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Rediscovery of Malmsten's integrals, their evaluation by contour integration methods and some related results

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Abstract This article is devoted to a family of logarithmic integrals recently treated in mathematical literature, as well as to some closely related results. First, it is shown that the problem is much older than usually reported. In particular, the so-called *Vardi's integral*, which is a particular case of the considered family of integrals, was first evaluated by Carl Malmsten and colleagues in 1842. Then, it is shown that under some conditions, the contour integration method may be successfully used for the evaluation of these integrals (they are called *Malmsten's integrals*). Unlike most modern methods, the proposed one does not require "heavy" special functions and is based solely on the Euler's Γ-function. A straightforward extension to an arctangent family of integrals is treated as well. Some integrals containing polygamma functions are also evaluated by a slight modification of the proposed method. Malmsten's integrals usually depend on several parameters including discrete ones. It is shown that Malmsten's integrals of a discrete real parameter may be represented by a kind of finite Fourier series whose coefficients are given in terms of the Γ-function and its logarithmic derivatives. By studying such orthogonal expansions, several interesting theorems concerning the values of the Γ-function at rational arguments are proven. In contrast, Malmsten's integrals of a continuous complex parameter are found to be connected with the generalized Stieltjes constants. This connection reveals to be useful for the determination of the first generalized Stieltjes constant at seven rational arguments in the range $(0, 1)$ by means of elementary functions, the Euler's constant γ, the first Stieltjes constant γ_1 and the Γ-function. However, it is not known if any first generalized Stieltjes constant at rational argument may be expressed in the same way. Useful in this regard, the multiplication theorem, the recurrence relationship and the reflection formula for the Stieltjes constants are provided as well. A part of the manuscript is devoted to certain logarithmic and trigonometric series

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related to Malmsten's integrals. It is shown that comparatively simple logarithmico–trigonometric series may be evaluated either via the Γ-function and its logarithmic derivatives, or via the derivatives of the Hurwitz ζ-function, or via the antiderivative of the first generalized Stieltjes constant. In passing, it is found that the authorship of the Fourier series expansion for the logarithm of the Γ-function is attributed to Ernst Kummer erroneously: Malmsten and colleagues derived this expansion already in 1842, while Kummer obtained it only in 1847. Interestingly, a similar Fourier series with the cosine instead of the sine leads to the second-order derivatives of the Hurwitz ζ-function and to the antiderivatives of the first generalized Stieltjes constant. Finally, several errors and misprints related to logarithmic and arctangent integrals were found in the famous Gradshteyn & Ryzhik's table of integrals as well as in the Prudnikov et al. tables.

Keywords Logarithmic integrals · Logarithmic series · Theory of functions of a complex variable · Contour integration · Rediscoveries · Malmsten · Vardi · Number theory · Gamma function · Zeta function · Rational arguments · Special constants · Generalized Euler's constants · Stieltjes constants · Otrhogonal expansions

Mathematics Subject Classification 33B15 · 30-02 · 30D10 · 30D30 · 11-02 · 11M35 · 11M06 · 01A55 · 97I30 · 97I80

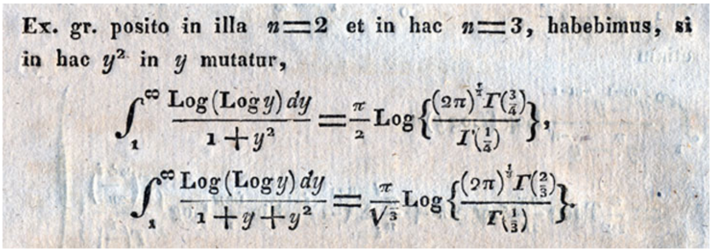
1 Introduction

1.1 Introductory remarks and history of the problem

In an article which appeared in the American Mathematical Monthly at the end of 1980s, Vardi [67] treats several interesting logarithmic integrals found in Gradshteyn and Ryzhik's tables [28]. His exposition begins with the integrals

$$\int_{\pi/4}^{\pi/2} \ln\ln\operatorname{tg} x\,dx = \int_0^1 \frac{\ln\ln\frac{1}{x}}{1+x^2}\,dx = \int_1^\infty \frac{\ln\ln x}{1+x^2}\,dx = \frac{1}{2}\int_0^\infty \frac{\ln x}{\operatorname{ch} x}\,dx$$
$$= \frac{\pi}{2}\ln\left\{\frac{\Gamma(3/4)}{\Gamma(1/4)}\sqrt{2\pi}\right\}, \tag{1}$$

which can be deduced one from another by a simple change of variable (these formulas are given in no. 4.229-7, 4.325-4 and 4.371-1 of [28]). Although the first results for such a kind of integrals may be found in the mathematical literature of the 19th century (e.g., in famous tables [62]), they continue to attract the attention of modern researchers and their evaluation still remains interesting and challenging. Vardi's paper [67] generated a new wave of interest to such logarithmic integrals and numerous works on the subject, including very recent ones, appeared since [67], [2], [13], [7], [47], [46], [68], [4], [8], [44]. On the other hand, since the subject is very old, it is hard to avoid rediscoveries. For the computation of the above mentioned integral (1), modern authors [2], [13, p. 237], [7, p. 160], [46], [4], [44], send the reader to the


Ex. gr. posito in illa $n=2$ et in hac $n=3$, habebimus, si in hac y^2 in y mutatur,

$$\int_1^\infty \frac{\mathrm{Log}(\mathrm{Log}\,y)\,dy}{1+y^2} = \frac{\pi}{2}\,\mathrm{Log}\left\{\frac{(2\pi)^{\frac{1}{2}}\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})}\right\},$$

$$\int_1^\infty \frac{\mathrm{Log}(\mathrm{Log}\,y)\,dy}{1+y+y^2} = \frac{\pi}{\sqrt{3}}\,\mathrm{Log}\left\{\frac{(2\pi)^{\frac{1}{3}}\Gamma(\frac{2}{3})}{\Gamma(\frac{1}{3})}\right\}.$$

Fig. 1 A fragment of p. 12 from the Malmsten et al.'s dissertation [40]

Vardi's paper [67].[1] Vardi, failing to identify the author of formula (1) and failing to locate its proof, proposed a method of proof based essentially on the use of the Dirichlet L-function. However, formula (1), in all four forms, was already known to David Bierens de Haan [62, Table 308-28, 148-1, 260-1], and if we go to a deeper exploration of this question, we find that integral (1) was first evaluated by Carl Johan Malmsten[2] and colleagues in 1842 in a dissertation written in Latin [40, p. 12], see Fig. 1. A part of this dissertation was later republished in the famous *Journal für die reine und angewandte Mathematik*[6] [41], see, e.g., p. 7 for integral (1). Moreover, two other logarithmic integrals,

$$\int\limits_0^1 \frac{\ln\ln\frac{1}{x}}{1+x+x^2}\,dx = \int\limits_1^\infty \frac{\ln\ln x}{1+x+x^2}\,dx = \frac{\pi}{\sqrt{3}}\ln\left\{\frac{\Gamma(2/3)}{\Gamma(1/3)}\sqrt[3]{2\pi}\right\} \tag{2a}$$

and

$$\int\limits_0^1 \frac{\ln\ln\frac{1}{x}}{1-x+x^2}\,dx = \int\limits_1^\infty \frac{\ln\ln x}{1-x+x^2}\,dx = \frac{2\pi}{\sqrt{3}}\ln\left\{\frac{\sqrt[6]{32\pi^5}}{\Gamma(1/6)}\right\} \tag{2b}$$

mentioned in [2, 67], [13, p. 238], [4, 70], were also first evaluated by Malmsten et al. [40, pp. 12 and 43] and [41, formulas (12) and (72)].[3] Malmsten and his colleagues evaluated many other beautiful logarithmic integrals[4] and series as well, but

[1]Only Bassett [8], an undergraduate student, remarked that solutions for integrals (1) and (2a, 2b) are much older than Vardi's paper [67].

[2]Carl Johan Malmsten, written also Karl Johan Malmsten (born April 9, 1814 in Uddetorp, died February 11, 1886 in Uppsala), was a Swedish mathematician and politician. He became Docent in 1840, and then, Professor of mathematics at the Uppsala University in 1842. He was elected a member of the Royal Swedish Academy of Sciences (*Kungliga Vetenskaps–akademien*) in 1844. He was also a minister without portfolio in 1859–1866 and Governor of Skaraborg County in 1866–1879. For further information, see [36, vol. 17, pp. 657–658].

[3]Both results are presented here in the original form, as they appear in the given sources. In fact, both formulas may be further simplified and written in terms of $\Gamma(1/3)$ only, see (45) and (44), respectively (see also exercise no. 32 where both integrals appear in a more general form). Surprisingly, the latter fact escaped the attention of Malmsten, of his colleagues and of many other researchers. Moreover, Vardi [67, p. 313] even wrote that "in (2a) the number 3 plays the 'key role' and in (2b) 6 is the 'magic number'". A more detailed criticism of the latter statement is given in exercise no. 30.

[4]Many of which were independently evaluated in [2] and in [44].

unfortunately, none of the above-mentioned contemporary authors mentioned them. Moreover, the latter even named integral (1) after Vardi (they call it *Vardi's integral*), and so did many well-known internet resources such as Wolfram MathWorld site [70] or OEIS Foundation site [57].

