4\end{cases}\end{aligned}$$

By the way, as concerns the divergent Stirling series (1), it can be readily derived from (64). By formally interchanging sum signs in (64), which is obviously not permitted because series are not absolutely convergent, we have

$$\sum_{n=1}^{\infty}\frac{1}{n\cdot n!}\sum_{l=0}^{\lfloor\frac{1}{2}n\rfloor}(-1)^l\frac{(2l)!\cdot|S_1(n,2l+1)|}{(2\pi z)^{2l+1}} \asymp \sum_{l=0}^{\infty}(-1)^l\frac{(2l)!}{(2\pi z)^{2l+1}}\underbrace{\sum_{n=1}^{\infty}\frac{|S_1(n,2l+1)|}{n\cdot n!}}_{\zeta(2l+2)} =$$

$$= \sum_{l=1}^{\infty}(-1)^{l-1}\frac{(2l-2)!}{(2\pi z)^{2l-1}}\cdot\zeta(2l) = \sum_{l=1}^{\infty}\frac{\pi\cdot B_{2l}}{2l(2l-1)z^{2l-1}}$$

where we first used (33) for $\zeta(2l+2)$, and then, Euler's formula

$$\zeta(2l) = (-1)^{l+1}\frac{(2\pi)^{2l}\cdot B_{2l}}{2\cdot(2l)!} = \frac{(2\pi)^{2l}\cdot|B_{2l}|}{2\cdot(2l)!}\,, \qquad l=1,2,3,\ldots$$

Further observations concern the convergence of the derived series and are treated in details in the next section.

3.1.2. Convergence analysis of the derived series

The complete study of the convergence of (64) is quite long and complicated, that is why we split it in two stages. First, we obtain the upper bound for the general term of (64), and then, derive an accurate approximation for it when n becomes sufficiently large. In what follows, we may suppose, without essential loss of generality, that z is real and positive. The general term of series (64) is given by the finite sum over l. This truncated sum has only odd terms, and hence, by elementary transformations, may be reduced to that containing both odd and even terms

$$\begin{aligned}\sum_{l=0}^{\lfloor\frac{1}{2}n\rfloor}(-1)^l\frac{(2l)!\cdot|S_1(n,2l+1)|}{(2\pi z)^{2l+1}} &= \sum_{l=0}^{\lfloor\frac{1}{2}n\rfloor}(-1)^{\frac{1}{2}(2l+1)-\frac{1}{2}}\frac{(2l+1)!\cdot|S_1(n,2l+1)|}{(2l+1)\cdot(2\pi z)^{2l+1}} = \\ &= \frac{1}{2}\sum_{l=1}^{n}\left[1-(-1)^l\right]\cdot(-1)^{\frac{1}{2}(l-1)}\cdot\frac{(l-1)!\cdot|S_1(n,l)|}{(2\pi z)^l} = \ldots\end{aligned} \tag{65}$$

Now, from Legendre's integral for the Euler Γ-function,[38] it follows that

$$\begin{cases}(-1)^{\frac{1}{2}(l-1)}\cdot\dfrac{(l-1)!}{(2\pi z)^l} = -i\displaystyle\int_0^{\infty}\left[\frac{i\,x}{2\pi z}\right]^l\cdot\frac{e^{-x}\,dx}{x}\\[2ex] (-1)^l\cdot(-1)^{\frac{1}{2}(l-1)}\cdot\dfrac{(l-1)!}{(2\pi z)^l} = -i\displaystyle\int_0^{\infty}\left[-\frac{i\,x}{2\pi z}\right]^l\cdot\frac{e^{-x}\,dx}{x}\end{cases}$$

Hence, expression (65) may be continued as follows

$$\ldots = \frac{i}{2}\int_0^{\infty}\sum_{l=1}^{n}\left\{\left[-\frac{i\,x}{2\pi z}\right]^l - \left[\frac{i\,x}{2\pi z}\right]^l\right\}|S_1(n,l)|\cdot\frac{e^{-x}\,dx}{x} =$$

[38] Namely, $(l+1)! = \int x^{l+1}e^{-x}dx$ taken over $x=[0,\infty)$.

$$
\begin{aligned}
&= \frac{i}{2}\int\limits_0^\infty \left\{\left(-\frac{i\,x}{2\pi z}\right)_n - \left(\frac{i\,x}{2\pi z}\right)_n\right\} \frac{e^{-x}\,dx}{x} = \\
&= \frac{i}{4\pi^2 z}\int\limits_0^\infty \operatorname{sh}\frac{x}{2z}\left\{\Gamma\!\left(\frac{i\,x}{2\pi z}\right)\Gamma\!\left(n-\frac{i\,x}{2\pi z}\right) - \Gamma\!\left(-\frac{i\,x}{2\pi z}\right)\Gamma\!\left(n+\frac{i\,x}{2\pi z}\right)\right\} e^{-x}\,dx = \\
&= -\frac{1}{2\pi^2 z}\int\limits_0^\infty \operatorname{sh}\frac{x}{2z}\cdot e^{-x}\cdot \operatorname{Im}\left[\Gamma\!\left(\frac{i\,x}{2\pi z}\right)\Gamma\!\left(n-\frac{i\,x}{2\pi z}\right)\right]dx
\end{aligned}
\tag{66}
$$

where at the final stage we, first, replaced Pochhammer symbols by Γ-functions, and then, used the well-known relationship $\Gamma(z)\Gamma(-z) = -(\pi/z)\csc\pi z$. The last integral in (66) is difficult to evaluate in a closed form, but its upper bound may be readily obtained. In view of the fact that $|\operatorname{Im}\Gamma(v)| \leqslant |\Gamma(v)| \leqslant |\Gamma(\operatorname{Re} v)|$, we have

$$
\begin{aligned}
&\frac{1}{2\pi^2 z}\left|\int\limits_0^\infty \operatorname{sh}\frac{x}{2z}\cdot e^{-x}\cdot \operatorname{Im}\left[\Gamma\!\left(\frac{i\,x}{2\pi z}\right)\Gamma\!\left(n-\frac{i\,x}{2\pi z}\right)\right]dx\right| \leqslant \\
&\qquad \leqslant \frac{\Gamma(n)}{2\pi^2 z}\int\limits_0^\infty \operatorname{sh}\frac{x}{2z}\cdot e^{-x}\cdot\left|\Gamma\!\left(\frac{i\,x}{2\pi z}\right)\right|dx = \frac{(n-1)!}{\pi\sqrt{2z}}\int\limits_0^\infty e^{-x}\sqrt{\operatorname{sh}\frac{x}{2z}}\cdot\frac{dx}{\sqrt{x}}
\end{aligned}
\tag{67}
$$

Whence, by making a change of variable in the latter integral $x = 2zt$, we have for any positive integer n (not necessarily large)

$$
\frac{1}{n\cdot n!}\left|\sum_{l=0}^{\lfloor\frac{1}{2}n\rfloor}(-1)^l\frac{(2l)!\cdot|S_1(n,2l+1)|}{(2\pi z)^{2l+1}}\right| \leqslant \frac{1}{n^2}\cdot\frac{1}{\pi}\int\limits_0^\infty\sqrt{\frac{\operatorname{sh}t}{t}}\cdot e^{-2zt}\,dt \tag{68}
$$

where the latter integral converges uniformly in the half-plane z which lies to the right of the line $\operatorname{Re} z = \frac{1}{4}$ (imaginary part of z contributes only to the bounded oscillations of the integrand). Consequently, series (64) converges at least in $\operatorname{Re} z > \frac{1}{4}$, and this at the same rate or better than Euler's series $\sum n^{-2}$.

Numerical simulations show, however, that the greater n, the greater the relative difference between the upper bound and the left-hand side in (68), see Fig. 3, and thus, this upper bound is relatively rough.[39] A more accurate description of the behavior of sum (65) at large n may be obtained by seeking its asymptotics. In order to find it, we proceed as follows. We first rewrite the second line of (66) as

$$
\frac{i}{2}\int\limits_0^\infty \left\{\left(-\frac{i\,x}{2\pi z}\right)_n - \left(\frac{i\,x}{2\pi z}\right)_n\right\} \frac{e^{-x}\,dx}{x} = \int\limits_0^\infty \operatorname{Im}\left[\left(\frac{i\,x}{2\pi z}\right)_n\right]\frac{e^{-x}\,dx}{x} \tag{69}
$$

Now, it is well-known that the function $1/\Gamma(z)$ is regular on the entire complex z-plane, and therefore, may be expanded into the MacLaurin series

$$
\frac{1}{\Gamma(z)} = z + \gamma z^2 + \left(\frac{\gamma^2}{2} - \frac{\pi^2}{12}\right)z^3 + \ldots \equiv \sum_{k=1}^\infty z^k a_k\,, \qquad |z| < \infty\,, \tag{70}
$$

[39] The error is mainly due to the use of inequality $|\operatorname{Im}\Gamma(v)| \leqslant |\Gamma(v)|$.

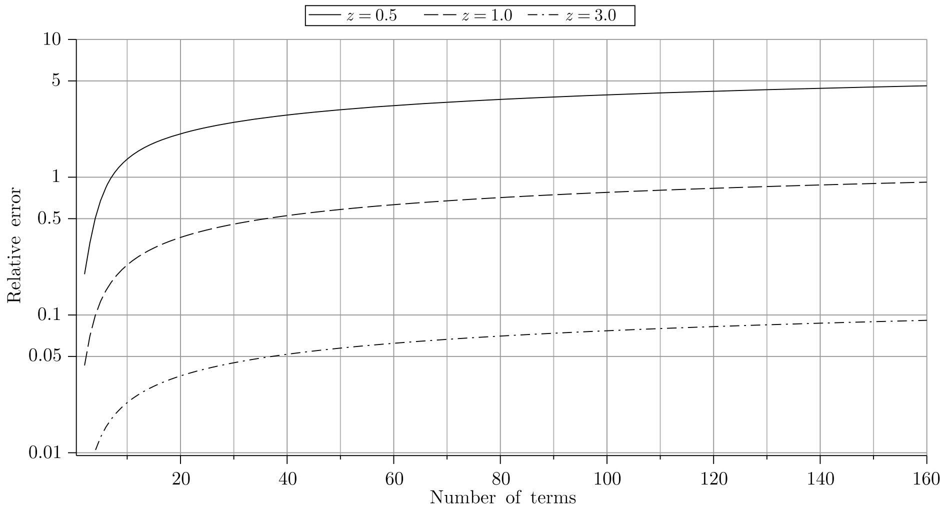

Fig. 3. Relative error between the upper bound and the left-hand side in (68) as a function of n for three different values of argument z, logarithmic scale.

where

$$a_k \equiv \frac{1}{k!}\cdot\left[\frac{1}{\Gamma(z)}\right]^{(k)}_{z=0} = \frac{(-1)^k}{\pi\, k!}\cdot\Big[\sin\pi x\cdot\Gamma(x)\Big]^{(k)}_{x=1} \tag{71}$$

the last representation for coefficients a_k, which follows from the reflection formula for the Γ-function, being often more suitable for computational purposes.[40] Using approximation (10) for the Pochhammer symbol and the above MacLaurin series for $1/\Gamma(z)$, we have for sufficiently large n

$$\begin{aligned}\operatorname{Im}\left[\left(\frac{i\,x}{2\pi z}\right)_n\right] &\sim \operatorname{Im}\left[\frac{n^{\frac{ix}{2\pi z}}\cdot\Gamma(n)}{\Gamma\left(\frac{ix}{2\pi z}\right)}\right] = (n-1)!\operatorname{Im}\left[\left(\cos\frac{x\ln n}{2\pi z}+i\sin\frac{x\ln n}{2\pi z}\right)\cdot\sum_{k=1}^{\infty}a_k\cdot\left(\frac{i\,x}{2\pi z}\right)^k\right] = \\ &= (n-1)!\left[\cos\frac{x\ln n}{2\pi z}\cdot\sum_{k=0}^{\infty}(-1)^k a_{2k+1}\left(\frac{x}{2\pi z}\right)^{2k+1}+\sin\frac{x\ln n}{2\pi z}\cdot\sum_{k=1}^{\infty}(-1)^k a_{2k}\left(\frac{x}{2\pi z}\right)^{2k}\right]\end{aligned} \tag{72}$$

the error due to considering only the first term in (10) being negligible with respect to logarithmic terms which will appear later. Inserting this expression into (69), performing the term-by-term integration and taking into account that[41]

$$\begin{cases}\displaystyle\int_0^\infty x^{s-1}e^{-zx}\cos ux\,dx = \frac{\Gamma(s)}{\left(z^2+u^2\right)^{s/2}}\cdot\cos\left[s\operatorname{arctg}\frac{u}{z}\right]\\[2ex] \displaystyle\int_0^\infty x^{s-1}e^{-zx}\sin ux\,dx = \frac{\Gamma(s)}{\left(z^2+u^2\right)^{s/2}}\cdot\sin\left[s\operatorname{arctg}\frac{u}{z}\right]\end{cases} \tag{73}$$

yield

[40] On the computation of a_k, see also [1, p. 256, n° 6.1.34], [158, pp. 344 & 349], [65].