On the other hand, it is understandable that, sometimes, it can be quite difficult to find the original source of a formula, especially because of the oldness of the result, because the chain of references may be too long and confusing, and because it could be published in many different languages. For example, Gradshteyn and Ryzhik wrote originally in Russian (their book [28] is a translation from Russian), Bierens de Haan published usually in Dutch or in French, Malmsten wrote in Swedish, French and Latin, and we now use mostly English. As the reference for (1), Gradshteyn and Ryzhik [28] as well as Vardi [67], cite the famous Bierens de Haan's tables [62]. In the latter, on the p. 207, Table 148, the reference for the integral (1) is given as "(IV, 265)".[5] This means that this result comes from the 4th volume of the Memoirs of the Royal Academy of Sciences of Amsterdam, which is entirely composed of the *Tables d'intégrales définies* by Bierens de Haan [61], and which is an old version of the well-known *Nouvelles tables d'intégrales définies* [62]. The old version [61] is essentially the same as the new one [62], except that it provides original sources (the new version [62] contains much less misprints and errors, but original references given in the old edition were removed). Thus, we may find in the old edition [61, pp. 264–265, Table 191-1] that integral (1) was evaluated by Malmsten in the work referenced as "Cr. 38. 1." This is often the most difficult part of the work, to understand what an old abbreviation may stand for. Bierens de Haan does not explain it, and unfortunately, neither do modern dictionaries nor encyclopedia. After several hours of search, we finally found that "Cr." stands for "Crelle's Journal", which is a jargon name for the *Journal für die reine und angewandte Mathematik*.[6] The number 38 stands for the volume's number, and 1 is not the issue's number but the number of the page from which the manuscript starts. Furthermore, a deeper study of Malmsten's works, see e.g. [63, p. 31], shows that this article is a concise and updated version of the collective dissertation [40] which was presented at the Uppsala University in April–June 1842.[7] Therefore, taking into account the undoubted Malmsten and colleagues' priority in the evaluation of the logarithmic integrals of the type (1) and (2a), (2b), we think that integral (1) should be called *Malmsten's integral* rather than *Vardi's integral*. Throughout the manuscript, integrals of kind (1) and (2a), (2b) are called Malmsten's integrals.

The aim of the present work is multifold; accordingly, the article is divided in three parts. In the first part (Sect. 2), we present Malmsten's original proof that Vardi and other modern researchers missed. The presentation of this proof may be of interest

[5] Another frequently encountered notation for references in Bierens de Haan's tables [61] and [62]—which may be not easy to understand for English-speaking readers—is "V. T." which stands for *voir tableau*, i.e. "see table" in English.

[6] English translation: "Journal for Pure and Applied Mathematics".

[7] However, we were surprised to see that Malmsten in the article [41] did not even mention the aforementioned dissertation [40]. In fact, integrals (1) and (2a), (2b) were very probably derived by one of his students or colleagues, but now it is almost impossible to know who exactly did it.

for a large audience of readers for multiple reasons. First, it may be quite difficult to find references [40] and [41], as well as cited works. Second, the manuscript was written in Latin, and references are in French. Latin, being discarded from the study program of most mathematical faculties, may be difficult to understand for many researchers. Third, Malmsten does not make use of special functions other than Γ-function; instead, he smartly employs elementary transformations, so that his proof may be understood even by a first-year student. Fourth, the work [41] contains numerous misprints in formulas and a proper presentation might be quite useful as well. At the end of the presentation, we briefly discuss further Malmsten et al.'s contributions, such as, for example, the Fourier series expansion for the logarithm of the Γ-function (obtained 5 years before Kummer) or the derivation of the reflection formula for two series closely related to ζ-functions (obtained 17 years before the famous Riemann's functional relationship for the ζ-function). Also, connections between the logarithm of the Γ-function and the digamma function, written by Malmsten as a kind of discrete cosine transform, are interesting and provide some further ideas that we later re-used in Sect. 4.5. At the end of this part, we remark that several widely known tables of integrals, such as Gradshteyn and Ryzhik's tables [28], Bierens de Haan's tables [61, 62], and probably, Prudnikov et al.'s tables [53], borrowed a large amount of Malmsten's results, but in most of them, original references to Malmsten were lost. Moreover, some of these results appear with misprints.

In the second part of the manuscript (Sect. 3), we introduce a family of logarithmic integrals of which integral (1) and many other Malmsten's integrals are simple particular cases. We propose an alternative method for the analytical evaluation of such a kind of integral. Unlike most modern methods, the proposed one does not require "heavy" special functions and is based on the methods of contour integration. A non-exhaustive condition under which considered family of integrals may be always expressed in terms of the Γ-function is provided. A straightforward extension to an arctangent family of integrals is treated as well. At the end of this part, we consider in detail examples of application of the proposed method to four most frequently encountered Malmsten's integrals.

The third part of this work (Sect. 4) is designed as a collection of original exercises containing new formulas and theorems, which can be derived directly or indirectly by the proposed method. The exercises and theorems have been grouped thematically:

- Logarithmic Malmsten's integrals containing hyperbolic functions and some closely related results are treated in Sect. 4.1. In particular, integrals, which can be evaluated by the direct application of the proposed method, are given in Sect. 4.1.1. These may be roughly divided in two parts: relatively simple Malmsten's integrals containing two or three parameters (e.g. exercises no. 1, 2, 4, 5, 6-a, 7, 8, 17, etc.) and complete Malmsten's integrals depending on three or more parameters, including discrete ones (e.g. exercises no. 3, 6-b,c,d, 9, 13, 11, 14). Simple Malmsten's integrals, some of which are evaluated up to order 20,[18] often lead to various special constants such as Euler's constant γ, $\Gamma(1/3)$, $\Gamma(1/4)$, $\Gamma(1/\pi)$, Catalan's constant G, Apéry's constant $\zeta(3)$ and others. As regards complete Malmsten's integrals, whose evaluation is carried out up to order 4, it is found that such integrals, when depending on a discrete real parameter, may be represented by a kind of finite Fourier series whose coefficients are given in terms of the Γ-function and its logarithmic derivatives. In contrast, when the considered discrete real parameter becomes continuous and complex, such integrals may be expressed by means of the first generalized Stieltjes constants (such exercises are placed in Sect. 4.5).

- The results closely related to logarithmic Malmsten's integral are placed in Sect. 4.1.2. These include the evaluation of integrals similar to Malmsten's ones and that of certain closely connected series. Most of these series are logarithmico–trigonometric and may be evaluated either via the Γ-function and its logarithmic derivatives, or via the derivatives of the Hurwitz ζ-function, or via the antiderivative of the first generalized Stieltjes constant (conversely, such series may be regarded as Fourier series expansions of the above-mentioned functions).
- In Sect. 4.2, we treat $\ln\ln$-integrals, most of which are obtained by a simple change of variable of integrals from Sect. 4.1. In the same section, we also show that Vardi's hypothesis about the relationship between the argument of the Γ-function and the degree in which the poles of the corresponding integrand are the roots of unity is not true in general (exercise no. 30).
- Section 4.3 is devoted to arctangent integrals. Similarly to logarithmic integrals, arctangent integrals can be roughly classified into several categories: comparatively simple, complete and closely related (such as, e.g., exercise no. 40 where an analog of the second Binet's formula for the logarithm of the Γ-function is derived).
- In Sect. 4.4, we show that some slight modifications of the method developed in Sect. 3 may be quite fruitful for the evaluation of certain integrals containing logarithm of the Γ-function and the polygamma functions.
- Lastly, in Sect. 4.5 we put exercises and theorems related to the values of the Γ-function at rational arguments and to the Stieltjes constants. Mostly, these results are deduced from precedent exercises. For instance, by means of finite orthogonal representations obtained for the Malmsten' integrals in Sect. 4.1, we prove several interesting theorems concerning the logarithm of the Γ-function at rational arguments, including some variants of Parseval's theorem (exercises no. 58–62). By the way, with the help of the same technique one can derive similar theorems implying polygamma functions. In the second part of Sect. 4.5, we show that some complete Malmsten's integrals, which were previously evaluated in Sect. 4.1, may be also expressed by means of the first generalized Stieltjes constants. This connection between Malmsten's integrals of a real discrete and of a continuous complex parameters is not only interesting in itself, but also permits evaluation of the first generalized Stieltjes constant $\gamma_1(p)$ at $p = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6}, \frac{2}{3}, \frac{3}{4}, \frac{5}{6}$ by means of elementary functions, the Euler's constant γ, the first Stieltjes constant γ_1 and the Γ-function (see exercise no. 64). However, it is still unknown if any first generalized Stieltjes constant at rational argument may be expressed in the same way (from this point of view, the evaluation of $\gamma_1(1/5)$ could be of special interest). In this framework, we also discovered that the sum of the first generalized Stieltjes constant $\gamma_1(p)$, $p \in (0, 1)$, with its reflected version $\gamma_1(1-p)$ may be expressed, at least for seven different rational values of p, in terms of elementary functions, the Euler's constant γ and the first Stieltjes constant γ_1. At the same time, it is not known if other sums $\gamma_1(p) + \gamma_1(1-p)$ share the same property. An alternative evaluation of integrals from exercises no. 65–66 could probably provide some light on this problem.

Finally, answers for all exercises were carefully verified numerically with Maple 12 (except exercises with Stieltjes constants which were verified with Wolfram Alpha Pro). By default, if nothing is explicitly said, the presented result coincides with the numerical one. Actually, only in few cases Maple 12 fails to correctly evaluate integrals. For instance, it fails both numerically and symbolically to evaluate the first integral on the left in (1). Maple 12.0 gives $(-\pi\ln 2 + i\pi^2)/4 \approx 0.544 + i\,2.467$, while it is clear that this integral has no imaginary part at all, and the real one is neither correctly evaluated. By the way, authors of [44] also reported incorrect numerical and symbolical evaluation of this integral by Mathematica 6.0. However, unlike [44], we will not specify wherever Maple 12 is able or unable to evaluate integrals analytically, because in almost all cases Maple 12 was unable to do it.