[41] These equalities are valid wherever the integrals on the left converge, see e.g. [28, p. 130], [96, p. 12], [85].

$$\int\limits_0^\infty \operatorname{Im}\left[\left(\frac{i\,x}{2\pi z}\right)_n\right]\frac{e^{-x}\,dx}{x} \sim (n-1)!\sum_{k=0}^\infty (-1)^k a_{2k+1}\frac{(2k)!}{(2\pi z)^{2k+1}}\cdot\frac{\cos\left[(2k+1)\operatorname{arctg}\dfrac{\ln n}{2\pi z}\right]}{\left[1+\dfrac{\ln^2 n}{4\pi^2 z^2}\right]^{k+\frac12}}+$$
$$+(n-1)!\sum_{k=1}^\infty (-1)^k a_{2k}\frac{(2k-1)!}{(2\pi z)^{2k}}\cdot\frac{\sin\left[2k\operatorname{arctg}\dfrac{\ln n}{2\pi z}\right]}{\left[1+\dfrac{\ln^2 n}{4\pi^2 z^2}\right]^{k}}$$

whence, the required asymptotics is

$$\begin{aligned}\frac{1}{n\cdot n!}\sum_{l=0}^{\lfloor\frac12 n\rfloor}(-1)^l\frac{(2l)!\cdot|S_1(n,2l+1)|}{(2\pi z)^{2l+1}} &= \\ &=\frac{1}{n^2}\sum_{k=0}^\infty\frac{(-1)^k}{(2k+1)\big(4\pi^2z^2+\ln^2 n\big)^{k+\frac12}}\cdot\cos\left[(2k+1)\operatorname{arctg}\frac{\ln n}{2\pi z}\right]\cdot\left[\frac{1}{\Gamma(x)}\right]^{(2k+1)}_{x=0}+ \\ &+\frac{1}{n^2}\sum_{k=1}^\infty\frac{(-1)^k}{2k\big(4\pi^2z^2+\ln^2 n\big)^{k}}\cdot\sin\left[2k\operatorname{arctg}\frac{\ln n}{2\pi z}\right]\cdot\left[\frac{1}{\Gamma(x)}\right]^{(2k)}_{x=0}+O\left(\frac{1}{n^3}\right)\end{aligned}\tag{74}$$

for sufficiently large n. Retaining first few terms, we have

$$\begin{aligned}\frac{1}{n\cdot n!}\sum_{l=0}^{\lfloor\frac12 n\rfloor}(-1)^l\frac{(2l)!\cdot|S_1(n,2l+1)|}{(2\pi z)^{2l+1}} &= \frac{2\pi z}{n^2}\left\{\frac{1}{4\pi^2z^2+\ln^2 n}-\frac{2\gamma\ln n}{\big(4\pi^2z^2+\ln^2 n\big)^2}\right\}+ \\ &+O\left(\frac{1}{n^2\ln^4 n}\right), \qquad n\to\infty\end{aligned}\tag{75}$$

Thus, for moderate values of z, series (64) converges approximately at the same rate as $\sum(n\ln n)^{-2}$, i.e. at the same rate as, for example, Fontana–Mascheroni's series (9), (40), see asymptotics (18).

3.1.3. Some important particular cases of the derived series

Let's now consider some applications of the formula (64). In the first instance, it is natural to obtain a series expansion for

$$\begin{aligned}\ln\Gamma\left(\frac{1}{\pi}\right) &= \left(1-\frac{1}{\pi}\right)\cdot\ln\pi-\frac{1}{\pi}+\frac12\ln 2+\frac{1}{2\pi}\sum_{n=1}^\infty\frac{1}{n\cdot n!}\cdot\left\{\sum_{l=0}^{\lfloor\frac12 n\rfloor}(-1)^l\frac{(2l)!\cdot|S_1(n,2l+1)|}{2^{2l}}\right\}= \\ &=\left(1-\frac{1}{\pi}\right)\cdot\ln\pi-\frac{1}{\pi}+\frac12\ln 2+\frac{1}{2\pi}\left\{1+\frac14+\frac{1}{12}+\frac{1}{32}+\frac{1}{75}+\frac{1}{144}+\frac{13}{2880}+\frac{157}{46\,080}+\ldots\right\}\end{aligned}\tag{76}$$

The graphical illustration of the convergence of this series is given in Fig. 4. With equal ease, we derive

$$\begin{aligned}\ln\Gamma\left(\frac{2}{\pi}\right) &= \left(1-\frac{2}{\pi}\right)\cdot\ln\pi-\frac{2}{\pi}+\frac{2}{\pi}\ln 2+\frac{1}{4\pi}\sum_{n=1}^\infty\frac{1}{n\cdot n!}\cdot\left\{\sum_{l=0}^{\lfloor\frac12 n\rfloor}(-1)^l\frac{(2l)!\cdot|S_1(n,2l+1)|}{2^{4l}}\right\}= \\ &=\left(1-\frac{2}{\pi}\right)\cdot\ln\pi-\frac{2}{\pi}+\frac{2}{\pi}\ln 2+\end{aligned}\tag{77}$$

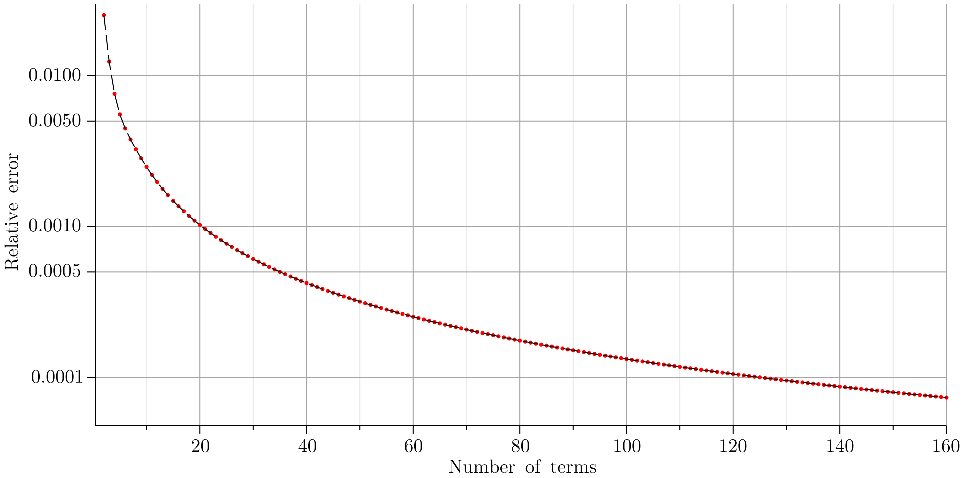

Fig. 4. Relative error of the series expansion for $\ln\Gamma(\pi^{-1})$ given by (76), logarithmic scale.

$$+\frac{1}{4\pi}\left\{1+\frac{1}{4}+\frac{5}{48}+\frac{7}{128}+\frac{631}{19\,200}+\frac{199}{9216}+\frac{19\,501}{1\,290\,240}+\frac{32\,707}{2\,949\,120}+\ldots\right\}$$

Many other similar expansions may be derived analogously. Let's now see how the series behaves outside the region of convergence. For this aim, we take $z=\frac{1}{2}\pi^{-1}$. Formula (64) yields

$$\begin{aligned}\ln\Gamma\left(\frac{1}{2\pi}\right) &\stackrel{?}{=} \left(1-\frac{1}{2\pi}\right)\cdot\ln 2\pi-\frac{1}{2\pi}+\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n\cdot n!}\cdot\left\{\sum_{l=0}^{\lfloor\frac{1}{2}n\rfloor}(-1)^l(2l)!\cdot|S_1(n,2l+1)|\right\}= \\ &=\left(1-\frac{1}{2\pi}\right)\cdot\ln 2\pi-\frac{1}{2\pi}+\frac{1}{\pi}\left\{1+\frac{1}{4}-\frac{1}{16}-\frac{11}{300}+\frac{1}{144}+\frac{17}{630}+\frac{101}{5760}-\frac{311}{102\,060}-\ldots\right\}\end{aligned} \tag{78}$$

At first sight, it might seem that this alternating series slowly converges to $\ln\Gamma(\frac{1}{2}\pi^{-1})\approx 1.765383194$: the summation of its first 3 terms gives the value $1.764207893\ldots$ which corresponds to the relative accuracy 6.6×10^{-4}, that of 18 terms gives $1.765525087\ldots$, i.e. the relative accuracy 8.0×10^{-5}, the summation of first 32 terms yields $1.765392783\ldots$ which corresponds to the relative error 5.4×10^{-6}.[42] Notwithstanding, further numerical simulations, see Fig. 5, leave no doubts: this series is divergent.

3.2. Second series expansion for the logarithm of the Γ-function

Rewrite formula (64) for $2z$ instead of z, and subtract the result from (64). In virtue of Legendre's duplication formula for the Γ-function $\ln\Gamma(2z)=(2z-1)\ln 2-\frac{1}{2}\ln\pi+\ln\Gamma(z)+\ln\Gamma(z+\frac{1}{2})$, we have

$$\begin{aligned}\ln\Gamma\left(\frac{1}{2}+z\right) &= z\ln z-z+\frac{1}{2}\ln 2\pi-\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n\cdot n!}\sum_{l=0}^{\lfloor\frac{1}{2}n\rfloor}(-1)^l\frac{(2l)!\cdot(2^{2l+1}-1)\cdot|S_1(n,2l+1)|}{(4\pi z)^{2l+1}}= \\ &= z\ln z-z+\frac{1}{2}\ln 2\pi-\frac{1}{\pi}\left\{\frac{1}{4\pi z}+\frac{1}{16\pi z}+\frac{1}{18}\left(\frac{1}{2\pi z}-\frac{7}{32\pi^3z^3}\right)+\frac{1}{32}\left(\frac{1}{2\pi z}-\frac{7}{16\pi^3z^3}\right)+\right. \\ &\quad\left.+\frac{1}{600}\left(\frac{6}{\pi z}-\frac{245}{32\pi^3z^3}+\frac{93}{128\pi^5z^5}\right)+\frac{1}{4320}\left(\frac{30}{\pi z}-\frac{1575}{32\pi^3z^3}+\frac{1395}{128\pi^5z^5}\right)+\ldots\right\}\end{aligned} \tag{79}$$

[42] We do not count the third term which is zero.

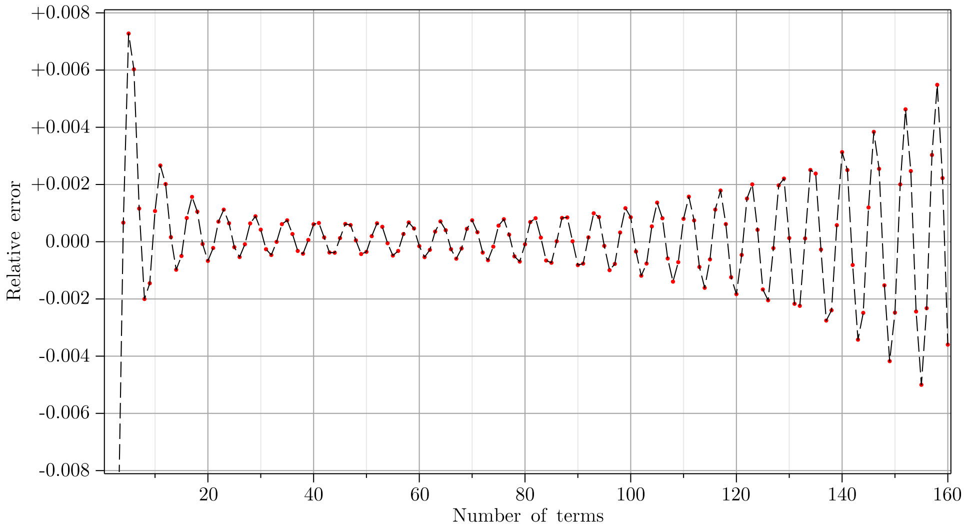

Fig. 5. Relative error of the series expansion for $\ln\Gamma(\frac{1}{2}\pi^{-1})$ given by (78), linear scale.

which holds in the green zone shown in Fig. 2. This expression allows to expand any value of the form $\ln\Gamma(\frac{1}{2}+\alpha\pi^{-1})$ into the series with rational coefficients if α is rational greater than $\frac{1}{6}\pi$. For example, putting $z=\pi^{-1}$, we have

$$\ln\Gamma\left(\frac{1}{2}+\frac{1}{\pi}\right) = -\frac{1+\ln\pi}{\pi}+\frac{1}{2}\ln 2\pi-\frac{1}{4\pi}\sum_{n=1}^{\infty}\frac{1}{n\cdot n!}\sum_{l=0}^{\lfloor\frac{1}{2}n\rfloor}(-1)^l\frac{(2l)!\cdot(2^{2l+1}-1)\cdot|S_1(n,2l+1)|}{2^{4l}} =$$
$$= -\frac{1+\ln\pi}{\pi}+\frac{1}{2}\ln 2\pi-\frac{1}{4\pi}\left\{1+\frac{1}{4}+\frac{1}{16}+\frac{1}{128}-\frac{119}{19\,200}-\frac{71}{9216}-\frac{7853}{1\,290\,240}-\frac{12\,611}{2\,949\,120}-\ldots\right\}$$

Furthermore, both series expansions (64) and (79), used together with the reflection formula and the recurrence relationship for the Γ-function, yield series with rational coefficients for any values of the form $\ln\Gamma(\frac{1}{2}n\pm\alpha\pi^{-1})$, where n is integer.