1.2 Notations

Throughout the manuscript, following abbreviated notations are used: $\gamma = 0.5772156649\ldots$ for the Euler's constant, γ_n for the nth Stieltjes constant, $\gamma_n(p)$ for the nth generalized Stieltjes constant at point p,[8] $\mathrm{G} = 0.9159655941\ldots$ for Catalan's constant, $\lfloor x \rfloor$ for the integer part of x, $\operatorname{tg} z$ for the tangent of z, $\operatorname{ctg} z$ for the cotangent of z, $\operatorname{ch} z$ for the hyperbolic cosine of z, $\operatorname{sh} z$ for the hyperbolic sine of z, $\operatorname{th} z$ for the hyperbolic tangent of z, $\operatorname{cth} z$ for the hyperbolic cotangent of z.[9] In order to avoid any confusion between compositional inverse and multiplicative inverse, inverse trigonometric and hyperbolic functions are denoted as arccos, arcsin, arctg, … and not as $\cos^{-1}$, $\sin^{-1}$, tg^{-1}, …. We write $\Gamma(z)$, $\Psi(z)$, $\Psi_1(z)$, $\Psi_2(z)$, $\Psi_3(z)$, …, $\Psi_n(z)$ to denote, respectively, gamma, digamma, trigamma, tetragamma, pentagamma, …, $(n-2)$th polygamma functions of argument z. The Riemann ζ-function, the η-function (known also as the alternating Riemann ζ-function) and the Hurwitz ζ-function are, respectively, defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \qquad \eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s}, \qquad \zeta(s,v) = \sum_{n=0}^{\infty} \frac{1}{(n+v)^s},$$

$v \neq 0, -1, -2, \ldots$, with $\operatorname{Re} s > 1$ for the ζ-functions and $\operatorname{Re} s > 0$ for the η-function. Where necessary, these definitions may be extended to other domains by the principle of analytic continuation. For example, one of the most known analytic continuations for the Hurwitz ζ-function is the so-called Hermite representation

$$\zeta(s,v) = \frac{v^{1-s}}{s-1} + \frac{v^{-s}}{2} + 2\int_0^{\infty} \frac{\sin\!\left(s \operatorname{arctg} \frac{x}{v}\right)}{(e^{2\pi x}-1)\left(v^2+x^2\right)^{s/2}}\, dx\,, \qquad \operatorname{Re} v > 0, \tag{3}$$

which extends $\zeta(s,v)$ to the entire complex plane except at $s=1$, see e.g. [9, vol. I, p. 26, Eq. 1.10(7)]. Note also that the η-function may be easily reduced to the Rie-

[8] We remark, in passing, that by convention $\gamma_n \equiv \gamma_n(1)$ for any natural n.

[9] Most of these notations come from Latin, e.g "ch" stands for *cosinus hyperbolicus*, "sh" stands for *sinus hyperbolicus*, etc.

mann ζ-function $\eta(s)=(1-2^{1-s})\zeta(s)$, while the Hurwitz ζ-function is an independent transcendent (except some particular values). Moreover, the alternating Hurwitz ζ-function $\eta(s,v)$ may be similarly reduced to the ordinary Hurwitz ζ-function

$$\eta(s,v)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+v)^s}=\lim_{z\to s}\left\{2^{1-z}\zeta\left(z,\frac{v}{2}\right)-\zeta(z,v)\right\}, \tag{4}$$

$v\neq 0,-1,-2,\ldots$, $\operatorname{Re}s>0$. $\operatorname{Re}z$ and $\operatorname{Im}z$ denote, respectively, real and imaginary parts of z. Natural numbers are defined in a traditional way as a set of positive integers, which is denoted by $\mathbb{N}$. Kronecker symbol of arguments l and k is denoted by $\delta_{l,k}$. Letter i is never used as index and is $\sqrt{-1}$. Complex integration over region $A\leqslant\operatorname{Im}z\leqslant B$ means that the complex line integral is taken around an infinitely long horizontal strip delimited by inequality $A\leqslant\operatorname{Im}z\leqslant B$, where $(A,B)\in\mathbb{R}^2$ (i.e. the integration contour is a rectangle with vertices at $[R+iA,\,R+iB,\,-R+iB,\,-R+iA]$ with $R\to\infty$. The notation $\operatorname{res}_{z=a}f(z)$ stands for the residue of the function $f(z)$ at the point $z=a$. Other notations are standard. Finally, we remark that the references to the formulas are given between parentheses "( )", those to the number of exercise from Sect. 4 are preceded by "no."; the bibliographic references are given in square brackets "[ ]".

2 Malmsten's method and its results

2.1 Malmsten's original proof of the integral formula (1)

The proof is presented in a way closest to the original [41]; we have only replaced the old notations by the new ones, as well as correcting numerous misprints in formulas (by the way, [40] contains much less misprints). In the effort to make it more accessible for the readers, several modern references were also added, but, of course, the old ones are also left. These references are marked with an * (only in this subsection).

Malmsten begins with the elementary integral

$$\int\limits_0^{\infty}e^{-xz}\sin vz\,dz=\frac{v}{x^2+v^2},\qquad x>0,\ v>0,$$

which, being integrated[10] over v from 0 to u, becomes

$$2\int\limits_0^{\infty}\frac{(1-\cos uz)\,e^{-xz}}{z}\,dz=\ln\bigl(x^2+u^2\bigr)-2\ln x. \tag{5}$$

[10] This operation is permitted because the considered improper integral is uniformly convergent with respect to v [34, p. 44, § 1.12]*, [58, vol. II, pp. 262–269]*, [10, pp. 175–179]*.

The latter logarithm is then replaced by one of Frullani's integrals [34, pp. 406–407, § 12.16]*, [50]*, [24, p. 455]*,

$$\ln x = \int_0^\infty \frac{e^{-z} - e^{-xz}}{z}\, dz, \tag{6}$$

which yields

$$\ln\bigl(x^2 + u^2\bigr) = 2\int_0^\infty \frac{e^{-z} - e^{-xz}\cos uz}{z}\, dz. \tag{7}$$

Multiply the last equality by $\dfrac{\operatorname{sh} au}{\operatorname{sh} \pi u}$, parameter a being in the range $(-\pi, +\pi)$, and then integrate it over all values of u from 0 to ∞

$$\int_0^\infty \frac{\operatorname{sh} au}{\operatorname{sh} \pi u} \ln\bigl(x^2 + u^2\bigr)\, du = 2\int_0^\infty \Biggl[\int_0^\infty \frac{\operatorname{sh} au}{\operatorname{sh} \pi u}\bigl(e^{-z} - e^{-xz}\cos uz\bigr) du\Biggr] \frac{dz}{z}. \tag{8}$$

In virtue of formulas (b″) and (a′) from [64, vol. II, p. 186][11]

$$\int_0^\infty \frac{\operatorname{sh} au}{\operatorname{sh} \pi u}\, du = \frac{1}{2}\operatorname{tg}\frac{a}{2} \qquad \text{and} \qquad \int_0^\infty \frac{\operatorname{sh} au}{\operatorname{sh} \pi u}\cos uz\, du = \frac{\sin a}{2(\operatorname{ch} z + \cos a)},$$

where $-\pi < a < \pi$ and $-\infty < z < \infty$, expression (8) may be rewritten as follows:

$$\int_0^\infty \frac{\operatorname{sh} au}{\operatorname{sh} \pi u} \ln\bigl(x^2 + u^2\bigr)\, du = \int_0^\infty \Biggl[\operatorname{tg}\frac{a}{2} - \frac{2e^{-xz}\sin a}{1 + 2e^{-z}\cos a + e^{-2z}}\Biggr] \frac{e^{-z}\, dz}{z}, \tag{9}$$

$-\pi < a < \pi$. Now make a change of variable in the last integral by putting $y = e^{-z}$. This yields

$$\int_0^\infty \frac{\operatorname{sh} au}{\operatorname{sh} \pi u} \ln\bigl(x^2 + u^2\bigr)\, du = \int_0^1 \Biggl[\operatorname{tg}\frac{a}{2} - \frac{2y^x \sin a}{1 + 2y\cos a + y^2}\Biggr] \frac{dy}{\ln \frac{1}{y}} \equiv T_a(x), \tag{10}$$

$-\pi < a < \pi$, where the last integral was designated by $T_a(x)$ for brevity. It can be easily demonstrated that if the parameter a is chosen so that $a = \pi m/n$, numbers m and n being positive integers such that $m < n$ (in other words, if a is a rational part of π), then, the integral on the right part of the last equation may be always expressed

[11] These formulas may be also derived by contour integration methods, see e.g. [59, pp. 186, 197–198]*, [69, p. 132]*, [23, pp. 276–277]*.

in terms of the Γ-function. The differentiation of $T_a(x)$ with respect to x gives

$$\frac{dT}{dx} = 2\int_0^1 \frac{y^x \sin a}{1+2y\cos a + y^2}\, dy. \tag{11}$$

But if the parameter a is a rational part of π, the latter integral, in virtue of what was established in [64, vol. II, pp. 163–165], is

$$\int_0^1 \frac{y^x \sin a}{1+2y\cos a+y^2}\, dy$$
$$= \begin{cases} \displaystyle\sum_{l=1}^{n-1} (-1)^{l-1} \sin(la) \left\{\Psi\left(\frac{x+n+l}{2n}\right) - \Psi\left(\frac{x+l}{2n}\right)\right\}, & \text{if } m+n \text{ is odd,} \\ \displaystyle\sum_{l=1}^{\lfloor\frac{1}{2}(n-1)\rfloor} (-1)^{l-1} \sin(la) \left\{\Psi\left(\frac{x+n-l}{n}\right) - \Psi\left(\frac{x+l}{n}\right)\right\}, & \text{if } m+n \text{ is even.} \end{cases}$$