As a final remark, we note that expression (79), written for z instead of $\frac{1}{2}+z$, straightforwardly produces another series expansion for the logarithm of the Γ-function

$$\ln\Gamma(z) = \left(z-\frac{1}{2}\right)\ln\left(z-\frac{1}{2}\right)-z+\frac{1}{2}+\frac{1}{2}\ln 2\pi -$$
$$-\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n\cdot n!}\sum_{l=0}^{\lfloor\frac{1}{2}n\rfloor}(-1)^l\frac{(2l)!\cdot(2^{2l+1}-1)\cdot|S_1(n,2l+1)|}{(4\pi)^{2l+1}\cdot\left(z-\frac{1}{2}\right)^{2l+1}} \tag{80}$$

which converges in the green zone given in Fig. 2 shifted by $\frac{1}{2}$ to the right. In particular, if z is real, it converges for any $z>\frac{2}{3}$.

Remark. Expansion (79) may be also derived if we replace in (59) Binet's formula by its analog with "conjugated" denominator

$$\int_0^\infty\frac{\operatorname{arctg} ax}{e^{bx}+1}\,dx = -\frac{\pi}{b}\ln\Gamma\left(\frac{1}{2}+\frac{b}{2\pi a}\right)-\frac{1}{2a}\left(1+\ln\frac{2\pi a}{b}\right)+\frac{\pi}{2b}\ln 2\pi$$

where $a>0$ and $\operatorname{Re} b>0$, see [13, Sect. 4, exercise n° 40-a], or if we replace it by the following formula

$$\int\limits_0^\infty \frac{\operatorname{arctg} ax}{\operatorname{sh} bx}\, dx \;=\; \frac{\pi}{b}\left\{\ln\Gamma\!\left(\frac{b}{2\pi a}\right) - \ln\Gamma\!\left(\frac{1}{2}+\frac{b}{2\pi a}\right) - \frac{1}{2}\ln\frac{2\pi a}{b}\right\}$$

derived in [13, Sect. 4, exercise n° 39-e]. Making a change of variable $x = -\frac{2}{b}\operatorname{arcth} u$, and then, proceeding analogously to (60)–(64), yields

$$\ln\Gamma(z) - \ln\Gamma\!\left(\frac{1}{2}+z\right) \;=\; -\frac{1}{2}\ln z + \frac{1}{\pi}\sum_{n=0}^{\infty}\frac{1}{2n+1}\sum_{k=1}^{2n+1}\frac{2^k}{k!}\binom{2n}{k-1}\sum_{l=0}^{n}(-1)^l\frac{(2l)!\cdot S_1(k,2l+1)}{(2\pi z)^{2l+1}} \tag{81}$$

which, being combined with (79), leads to a rearranged version of (64).

3.3. Series expansion for the polygamma functions

By differentiating expressions (64) and (80), one may easily deduce similar series expansions for the polygamma functions. Differentiating the former expansion yields the following series representations for the digamma and trigamma functions

$$\begin{aligned}\Psi(z) \;&=\; \ln z - \frac{1}{2z} - \frac{1}{\pi z}\sum_{n=1}^{\infty}\frac{1}{n\cdot n!}\sum_{l=0}^{\lfloor\frac{1}{2}n\rfloor}(-1)^l\frac{(2l+1)!\cdot|S_1(n,2l+1)|}{(2\pi z)^{2l+1}} = \\ &= \ln z - \frac{1}{2z} - \frac{1}{\pi z}\left\{\frac{1}{2\pi z}+\frac{1}{8\pi z}+\frac{1}{18}\left(\frac{1}{\pi z}-\frac{3}{4\pi^3z^3}\right)+\frac{1}{32}\left(\frac{1}{\pi z}-\frac{3}{2\pi^3z^3}\right)+\right. \\ &\quad\left.+\frac{1}{600}\left(\frac{12}{\pi z}-\frac{105}{4\pi^3z^3}+\frac{15}{4\pi^5z^5}\right)+\frac{1}{4320}\left(\frac{60}{\pi z}-\frac{675}{4\pi^3z^3}+\frac{225}{4\pi^5z^5}\right)+\ldots\right\}\end{aligned} \tag{82}$$

and

$$\begin{aligned}\Psi_1(z) \;&=\; \frac{1}{2z^2}+\frac{1}{z}+\frac{1}{\pi z^2}\sum_{n=1}^{\infty}\frac{1}{n\cdot n!}\sum_{l=0}^{\lfloor\frac{1}{2}n\rfloor}(-1)^l\frac{(2l+2)!\cdot|S_1(n,2l+1)|}{(2\pi z)^{2l+1}} = \\ &= \frac{1}{2z^2}+\frac{1}{z}+\frac{1}{\pi z^2}\left\{\frac{1}{\pi z}+\frac{1}{4\pi z}+\frac{1}{18}\left(\frac{2}{\pi z}-\frac{3}{\pi^3z^3}\right)+\frac{1}{16}\left(\frac{1}{\pi z}-\frac{3}{\pi^3z^3}\right)+\right. \\ &\quad\left.+\frac{1}{600}\left(\frac{24}{\pi z}-\frac{105}{\pi^3z^3}+\frac{45}{2\pi^5z^5}\right)+\frac{1}{4320}\left(\frac{120}{\pi z}-\frac{675}{\pi^3z^3}+\frac{675}{2\pi^5z^5}\right)+\ldots\right\}\end{aligned} \tag{83}$$

respectively. More generally, by differentiating k times with respect to z the above series for $\Psi(z)$, we obtain a series expansion for the kth polygamma function