By substituting these formulas into the right part of (11), and by calculating the antiderivative, we obtain

$$T_a(x) = \begin{cases} C_1 + 2\displaystyle\sum_{l=1}^{n-1} (-1)^{l-1} \sin(la) \ln\left\{\frac{\Gamma\left(\frac{x+n+l}{2n}\right)}{\Gamma\left(\frac{x+l}{2n}\right)}\right\}, & \text{if } m+n \text{ is odd,} \\ \\ C_2 + 2\displaystyle\sum_{l=1}^{\lfloor\frac{1}{2}(n-1)\rfloor} (-1)^{l-1} \sin(la) \ln\left\{\frac{\Gamma\left(\frac{x+n-l}{n}\right)}{\Gamma\left(\frac{x+l}{n}\right)}\right\}, & \text{if } m+n \text{ is even,} \end{cases} \tag{12}$$

where C_1 and C_2 are constants of integration. In order to find them, the following procedure is adopted. Put in the last formula, first $x = r$, and then $x = s$. Subtracting one from another and dividing by minus two, we have $\Delta_a(r,s) \equiv \frac{1}{2}[T_a(s) - T_a(r)] =$

$$\int_0^1 \frac{y^r(1-y^{s-r})\sin a}{1+2y\cos a+y^2} \cdot \frac{dy}{\ln\frac{1}{y}}$$
$$= \begin{cases} \displaystyle\sum_{l=1}^{n-1} (-1)^{l-1} \sin(la) \ln\left\{\frac{\Gamma\left(\frac{s+n+l}{2n}\right)\Gamma\left(\frac{r+l}{2n}\right)}{\Gamma\left(\frac{r+n+l}{2n}\right)\Gamma\left(\frac{s+l}{2n}\right)}\right\}, & \text{if } m+n \text{ is odd,} \\ \\ \displaystyle\sum_{l=1}^{\lfloor\frac{1}{2}(n-1)\rfloor} (-1)^{l-1} \sin(la) \ln\left\{\frac{\Gamma\left(\frac{s+n-l}{n}\right)\Gamma\left(\frac{r+l}{n}\right)}{\Gamma\left(\frac{r+n-l}{n}\right)\Gamma\left(\frac{s+l}{n}\right)}\right\}, & \text{if } m+n \text{ is even.} \end{cases}$$

The difference between $\Delta_a(0,1)$ and $\Delta_a(1,2)$ yields

$$\int_0^1 \frac{(1-2y+y^2)\sin a}{1+2y\cos a+y^2}\cdot\frac{dy}{\ln\frac{1}{y}}$$
$$=\begin{cases} \displaystyle\sum_{l=1}^{n-1}(-1)^{l-1}\sin(la)\ln\left\{\frac{\Gamma^2\left(\frac{n+l+1}{2n}\right)\Gamma\left(\frac{l+2}{2n}\right)\Gamma\left(\frac{l}{2n}\right)}{\Gamma^2\left(\frac{l+1}{2n}\right)\Gamma\left(\frac{n+l}{2n}\right)\Gamma\left(\frac{n+l+2}{2n}\right)}\right\}, \\ \qquad \text{if } m+n \text{ is odd,} \\ \displaystyle\sum_{l=1}^{\lfloor\frac{1}{2}(n-1)\rfloor}(-1)^{l-1}\sin(la)\ln\left\{\frac{\Gamma^2\left(\frac{n-l+1}{n}\right)\Gamma\left(\frac{l+2}{n}\right)\Gamma\left(\frac{l}{n}\right)}{\Gamma^2\left(\frac{l+1}{n}\right)\Gamma\left(\frac{n-l}{n}\right)\Gamma\left(\frac{n-l+2}{n}\right)}\right\}, \\ \qquad \text{if } m+n \text{ is even.} \end{cases}$$

Over and over again from (12) for $x=1$, we get

$$\int_0^1 \frac{(1-2y+y^2)\sin a}{1+2y\cos a+y^2}\cdot\frac{dy}{\ln\frac{1}{y}} = (1+\cos a)$$
$$\times\begin{cases} \displaystyle C_1+2\sum_{l=1}^{n-1}(-1)^{l-1}\sin(la)\ln\left\{\frac{\Gamma\left(\frac{1+n+l}{2n}\right)}{\Gamma\left(\frac{1+l}{2n}\right)}\right\}, & \text{if } m+n \text{ is odd,} \\ \displaystyle C_2+2\sum_{l=1}^{\lfloor\frac{1}{2}(n-1)\rfloor}(-1)^{l-1}\sin(la)\ln\left\{\frac{\Gamma\left(\frac{1+n-l}{n}\right)}{\Gamma\left(\frac{1+l}{n}\right)}\right\}, & \text{if } m+n \text{ is even.} \end{cases}$$

By comparing last two expressions, one may easily identify both constants of integration:

$$\begin{cases} C_1 = \dfrac{\sin a\cdot\ln 2n}{1+\cos a} = \operatorname{tg}\dfrac{a}{2}\cdot\ln 2n, \\ C_2 = \dfrac{\sin a\cdot\ln n}{1+\cos a} = \operatorname{tg}\dfrac{a}{2}\cdot\ln n. \end{cases}$$

In the final analysis, the substitution of these values into (12) yields

$$T_a(x) = \int_0^\infty \frac{\operatorname{sh} au}{\operatorname{sh}\pi u}\ln\left(x^2+u^2\right)du$$

$$
= \begin{cases} \operatorname{tg}\dfrac{\pi m}{2n}\cdot\ln 2n + 2\displaystyle\sum_{l=1}^{n-1}(-1)^{l-1}\sin\dfrac{\pi m l}{n}\cdot\ln\left\{\dfrac{\Gamma\left(\frac{1}{2}+\frac{x+l}{2n}\right)}{\Gamma\left(\frac{x+l}{2n}\right)}\right\}, \\ \text{if } m+n \text{ is odd,} \\ \\ \operatorname{tg}\dfrac{\pi m}{2n}\cdot\ln n + 2\displaystyle\sum_{l=1}^{\lfloor\frac{1}{2}(n-1)\rfloor}(-1)^{l-1}\sin\dfrac{\pi m l}{n}\cdot\ln\left\{\dfrac{\Gamma\left(1-\frac{l-x}{n}\right)}{\Gamma\left(\frac{l+x}{n}\right)}\right\}, \\ \text{if } m+n \text{ is even,} \end{cases} \tag{13}
$$

where $a \equiv \pi m/n$. Now, one can easily deduce formula (1).[12] Putting $m = 1$ and $n = 2$ in (13), we obtain

$$
\int_0^\infty \frac{\ln(u^2+x^2)}{2\operatorname{ch}\left(\frac{1}{2}\pi u\right)}\,du = 2\ln\left\{\frac{2\Gamma\left(\frac{x+3}{4}\right)}{\Gamma\left(\frac{x+1}{4}\right)}\right\}.
$$

By making a suitable change of variable in the above integral, and by taking into account that the definite integral of $\operatorname{ch}^{-1}x$ over $x \in [0,\infty)$ equals $\pi/2$, the latter equation takes the form

$$
\int_0^\infty \frac{\ln(u^2+x^2)}{\operatorname{ch} u}\,du = 2\pi\ln\left\{\frac{\sqrt{2\pi}\,\Gamma\left(\frac{x}{2\pi}+\frac{3}{4}\right)}{\Gamma\left(\frac{x}{2\pi}+\frac{1}{4}\right)}\right\}. \tag{14}
$$

Setting $x = 0$ yields immediately formula (1) in its hyperbolic form. □

By using a similar procedure and with the help of previous results, Malmsten also evaluated

$$
L_a(x) \equiv \int_0^\infty \frac{\operatorname{ch} au}{\operatorname{ch}\pi u}\ln\left(x^2+u^2\right)du \tag{15}
$$

$$
= \begin{cases} \sec\dfrac{\pi m}{2n}\cdot\ln 2n + 2\displaystyle\sum_{l=1}^{n}(-1)^{l-1}\cos\dfrac{(2l-1)m\pi}{2n}\cdot\ln\left\{\dfrac{\Gamma\left(\frac{1}{2}+\frac{2x+2l-1}{4n}\right)}{\Gamma\left(\frac{2x+2l-1}{4n}\right)}\right\}, \\ \text{if } m+n \text{ is odd,} \\ \\ \sec\dfrac{\pi m}{2n}\cdot\ln n + 2\displaystyle\sum_{l=1}^{\lfloor\frac{1}{2}(n-1)\rfloor}(-1)^{l-1}\cos\dfrac{(2l-1)m\pi}{2n}\cdot\ln\left\{\dfrac{\Gamma\left(1-\frac{2l-1-2x}{2n}\right)}{\Gamma\left(\frac{2l-1+2x}{2n}\right)}\right\}, \\ \text{if } m+n \text{ is even,} \end{cases}
$$

[12] From here, we shorten Malmsten's proof since the result is almost straightforward. Malmsten's proof was actually longer because he aimed for more general formulas.

where $a \equiv \pi m/n$. The reader can find this result on the p. 41 [40, formula (46)] and on the p. 28 [41, formula (70)].