$$\begin{aligned}\Psi_k(z) \;&=\; (-1)^{k+1}\frac{k!}{2z^{k+1}}+(-1)^{k+1}\frac{(k-1)!}{z^k}+\frac{(-1)^{k+1}}{\pi z^{k+1}}\sum_{n=1}^{\infty}\frac{1}{n\cdot n!}\sum_{l=0}^{\lfloor\frac{1}{2}n\rfloor}(-1)^l\frac{(2l+k+1)!\cdot|S_1(n,2l+1)|}{(2\pi z)^{2l+1}} = \\ &= (-1)^{k+1}\frac{k!}{2z^{k+1}}+(-1)^{k+1}\frac{(k-1)!}{z^k}+\frac{(-1)^{k+1}}{\pi z^{k+1}}\left\{\frac{(k+1)!}{2\pi z}+\frac{(k+1)!}{8\pi z}+\frac{1}{18}\left[\frac{(k+1)!}{\pi z}-\frac{(k+3)!}{8\pi^3z^3}\right]+\right. \\ &\quad\left.+\frac{1}{32}\left[\frac{(k+1)!}{\pi z}-\frac{(k+3)!}{4\pi^3z^3}\right]+\frac{1}{600}\left[\frac{12\,(k+1)!}{\pi z}-\frac{35\,(k+3)!}{8\pi^3z^3}+\frac{(k+5)!}{32\pi^5z^5}\right]+\ldots\right\}\end{aligned} \tag{84}$$

where $k = 1, 2, 3, \ldots$. Convergence analysis of these series is analogous to that performed in Section 3.1.2, and we omit the details because the calculations are a little bit long. This analysis reveals that the general term of these series may be always bounded by $\alpha_k(z)n^{-2}$, where $\alpha_k(z)$ depends solely on z and on the order of the polygamma function k. At large n, the general term of these series is of the same order as $\big(n\ln^m n\big)^{-2}$, where $m=1$ for $\Psi(z)$, $m=2$ for $\Psi_1(z)$ and $\Psi_2(z)$, $m=3$ for $\Psi_3(z)$ and $\Psi_4(z)$, and so on.

We now give several particular cases of the above expansions. From (82)–(84), it follows that at π^{-1}, the polygamma functions have the following series representations

$$\begin{aligned}\Psi\left(\frac{1}{\pi}\right) &= -\ln\pi-\frac{\pi}{2}-\sum_{n=1}^{\infty}\frac{1}{n\cdot n!}\sum_{l=0}^{\lfloor\frac{1}{2}n\rfloor}(-1)^l\,\frac{(2l+1)!\cdot|S_1(n,2l+1)|}{2^{2l+1}}=\\ &=-\ln\pi-\frac{\pi}{2}-\frac{1}{2}-\frac{1}{8}-\frac{1}{72}+\frac{1}{64}+\frac{7}{400}+\frac{7}{576}+\frac{643}{94\,080}+\frac{103}{30\,720}+\ldots\end{aligned} \tag{85}$$

$$\begin{aligned}\Psi_1\left(\frac{1}{\pi}\right) &= \frac{\pi^2}{2}+\pi+\pi\sum_{n=1}^{\infty}\frac{1}{n\cdot n!}\sum_{l=0}^{\lfloor\frac{1}{2}n\rfloor}(-1)^l\,\frac{(2l+2)!\cdot|S_1(n,2l+1)|}{2^{2l+1}}=\\ &=\frac{\pi^2}{2}+\pi+\pi\left\{1+\frac{1}{4}-\frac{1}{18}-\frac{1}{8}-\frac{39}{400}-\frac{29}{576}-\frac{353}{23\,520}+\frac{11}{3840}+\ldots\right\}\end{aligned} \tag{86}$$

and, more generally, for $k=1,2,3,\ldots$

$$\begin{aligned}\Psi_k\left(\frac{1}{\pi}\right) &= (-1)^{k+1}\pi^k\cdot\left\{\frac{\pi\,k!}{2}+(k-1)!+\sum_{n=1}^{\infty}\frac{1}{n\cdot n!}\sum_{l=0}^{\lfloor\frac{1}{2}n\rfloor}(-1)^l\,\frac{(2l+k+1)!\cdot|S_1(n,2l+1)|}{2^{2l+1}}\right\}=\\ &=(-1)^{k+1}\pi^k\cdot\left\{\frac{\pi\,k!}{2}+(k-1)!+\frac{(1+k)!}{2}+\frac{(1+k)!}{8}+\frac{1}{18}\left[(k+1)!-\frac{(k+3)!}{8}\right]+\right.\\ &\quad\left.+\frac{1}{32}\left[(k+1)!-\frac{(k+3)!}{4}\right]+\frac{1}{600}\left[12\,(k+1)!-\frac{35\,(k+3)!}{8}+\frac{(k+5)!}{32}\right]+\ldots\right\}\end{aligned} \tag{87}$$

Fig. 6 shows the rate of convergence of first two series.

Second variant of the series expansions for the polygamma functions follows from (80). Differentiating the latter with respect to z yields

$$\Psi(z)=\ln\left(z-\frac{1}{2}\right)+\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n\cdot n!}\sum_{l=0}^{\lfloor\frac{1}{2}n\rfloor}(-1)^l\,\frac{(2l+1)!\cdot(2^{2l+1}-1)\cdot|S_1(n,2l+1)|}{(4\pi)^{2l+1}\cdot\left(z-\frac{1}{2}\right)^{2l+2}} \tag{88}$$

and

$$\Psi_k(z)=\frac{(-1)^{k+1}(k-1)!}{\left(z-\frac{1}{2}\right)^k}+\frac{(-1)^k}{\pi}\sum_{n=1}^{\infty}\frac{1}{n\cdot n!}\sum_{l=0}^{\lfloor\frac{1}{2}n\rfloor}(-1)^l\,\frac{(2l+k+1)!\cdot(2^{2l+1}-1)\cdot|S_1(n,2l+1)|}{(4\pi)^{2l+1}\cdot\left(z-\frac{1}{2}\right)^{2l+k+2}} \tag{89}$$

In particular,

$$\begin{aligned}\Psi\left(\frac{1}{2}+\frac{1}{\pi}\right) &= -\ln\pi+\sum_{n=1}^{\infty}\frac{1}{n\cdot n!}\sum_{l=0}^{\lfloor\frac{1}{2}n\rfloor}(-1)^l\,\frac{(2l+1)!\cdot(2^{2l+1}-1)\cdot|S_1(n,2l+1)|}{2^{4l+2}}=\\ &=-\ln\pi+\frac{1}{4}+\frac{1}{16}-\frac{5}{576}-\frac{13}{512}-\frac{569}{25\,600}-\frac{539}{36\,864}-\frac{98\,671}{12\,042\,240}-\ldots\end{aligned} \tag{90}$$

Similarly to expansions for $\ln\Gamma(z)$, expansions (84) and (89), combined with the reflection formula and the recurrence relationship for polygamma functions, give series with rational coefficients for any polygamma function of the argument $\frac{1}{2}n\pm\alpha\pi^{-1}$, where α is rational greater than $\frac{1}{6}\pi$ and n is integer.

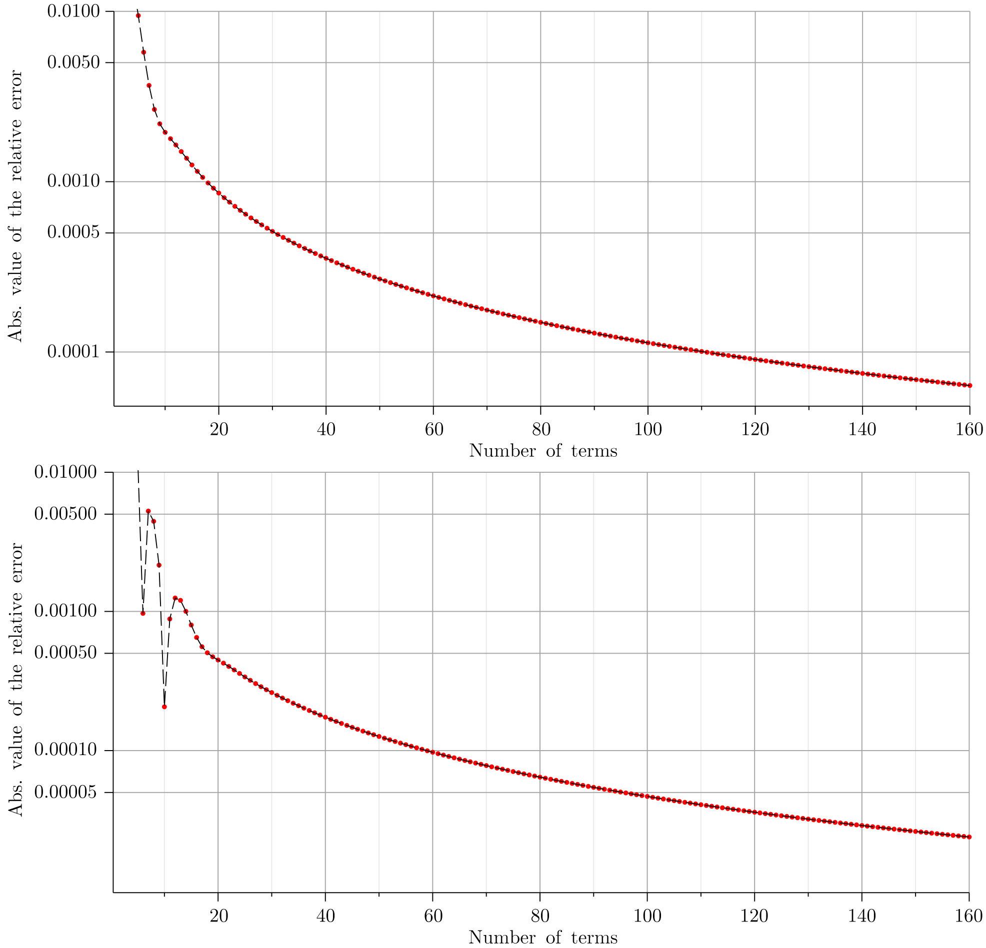

Fig. 6. *Top:* Relative error of the series expansion for $\Psi(\pi^{-1})$ given by (85). *Bottom:* Relative error of the series expansion for $\Psi_1(\pi^{-1})$ given by (86). For better visibility, both errors are presented in absolute values and logarithmic scales.

Final remark. Series which we discovered in the present work are very interesting especially because of the implication of combinatorial numbers $S_1(n,l)$. In this context, it seems appropriate to note that series of a similar nature for $\ln\Gamma(z)$ and $\Psi_k(z)$ were already obtained, but remain little-known and practically not mentioned in modern literature. For instance, in 1839, Jacques Binet [12, pp. 231–236, 257, 237, 235, 335–339, Eqs. (63), (81)] obtained several, rapidly convergent for large $|z|$, expansions

$$\begin{aligned}
\ln\Gamma(z) &= \left(z-\frac{1}{2}\right)\ln z - z + \frac{1}{2}\ln 2\pi + \frac{1}{2}\sum_{n=1}^{\infty}\frac{I(n)}{n}\cdot\frac{1}{(z+1)_n} \\
\ln\Gamma(z) &= \left(z-\frac{1}{2}\right)\ln z - z + \frac{1}{2}\ln 2\pi - \frac{1}{2}\sum_{n=1}^{\infty}\frac{I'(n)}{n}\cdot\frac{1}{(z)_n} \\
\Psi(z) &= \ln z - \frac{1}{2z} - \frac{1}{2}\sum_{n=2}^{\infty}\frac{K(n)}{n}\cdot\frac{1}{(z+1)_n} \\
\Psi(z) &= \ln z - \frac{1}{2z} - \frac{1}{2}\sum_{n=2}^{\infty}\frac{K(n)-nK(n-1)}{n}\cdot\frac{1}{(z)_n}
\end{aligned} \tag{91}$$

where numbers $I(n)$, $I'(n)$ and $K(n)$ are rational and may be given in terms of the Stirling numbers of the first kind. Binet recognized these numbers, denoted them by a capital H, referenced Stirling's treatise [145] and even corrected Stirling's error: in the table on p. 11 [145] he noticed that the value of $|S_1(9,3)| = 118\,124$ and not $105\,056$. In our notations, Binet's formulæ for $I(n)$, $I'(n)$ and $K(n)$ read

$$
\begin{aligned}
I(n) &= \int_0^1 (2x-1)(x)_n\,dx = \sum_{l=1}^{n} \frac{l\cdot\bigl|S_1(n,l)\bigr|}{(l+1)(l+2)} = 2\sum_{l=1}^{n} \frac{\bigl|S_1(n,l)\bigr|}{l+2} - C_{2,n} \\
I'(n) &= \int_0^1 (2x-1)(1-x)(x)_{n-1}\,dx = \sum_{l=1}^{n-2} \frac{l\cdot\bigl|S_1(n-1,l+1)\bigr|}{(l+2)(l+3)(l+4)} \\
K(n) &= n! - 2\int_0^1 (x)_n\,dx = n! - 2\sum_{l=1}^{n} \frac{\bigl|S_1(n,l)\bigr|}{l+1} = n! - 2C_{2,n}
\end{aligned}
\tag{92}
$$