In some particular cases, right parts of (13) and (15) may be largely simplified, and sometimes, they even can be expressed in terms of only elementary functions. For example,

$$T_{\frac{\pi}{2}}(1) = \int\limits_0^\infty \frac{\ln(1+u^2)}{\operatorname{ch}\left(\frac{1}{2}\pi u\right)}\,du = 4\ln\frac{2\Gamma(1)}{\Gamma(1/2)} = 2\ln\frac{4}{\pi}, \tag{16}$$

or

$$\begin{aligned} T_{\frac{2\pi}{3}}\left(\frac{3}{2}\right) &= \int\limits_0^\infty \frac{\operatorname{sh}\left(\frac{2}{3}\pi u\right)}{\operatorname{sh}\pi u}\ln\left(\frac{9}{4}+u^2\right)du = \sqrt{3}\left\{\ln 6 + \ln\frac{\Gamma\left(\frac{11}{12}\right)\Gamma\left(\frac{13}{12}\right)}{\Gamma\left(\frac{5}{12}\right)\Gamma\left(\frac{7}{12}\right)}\right\} \\ &= \sqrt{3}\ln\left(\frac{1}{2}\operatorname{ctg}\frac{\pi}{12}\right), \end{aligned}$$

or more general integrals

$$\begin{aligned} T_{\frac{m\pi}{n}}\left(\frac{n}{2}\right) &= \int\limits_0^\infty \frac{\operatorname{sh}\left(\frac{m}{n}\pi u\right)}{\operatorname{sh}\pi u}\ln\left(\frac{n^2}{4}+u^2\right)du \\ &= \sum_{l=1}^{n-1}(-1)^{l-1}\sin\frac{\pi m l}{n}\cdot\ln\left\{\left(\frac{n}{2}-l\right)\operatorname{ctg}\left(\frac{\pi}{4}-\frac{\pi l}{2n}\right)\right\}, \end{aligned}$$

and

$$\begin{aligned} L_{\frac{m\pi}{n}}\left(\frac{n}{2}\right) &= \int\limits_0^\infty \frac{\operatorname{ch}(\frac{m}{n}\pi u)}{\operatorname{ch}\pi u}\ln\left(\frac{n^2}{4}+u^2\right)du \\ &= \sum_{l=1}^{n}(-1)^{l-1}\cos\frac{(2l-1)m\pi}{2n}\cdot\ln\left\{\left(\frac{n+1}{2}-l\right)\operatorname{ctg}\left(\frac{\pi}{4}-\frac{\pi(2l-1)}{4n}\right)\right\}, \end{aligned}$$

which hold both only for $m+n$ odd and where logarithms in right parts should be regarded as limits when the index $l = n/2$ and $l = (n+1)/2$ respectively, see [40, formulas (13), (21), (26), (55)],[13] [41, formulas (8), (18), (23), (79)], [62, Table 258-1,6,10,9], [61, Table 275-1,10,16,15]. Many other logarithmic integrals—which can be expressed in a closed form—may be also found in [40] and [41].

At the end of this historical excursion, it may be of interest to remark that Legendre was not far from Malmsten's formula (1) in its hyperbolic form. On p. 190 [64, vol. II]

[13] There is a misprint in formula (13): the sign "−" in the denominator of the integrand should be replaced by "+".

we find

$$\int_0^\infty \frac{dx}{(e^{\pi x}+e^{-\pi x})(m^2+x^2)} = \frac{1}{4m}\left\{\Psi\left(\frac{m}{2}+\frac{3}{4}\right)-\Psi\left(\frac{m}{2}+\frac{1}{4}\right)\right\},$$

where m is not necessarily integer. Multiplying both sides by $2m$ and computing the antiderivative yields

$$\int_0^\infty \frac{\ln(m^2+x^2)}{e^{\pi x}+e^{-\pi x}}\,dx = \ln\Gamma\left(\frac{m}{2}+\frac{3}{4}\right)-\ln\Gamma\left(\frac{m}{2}+\frac{1}{4}\right)+C.$$

Notwithstanding, the constant of integration C is not easy to determine, and this task was achieved, albeit differently, by Malmsten and colleagues ($C=\frac{1}{2}\ln 2$), see also comments on pp. 25–26 in [40]. In like manner, the second Binet's formula for the logarithm of the Γ-function may be also derived from Legendre's work [64] (see, for more details, exercise no. 40).

2.2 Brief discussion of other results obtained by Malmsten and his colleagues

Many other similar results were obtained by Malmsten and colleagues. Of course, they noticed that the hyperbolic form of the integrand could be transformed into that containing $\ln\ln x$ or $\ln\ln\frac{1}{x}$ in the numerator, and many valuable results for such integrals were obtained as well. Among these results, the most magnificent and remarkable are perhaps these two:

$$\int_0^1 \frac{x^{n-2}\ln\ln\frac{1}{x}}{1+x^2+x^4+\cdots+x^{2n-2}}\,dx = \int_1^\infty \frac{x^{n-2}\ln\ln x}{1+x^2+x^4+\cdots+x^{2n-2}}\,dx \tag{17}$$

$$= \begin{cases} \dfrac{\pi}{2n}\operatorname{tg}\dfrac{\pi}{2n}\ln 2\pi + \dfrac{\pi}{n}\displaystyle\sum_{l=1}^{n-1}(-1)^{l-1}\sin\dfrac{\pi l}{n}\cdot\ln\left\{\dfrac{\Gamma\left(\frac{1}{2}+\frac{l}{2n}\right)}{\Gamma\left(\frac{l}{2n}\right)}\right\}, & n=2,4,6,\ldots \\[4ex] \dfrac{\pi}{2n}\operatorname{tg}\dfrac{\pi}{2n}\ln \pi + \dfrac{\pi}{n}\displaystyle\sum_{l=1}^{\frac{1}{2}(n-1)}(-1)^{l-1}\sin\dfrac{\pi l}{n}\cdot\ln\left\{\dfrac{\Gamma\left(1-\frac{l}{n}\right)}{\Gamma\left(\frac{l}{n}\right)}\right\}, & n=3,5,7,\ldots \end{cases}$$

see [40, p. 12], [41, p. 7] or [61, Table 191-5], [62, Table 148-4], [28, no. 4.325-9], and

$$\int_0^1 \frac{x^{n-2}\ln\ln\frac{1}{x}}{1-x^2+x^4-\cdots+x^{2n-2}}\,dx = \int_1^\infty \frac{x^{n-2}\ln\ln x}{1-x^2+x^4-\cdots+x^{2n-2}}\,dx \tag{18}$$

$$= \frac{\pi}{2n}\sec\frac{\pi}{2n}\cdot\ln\pi + \frac{\pi}{n}\cdot\sum_{l=1}^{\frac{1}{2}(n-1)}(-1)^{l-1}\cos\frac{(2l-1)\pi}{2n}\cdot\ln\left\{\frac{\Gamma\left(1-\frac{2l-1}{2n}\right)}{\Gamma\left(\frac{2l-1}{2n}\right)}\right\}$$

holding for $n = 3, 5, 7, \ldots$, see [40, p. 42, Eq. (48)] (the latter result does not appear in other sources). Particular cases of these formulas were rediscovered several times in different forms by various authors, see e.g. no. 3.1–3.2, 3.5–3.6[14] [2] or examples 7.3, 7.5 [44]. Moreover, Malmsten's integrals (1) and (2a), (2b) are themselves particular cases of the above integrals.

Malmsten and colleagues also treated some integrals having continuous powers of the logarithm in the numerator and denominator. Evidently, only in few cases could authors evaluate such integrals in a closed-form. However, such integrals permitted, *inter alia*, to obtain the $\ln\ln$-integrals as a derivative with respect to the power of the logarithm:

$$\int_0^1 \frac{P(x)\ln\ln\frac{1}{x}}{Q(x)}\,dx = \lim_{a\to 0}\left\{\frac{d}{da}\int_0^1 \frac{P(x)\ln^a\frac{1}{x}}{Q(x)}\,dx\right\},$$

where $P(x)$ and $Q(x)$ denote polynomials in x.

An important part of both Malmsten's works [40] and [41] is also devoted to certain logarithmic series, to series related to ζ-functions and to some infinite products. Among the results concerning logarithmic series, the most striking is, with no doubts, the evaluation of the series of the kind

$$\sum_{n=1}^{\infty} \frac{\sin an \cdot \ln n}{n}, \qquad 0 < a < 2\pi, \tag{19}$$