In particular, the first few coefficients are $I(1) = \frac{1}{6}$, $I(2) = \frac{1}{3}$, $I(3) = \frac{59}{60}$, $I(4) = \frac{58}{15}$, $I(5) = \frac{533}{28}$, $I(6) = \frac{1577}{14}$, $\ldots$, $I'(1) = -\frac{1}{6}$, $I'(2) = 0$, $I'(3) = +\frac{1}{60}$, $I'(4) = +\frac{1}{15}$, $I'(5) = +\frac{25}{84}$, $I'(6) = +\frac{11}{7}$, $\ldots$ and $K(2) = \frac{1}{3}$, $K(3) = \frac{3}{2}$, $K(4) = \frac{109}{15}$, $K(5) = \frac{245}{6}$, $K(6) = \frac{11\,153}{42}$, $K(7) = \frac{23\,681}{12}$, $\ldots$.[43] Strictly speaking, Binet only found first four coefficients for each of these series and incorrectly calculated some of them (e.g. for $I(4)$ he took $\frac{227}{60}$ instead of $\frac{232}{60} = \frac{58}{15}$, for $K(5)$ he took $\frac{245}{3}$ instead of $\frac{245}{6}$), but otherwise his method and derivations are correct. Binet also remarked that

$$
\frac{I'(n)}{n}\cdot\frac{1}{(z)_n} = O\left(\frac{1}{n^{z+1}\ln n}\right), \qquad n\to\infty\,,
\tag{93}
$$

which implies that he knew the first-order approximation for the Cauchy numbers of the second kind as early as 1839.[44] In 1923 Niels E. Nørlund [111, pp. 243–244], [151, p. 335] obtained two series of a similar nature for the polygamma functions. In particular, for the Digamma function, he provided following expressions

$$
\begin{aligned}
\Psi(z) &= \ln z - \frac{1}{z} + \sum_{n=1}^{\infty} \frac{C_{2,n}}{n}\cdot\frac{1}{(z+1)_n} \\
\Psi(z) &= \ln z - \frac{1}{2z} - \sum_{n=2}^{\infty} \frac{\bigl|G_n\bigr|\cdot(n-1)!}{(z)_n}
\end{aligned}
\tag{94}
$$

A careful inspection of both formulæ reveals that they actually are rewritten versions of the foregoing expansions for $\Psi(z)$ given by Binet 84 years earlier.[45] One may also notice that Fontana–Mascheroni series (9), (40), is a particular case of the latter formula when $z = 1$. In contrast, the former expression at $z = 1$ yields a not particularly well-known series for Euler's constant

$$
\gamma = 1 - \sum_{n=1}^{\infty} \frac{C_{2,n}}{n\cdot(n+1)!} = 1 - \frac{1}{4} - \frac{5}{72} - \frac{1}{32} - \frac{251}{14\,400} - \frac{19}{1728} - \frac{19\,087}{2\,540\,160} - \ldots
\tag{95}
$$

which is, in fact, closely related to the above-mentioned Fontana–Mascheroni series and may be reduced to the latter by means of the recurrence relation

---

[43] Values $I'(1) = -\frac{1}{6}$, $I'(2) = 0$ are found from the integral formula, their definition via the sum with the Stirling numbers of the first kind being valid only for $n \geqslant 3$.

[44] Note, however, that Binet stated this result without proof (he wrote *Je ne développe pas ici ces résultats, parce que les détails sont un peu longs*).

[45] In order to reduce first Binet's series to first Nørlund's series, it suffices to remark that $\sum\limits_{n=2}^{\infty}\frac{(n-1)!}{(z+1)_n} = \sum\limits_{n=1}^{\infty}\frac{1\cdot 2\cdots n}{(z+1)(z+2)\cdots(z+n+1)} = \frac{1}{z} - \frac{1}{z+1}$ and $C_{2,1} = \frac{1}{2}$. The equivalence between second Binet's series and second Nørlund's series follows from the fact that $\frac{1}{2}\bigl[K(n) - nK(n-1)\bigr] = nC_{2,n-1} - C_{2,n} = \bigl|C_{1,n}\bigr| \equiv \bigl|G_n\bigr|\cdot n!$, where $K(1) = 0$ and $n = 2, 3, 4, \ldots$.

$$nC_{2,n-1} - C_{2,n} = \big|C_{1,n}\big| \equiv \big|G_n\big| \cdot n!\,, \qquad C_{2,0} = 1\,, \quad n = 1, 2, 3, \ldots \tag{96}$$

Namely, by partial fraction decomposition, (95) becomes

$$\begin{aligned}\gamma &= 1 - \sum_{n=1}^{\infty} \frac{C_{2,n}}{n!} \cdot \frac{1}{n(n+1)} = 1 - \sum_{n=1}^{\infty} \frac{C_{2,n}}{n \cdot n!} + \sum_{n=1}^{\infty} \frac{C_{2,n}}{(n+1)!} = \\ &= 1 - \sum_{n=1}^{\infty} \frac{C_{2,n-1} - \big|G_n\big| \cdot (n-1)!}{n!} + \sum_{n=1}^{\infty} \frac{C_{2,n}}{(n+1)!} = \sum_{n=1}^{\infty} \frac{\big|G_n\big|}{n}\end{aligned}$$

It is interesting to note that series (95) converges at the same rate as $\sum n^{-2} \ln^{-1} n$, while Fontana–Mascheroni series (9), (40) converges slightly faster, as $\sum n^{-2} \ln^{-2} n$, see (18). It seems also appropriate to note here, that apart from Nørlund, the series expansions equivalent or similar to those derived by Binet in 1839, were also obtained (sometimes simply rediscovered, sometimes generalized) by various contemporaneous writers, see e.g. [34, p. 2052, Eq. (1.17)], [35, p. 11], [106], [164, pp. 4005–4007].

4. A note on the history of this article

Various internet searches may indicate that this article was first expected to be published by the journal "Mathematics of Computation" (article submitted on 18 August 2014 and accepted for publication on 3 December 2014).[46] However, due to a disagreement with the managing editor of this journal during the production of this paper, I decided to withdraw it.

Note also that the present article was written before the recently published paper [14], which is an extension of the present work to generalized Euler's constants (Stieltjes constants).

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[46] A more complete description of the publication history may be traced by consulting the arXiv version of this paper arXiv: 1408.3902.

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