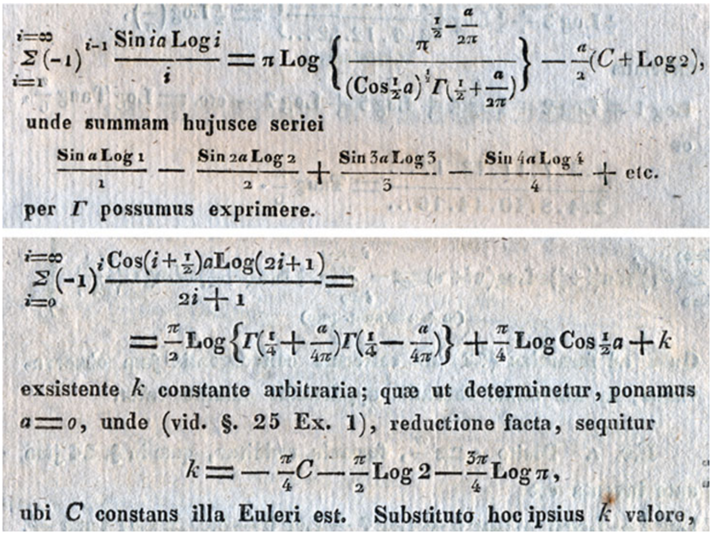
see Fig. 2 (for more details, see also exercise no. 20). This result, known as the Fourier series expansion for the logarithm of the Γ-function, is usually (and erroneously) attributed to Ernst Kummer who derived this expansion only in 1847, i.e. 5 years later. Another important result in the field of series concerns certain infinite sums related to ζ-functions. The famous reflection formula for the ζ-function

$$\zeta(1-s) = 2\zeta(s)\Gamma(s)(2\pi)^{-s}\cos\frac{\pi s}{2}, \qquad s \neq 0. \tag{20}$$

is well-known and is usually attributed to Riemann who derived it in 1859 [54], [19, p. 861], [31, p. 23], [71, p. 269]. At the same time, it is much less known that Malmsten and colleagues derived analogous relationships for two other "similar" series

$$\begin{cases} M(s) \equiv \dfrac{2}{\sqrt{3}}\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^s}\sin\frac{\pi n}{3}, & M(1-s) = \dfrac{2}{\sqrt{3}}M(s)\Gamma(s)3^s(2\pi)^{-s}\sin\dfrac{\pi s}{2}, \\ L(s) \equiv \displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^s}, & L(1-s) = L(s)\Gamma(s)2^s\pi^{-s}\sin\dfrac{\pi s}{2}, \end{cases} \tag{21}$$

[14] It seems, however, that there is a misprint in exercise no. 3.6 [2]. In the first line, the term x^5 should be removed from the denominator of the integrand.



$$\sum_{i=1}^{i=\infty}(-1)^{i-1}\frac{\operatorname{Sin} ia \operatorname{Log} i}{i} = \pi \operatorname{Log}\left\{\frac{\pi^{\frac{1}{2}-\frac{a}{2\pi}}}{(\operatorname{Cos}\frac{1}{2}a)^{\frac{1}{2}}\Gamma(\frac{1}{2}+\frac{a}{2\pi})}\right\} - \frac{a}{2}(C+\operatorname{Log} 2),$$

unde summam hujusce seriei

$$\frac{\operatorname{Sin} a \operatorname{Log} 1}{1} - \frac{\operatorname{Sin} 2a \operatorname{Log} 2}{2} + \frac{\operatorname{Sin} 3a \operatorname{Log} 3}{3} - \frac{\operatorname{Sin} 4a \operatorname{Log} 4}{4} + \text{etc.}$$

per Γ possumus exprimere.

$$\sum_{i=0}^{i=\infty}(-1)^{i}\frac{\operatorname{Cos}(i+\frac{1}{2})a \operatorname{Log}(2i+1)}{2i+1} =$$

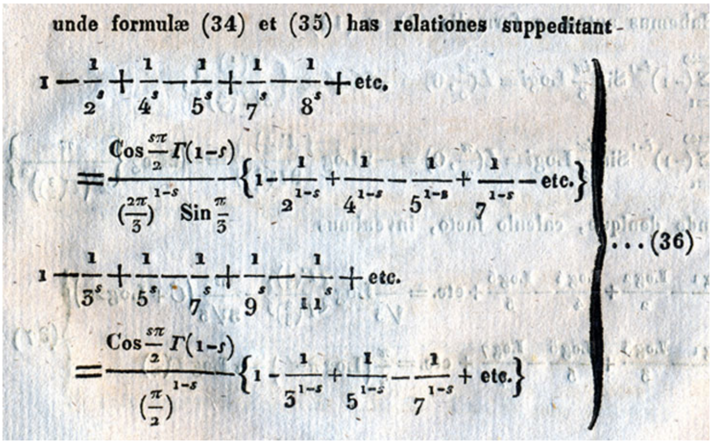
$$= \frac{\pi}{2}\operatorname{Log}\left\{\Gamma(\tfrac{1}{4}+\tfrac{a}{4\pi})\Gamma(\tfrac{1}{4}-\tfrac{a}{4\pi})\right\} + \frac{\pi}{4}\operatorname{Log}\operatorname{Cos}\tfrac{1}{2}a + k$$

exsistente k constante arbitraria; quæ ut determinetur, ponamus $a=0$, unde (vid. §. 25 Ex. 1), reductione facta, sequitur

$$k = -\frac{\pi}{4}C - \frac{\pi}{2}\operatorname{Log} 2 - \frac{3\pi}{4}\operatorname{Log}\pi,$$

ubi C constans illa Euleri est. Substituto hoc ipsius k valore,

Fig. 2 Fragments of pp. 62 (*top*) and 74 (*bottom*) from the Malmsten et al.'s dissertation [40], C designating the Euler's constant γ. Writing $a-\pi$ instead of a in the former series yields (19), which is the principal term in the Fourier series expansion of the logarithm of the Γ-function



unde formulæ (34) et (35) has relationes suppeditant

$$\left.\begin{aligned}
&1-\frac{1}{2^s}+\frac{1}{4^s}-\frac{1}{5^s}+\frac{1}{7^s}-\frac{1}{8^s}+\text{etc.}\\
&=\frac{\operatorname{Cos}\frac{s\pi}{2}\Gamma(1-s)}{\left(\frac{2\pi}{3}\right)^{1-s}\operatorname{Sin}\frac{\pi}{3}}\left\{1-\frac{1}{2^{1-s}}+\frac{1}{4^{1-s}}-\frac{1}{5^{1-s}}+\frac{1}{7^{1-s}}-\text{etc.}\right\}\\
&1-\frac{1}{3^s}+\frac{1}{5^s}-\frac{1}{7^s}+\frac{1}{9^s}-\frac{1}{11^s}+\text{etc.}\\
&=\frac{\operatorname{Cos}\frac{s\pi}{2}\Gamma(1-s)}{\left(\frac{\pi}{2}\right)^{1-s}}\left\{1-\frac{1}{3^{1-s}}+\frac{1}{5^{1-s}}-\frac{1}{7^{1-s}}+\text{etc.}\right\}
\end{aligned}\right\}\ldots(36)$$

Fig. 3 Bottom of p. 23 from the Malmsten et al.'s dissertation [40]. *Top*, we see the reflection formula for $M(s)$; *bottom*, for $L(s)$. Moreover, similar reflection formulas were later derived by Malmsten in [41] for other series similar to the η-function

$0 < s < 1$, already in 1842, see Fig. 3. The function $L(s)$ is directly related to the alternating Hurwitz ζ-function $L(s) = 2^{-s}\eta(s, \frac{1}{2})$, and therefore, Malmsten's functional equation may be rewritten as

$$\eta\left(1-s,\frac{1}{2}\right) = 2\eta\left(s,\frac{1}{2}\right)\Gamma(s)(2\pi)^{-s}\sin\frac{\pi s}{2},$$

which holds even when $s \to 0$, if the right part is regarded as a limit. The similitude to

(20) is striking, although $\eta(s, \frac{1}{2})$ is an independent transcendent of $\zeta(s)$. By the way, the above reflection formula (21) for $L(s)$ was also obtained by Oscar Schlömilch; in 1849 he presented it as an exercise for students [55], and then, in 1858, he published the proof [56]. Yet, it should be recalled that an analog of formula (20) for the alternating ζ-function $\eta(s)$ and formula (21) for $L(s)$ were already conjectured in 1749 by Euler, see pp. 94 and 105 of [20] respectively. However, Euler's results are usually not considered as rigorous proofs. Euler, first, studied the ratio $\eta(1-n)/\eta(n)$ for $n = 1, 2, \ldots, 10$. Then, by the method of mathematical induction, he conjectured that in general

$$\frac{\eta(1-n)}{\eta(n)} = -\frac{(2^n-1)(n-1)!}{(2^{n-1}-1)\pi^n}\cos\frac{\pi n}{2}, \qquad n = 1, 2, 3, \ldots \tag{22}$$

Next, Euler showed that this formula remains valid for negative n as well. Finally, he verified it analytically for $n = \frac{1}{2}$ and numerically for $n = \frac{3}{2}$.[15] As regards the function $L(s)$, Euler contented himself with the statement of the reflection formula (21) and added that it can be derived analogously. By the way, for Euler, formula (22) was not only interesting in itself, but was also a means by which could probably help in the closed-form evaluation of $\eta(n)$ for $n = 3, 5, 7, \ldots$ But since $\eta(1-n)$ for such n vanishes and so does $\cos\frac{1}{2}\pi n$, he faced (after the performance using l'Hôpital's rule) a more difficult series $\sum(-1)^{k-1}k^{n-1}\ln k$, $k \geqslant 1$. Note that the main advantage of Euler's (22) and Malmsten's (21) reflection formulas is that they can be verified numerically because in each both sides converge for $0 < s < 1$, while Riemann's formula (20) requires the notion of analytic continuation.

Malmsten and colleagues also studied integrals containing an arctangent with hyperbolic functions. These studies resulted in an interesting relationship between the Γ-function and the Ψ-function of a rational argument:

$$\Psi\left(\frac{m}{n}\right) = -\gamma - \ln 2\pi n - \frac{\pi}{2}\operatorname{ctg}\frac{\pi m}{n} - 2\sum_{l=1}^{n-1}\cos\frac{2\pi ml}{n}\cdot\ln\Gamma\left(\frac{l}{n}\right), \qquad m < n \tag{23}$$

where m and n are positive integers [40, p. 57, and see also pp. 56 and 72]. Malmsten and colleagues noticed that this relationship had not received sufficiently attention of mathematicians. On the one hand, the reader may remark that the right part of (23) may be easily transformed into elementary functions with the help of the reflection formula for the Γ-function. The pairwise summation of all terms in the sum over l (the first term with the last one, the second term with that before last, and so on) makes the Γ-functions in the right part totally vanish:

$$\begin{aligned} a_l + a_{n-l} &= \cos\frac{2\pi ml}{n}\cdot\left\{\ln\Gamma\left(\frac{l}{n}\right) + \ln\Gamma\left(1-\frac{l}{n}\right)\right\} \\ &= \cos\frac{2\pi ml}{n}\cdot\left\{\ln\pi - \ln\sin\frac{\pi l}{n}\right\}, \end{aligned} \tag{24}$$

[15] A sketch of the Euler's proof of formula (22) may be also found in [31, pp. 23–26].

where a_l denotes the lth term of the aforementioned sum.[16] In the case in which n is even, there will be one term which will not be concerned by the above simplification: the middle term $\ln\Gamma(1/2)$, but its value is well known and does not provide any additional information about the Γ-function. Consequently, formula (23) may be regarded as a variant of Gauss' digamma theorem.[17] On the other hand, (23) represents also a kind of discrete cosine transform (and more generally, a kind of finite Fourier transform), which have a huge quantity of applications in engineering, especially in signal processing and related disciplines. From this point of view, Malmsten was right when saying that these formulas were not sufficiently studied. In our work, we will derive many similar formulas (see exercises no. 3, 6, 9, 13, 11, 14, 25, 41, 46, 48, 49 in Sect. 4). In particular, Malmsten's integrals of the first order[18] can be often expanded in such a kind of finite Fourier series having logarithm of the Γ-function as coefficients. As a consequence, inverse transform may be also derived; the latter, *inter alia*, provides values of the Γ-function at rational argument as functions of Malmsten's integrals (see exercises no. 58–61). We will also give Parseval's theorem for such expansions (exercise no. 62). In addition, we will show that similar connections exist also for higher polygamma functions (exercise no. 48-b), as well as for Stieltjes constants (exercises no. 63–67).

Finally, we remark that in [40] some quantity of results were derived by means of divergent series,[19] but they were later re-obtained by Malmsten [41] by using other methods.

2.3 Malmsten's results and the Gradshteyn and Ryzhik's tables

The famous Gradshteyn and Ryzhik's tables [28] contains more than 20 formulas due to Malmsten and his colleagues. They were borrowed via Bierens de Haan's tables [62] and [61]. These are, for example, formulas no. 4.325-3,4,5,6,7,8,9, both formulas in no. 4.332, all formulas in no. 4.371 and 4.372, first three formulas in no. 4.373, formula no. 4.267-3 and some others. By the way, several errors crept into no. 4.372: in no. 4.372-1 and 4.372-2, the lower bound of both integrals should be 0 instead of 1. The same errors appear in Prudnikov et al.'s tables no. 2.6.29-1 and no. 2.6.29-2 [53, vol. I].[20] Also, the upper bound of the sum for the case "$m+n$ is even" $\frac{1}{2}(n-1)$ should be replaced by the integer part $\lfloor\frac{1}{2}(n-1)\rfloor$ (because when $m+n$ is even, n is not necessarily odd). Moreover, one should also add $m<n$ (otherwise, integrals on the left diverge). In no. 4.325-7, as showed in Malmsten's et al. works, parameter t should be in the range $(-\pi, +\pi)$. This is because the integral on the left is 2π-periodic, while the right part is not periodic; both parts coincide only if $t \in (-\pi, +\pi)$.

[16] We performed similar simplification for the Ψ-function in exercise no. 11, formula (48).

[17] Gauss presented the proof of this theorem in January 1812 [26].

[18] By the order of Malmsten's integral we mean the order of poles of the corresponding integrands.

[19] The use of divergent series was especially common in the 18th century, see, for instance, the excellent monograph [31].

[20] One of these two errors comes from the Bierens de Haan's tables (latter borrowed them in part from misprints in Malmsten's work [41]; Bierens de Haan even complained about the number of misprints in this work, see [61, p. 265]). For example, integrals' bounds are incorrect in [62, Table 148-1,2,3,4], [61, Table 191-1,2,3,4,5,6], [41, Eq. (10), (12) for both integrals]. The reader should be also careful with these sources since integrands in Gradshteyn and Ryzhik's tables are presented in other form.

Another error crept into no. 4.332-2. This integral is a rewritten version of (2a), and hence, its right-hand side must be exactly as in (2a), i.e. $\sqrt{2\pi}$ should be replaced by $\sqrt[3]{2\pi}$ (this misprint was previously pointed out by the authors of [44]). All these errors are present in the last 7th edition of the Gradshteyn and Ryzhik tables and in all other editions, including the original Russian edition.

3 The proposed method

3.1 Preliminaries

Malmsten and colleagues used for their derivations elementary functions and the Γ-function, without resorting to other special functions. The proposed method neither uses special functions, except the Γ-function, and is based on the theory of functions of a complex variable, and more precisely, on the contour integration technique. On the one hand, such a method turns out to be much more straightforward than Malmsten's method and, at the same time, it has wider applicability. On the other hand, it is simpler than most modern methods which tend to resort to "heavy" special functions. Moreover, the proposed method allows one to evaluate not only Malmsten's integrals from [40] and [41], but also many others (such results are given in Sect. 4).

The theory of functions of a complex variable, also known as *complex analysis*, is one of the most beautiful and useful branch of mathematics having many versatile applications. One such applications is the evaluation of integrals by means of Cauchy's residue theorem. This application is also known as the contour integration method. Cauchy's residue theorem states that for the function $f(z)$—which is analytic and single-valued inside and on a simple closed curve L except possibly for a finite number of isolated singularities—the contour integral

$$\oint\limits_L f(z)\,dz = \begin{cases} 0, & \text{if } f(z) \text{ has no singularities inside } L, \\ 2\pi i \sum\limits_{l=1}^{m} \operatorname*{res}\limits_{z=z_l} f(z), & \text{otherwise,} \end{cases} \tag{25}$$

where $\{z_l\}_{l=1}^m$ are the isolated singularities of the function $f(z)$ enclosed by the contour L. Technically, the application of the residue theorem for the evaluation of a given integral is done in two stages. First, by decomposition of the integration path L the line integral on the left in (25) is reduced to the evaluated integral (it can be done in many different ways; often, the latter appears as by-product). Then, one calculates the sum of the residues on the right in (25). The final result is obtained by equating both parts of (25). Numerous examples of such evaluations may be found in classical complex analysis literature [3, 16, 22, 23, 25, 32, 34, 42, 43, 59, 60, 69], [58, vol. III, part 2], [33, 65].

A major difficulty encountered when evaluating logarithmic integrals[21] by Cauchy's residue theorem consists in the following trade-off. On the one hand, the logarithm, being a typical multiple-valued function, has branch points, which should

[21] We will not consider here indirect methods, such as, for example, evaluation of logarithmic integrals based on the differentiation of the integrand (which does not contain a logarithm) with respect to a parameter.

not lie within the integration contour. On the other hand, not any integration path will lead to the wanted integral. In simple cases, e.g.,

$$\int\limits_0^\infty R(x)\ln^n x\,dx\,,\qquad \int\limits_0^\infty R(x)\ln\bigl(x^2+a^2\bigr)\,dx\,,\qquad \int\limits_0^\infty \frac{R(x)}{\ln^2 x+\pi^2}\,dx\,,$$

where $a>0$, $n\in\mathbb{N}$ and $R(x)$ denotes an arbitrary rational function of x (even in the first two cases), the evaluation may be succeed by considering respectively

$$\oint R(z)\ln^n(z)\,dz,\qquad \oint R(z)\ln(z+ia)\,dz,\qquad \oint \frac{R(z)}{\ln z-\pi i}\,dz,$$

taken around simple integration contours (e.g., semi-circle, circle with a cut) with the help of Jordan's lemma, see e.g. [59, pp. 187–188, 193–194, 197–198],[22] [69, pp. 129–132], [22, chapter VI, § 3], [23, pp. 281–296]. But the evaluation of more complicated logarithmic integrals may become a difficult task, because it may be very hard (or even impossible) to find an appropriate line integral. A typical example of such logarithmic integrals is the simplest Malmsten's integral (1). Consider, for example, its hyperbolic form. On the one hand, Jordan's lemma cannot be applied to this integral (it is sufficient to note that the integrand remains unbounded on the imaginary axis). On the other hand, the integrand has infinitely many poles in both semi-circles, which leads to an infinite series on the right part of (25). As regards the $\ln\ln$-form of Malmsten's integral (1), the integrand fulfills conditions of Jordan's lemma in both half-planes, but on the one hand, it has two branch points, 0 and 1, which should be properly indented, and on the other hand, the integral over $(-\infty,0]$ can be hardly reduced to that over $[0,+\infty)$. Thus, in order to evaluate such kinds of integrals, one may be led to consider unusual integration paths and more sophisticated forms of the integrand.

3.2 Introduction

Consider the following general family of logarithmic integrals:

$$\int\limits_{-\infty}^{+\infty} R\bigl(e^x\bigr)\ln\bigl(x^2+a^2\bigr)\,dx\,,\qquad a\in\mathbb{R}, \tag{26}$$

where $R(\cdot)$ denotes a rational function, and its particular case $a=0$

$$\int\limits_{-\infty}^{+\infty} R\bigl(e^x\bigr)\ln|x|\,dx=\int\limits_1^\infty \frac{R(u)+R(u^{-1})}{u}\ln\ln u\,du=\int\limits_0^1 \frac{R(y)+R(y^{-1})}{y}\ln\ln\frac{1}{y}\,dy\,,$$

[22] The readers of this book should beware of misprints and of some incorrect results. Answers in exercises no. 84, 88, 91 are incorrect; on the p. 189, $2\pi i$ is forgotten in the right part of equation (1). Several errors were corrected in the recent second edition of this book, but the few ones are still present, e.g. answer in no. 7.91 is incorrect, the above-mentioned coefficient $2\pi i$ is absent.

where we made two consecutive changes of variable $x = \ln u$ and $u = y^{-1}$. Denoting for brevity

$$Q(u) \equiv \frac{1}{u}\bigl\{R(u) + R\bigl(u^{-1}\bigr)\bigr\} \tag{27}$$

the last line takes the form

$$\int_{-\infty}^{+\infty} R\bigl(e^x\bigr)\ln|x|\,dx = \int_1^{\infty} Q(u)\ln\ln u\,du = \int_0^1 Q(y)\ln\ln\frac{1}{y}\,dy. \tag{28}$$

Thus, for any integral of the form (28), we may formulate the following statement: *if it is possible to find such a rational function $R(u)$ that the function $Q(u)$ may be represented in the form* (27)*, then last two integrals in* (28) *may be evaluated via integral* (26)*. The latter, as we come to see later, may be always expressed in terms of the Γ-function and its logarithmic derivatives.* Equation (27) implies also that function $Q(u)$ obeys the following functional relationship: $Q(u^{-1}) = u^2 Q(u)$. One of the consequences of this property is that the integrand (without $\ln\ln$ part) remains unchanged when bounds $[1, \infty)$ are replaced with $[0, 1]$. Consequently, the above statement may be reformulated as follows: *if the integrand from* (28) *(that of the integral in the middle) without* $\ln\ln$*-part, after a change of variable $u = y^{-1}$, remains invariant and only bounds* $[1, \infty)$ *are replaced with* $[0, 1]$*, i.e. if*

$$\int_1^{\infty} Q(u)\ln\ln u\,du = \int_0^1 Q(y)\ln\ln\frac{1}{y}\,dy,$$

then it can be always evaluated by the proposed method, and the result may be expressed in terms of the Γ-function and its logarithmic derivatives.

Now, one can remark that all $\ln\ln$-integrals that Malmsten and colleagues evaluated in [40] and [41] are particular cases of integrals (28) and fulfill the above condition on $Q(u)$. For example, the simplest Malmsten's integral (1) is obtained by taking $R(e^x) = \frac{1}{4}\operatorname{ch}^{-1}x$, which gives $Q(u) = 1/(1+u^2)$. By the way, it is curious that Malmsten did not notice that integrands of all his integrals obey $Q(u^{-1}) = u^2 Q(u)$. It is also obvious that if $Q(u^{-1}) \neq u^2 Q(u)$, then

$$\int_1^{\infty} Q(u)\ln\ln u\,du \neq \int_0^1 Q(y)\ln\ln\frac{1}{y}\,dy.$$

For instance, the following integral

$$\int_1^{\infty} \frac{\ln\ln u}{1+u}\,du \neq \int_0^1 \frac{\ln\ln\frac{1}{y}}{1+y}\,dy = \int_1^{\infty} \frac{\ln\ln u}{u^2+u}\,du = \int_0^{\infty} \frac{\ln x}{e^x+1}\,dx = -\frac{1}{2}\ln^2 2$$

which appears as no. 4.325-1 in [28], cannot be evaluated by the proposed method, at least directly. However, such kind of integrals can be often evaluated by other methods. For example, in some cases, the series expansions method turns out to be very useful for this aim, see e.g. exercises no. 18–19 in Sect. 4. There are also other methods that merit being mentioned in this context, but most of them resort to higher transcendences than the Γ-function. For instance, Adamchik [2] used the Hurwitz ζ-function and showed that if $Q(y)$ in (28) is a cyclotomic polynomial, then the corresponding integral may be always expressed in terms of derivatives of the Hurwitz ζ-function. Vardi [67], Vilceanu [68] and Bassett [8] suggested the use of the Dirichlet L-function. As regards Medina et al. [44], the authors evaluated these integrals with the help of polylogarithms.

3.3 The method for the evaluation of logarithmic and arctangent integrals

The Γ-function

$$\Gamma(z) = \int\limits_0^\infty t^{z-1} e^{-z}\, dt$$

is one of the oldest special functions which was introduced in analysis by Leonhard Euler. It was studied in detail by Euler himself, as well as by many other great mathematicians such as Carl Friedrich Gauss, Adrien-Marie Legendre, Karl Weierstrass, Otto Hölder and many others.[23] Classically, the Γ-function was defined only for positive values of its argument z, but it is now well known that it can be analytically continued to the entire complex plane except for simple poles at the points $0, -1, -2, \ldots$. In addition, since for each integer n, $\Gamma(n) = (n-1)! \neq 0$, by virtue of the reflection formula, one may easily establish that Γ-function has no zeros at all. As a consequence, $\ln\Gamma(z)$ has no branch points. From the recurrence relationship for the Γ-function $\Gamma(z+1) = z\,\Gamma(z)$, one can easily deduce an analogous functional relationship for the logarithm of the Γ-function, denoted $\Lambda(z)$ for brevity:

$$\Lambda(z+1) - \Lambda(z) = \ln z \tag{29}$$

It is therefore apparent that the use of the logarithm of the Γ-function may lead to the appearance of the logarithm. But why should one chose the logarithm of the Γ-function rather than simply the logarithm? The main advantage of the $\Lambda(z)$ function over the logarithm is that the former has no branch points at all, which allows one to use Cauchy's residue theorem with much less restriction to the choice of the integration contour.

Let $R(\cdot)$ be a real rational function of e^x, $x \in \mathbb{R}$, such that for some $\beta > 0$, we always have

$$\beta \ln\beta \cdot \max_{\varphi\in[0,2\pi]} \left|R\left(e^{\pm\beta+i\varphi}\right)\right| \to 0 \qquad \text{as} \quad \beta\to\infty. \tag{30}$$

Consider now the line integral

$$\ointctrclockwise_{L_\beta} R\left(e^z\right) \Lambda\left(\frac{z}{2\pi i} + \alpha\right) dz, \qquad \alpha \geqslant 0$$

[23] An interesting and quite well-written historical overview on the Γ-function is given in [19]. Motivated readers who are not afraid of French and German are also invited to take the look at these classic books [15, 27, 49] and [6].

taken around a rectangle with vertices at $[(-\beta,0),(+\beta,0),(+\beta,2\pi i),(-\beta,2\pi i)]$ designated by L_β. Bearing in mind that the positive direction is counterclockwise, the above contour integral may be split in four integrals as follows:

$$\oint\limits_{L_\beta} R\big(e^z\big)\,\Lambda\bigg(\frac{z}{2\pi i}+\alpha\bigg)\,dz=\int\limits_{-\beta}^{+\beta}\ldots\,dz+\int\limits_{\beta}^{\beta+2\pi i}\ldots\,dz+\int\limits_{\beta+2\pi i}^{-\beta+2\pi i}\ldots\,dz+\int\limits_{-\beta+2\pi i}^{-\beta}\ldots\,dz \tag{31}$$

where the integrands on the right were omitted for brevity. Now let $\beta\to\infty$. Taking all necessary precautions, the last equation becomes

$$\oint\limits_{L_\infty} R\big(e^z\big)\Lambda\bigg(\frac{z}{2\pi i}+\alpha\bigg)\,dz$$
$$=\lim_{\beta\to\infty}\left\{\int\limits_{-\beta}^{+\beta}\ldots\,dz+\int\limits_{\beta+2\pi i}^{-\beta+2\pi i}\ldots\,dz\right\}+\lim_{\beta\to\infty}\int\limits_{\beta}^{\beta+2\pi i}\ldots\,dz+\lim_{\beta\to\infty}\int\limits_{-\beta+2\pi i}^{-\beta}\ldots\,dz \tag{32}$$

Now, if (30) holds, then it can be shown that first limit in (32) converges to some finite non-zero quantity, while second and third limits equal zero. The latter may be proved in the following manner. We first notice that the behavior of the logarithm of the Γ-function in the sector $|\arg z|<\pi/2$ when $|z|\to\infty$ may be described by the following asymptotic formula[24]

$$\Lambda(z)=\bigg(z-\frac{1}{2}\bigg)\ln z-z+\frac{1}{2}\ln 2\pi+\underbrace{2\int\limits_0^\infty\frac{\operatorname{arctg}(x/z)}{e^{2\pi x}-1}\,dx}_{O(z^{-1})}=z\ln z+O(z), \tag{33}$$

see e.g. [42, vol. II, pp. 315–321], [39, pp. 88–89], [22, chapter VI, § 6], [1, no. 6.1.40–6.1.41, 6.1.43–6.1.44, 6.1.50], [53, vol. I, no. 2.7.5-6], [26, p. 33]. Hence, the integral in the second limit on the right in (32), after a change of variable $z=\beta+i\varphi$, may be estimated, for sufficiently large β, in the following manner:

$$\left|\int\limits_{\beta}^{\beta+2\pi i} R\big(e^z\big)\Lambda\bigg(\frac{z}{2\pi i}+\alpha\bigg)\,dz\right|\leqslant 2\pi\max_{\varphi\in[0,2\pi]}\left|R\big(e^{\beta+i\varphi}\big)\Lambda\bigg(\frac{\beta+i\varphi}{2\pi i}+\alpha\bigg)\right|$$
$$\leqslant 2\pi\cdot\max_{\varphi\in[0,2\pi]}\left|\Lambda\bigg(\frac{\beta+i\varphi}{2\pi i}+\alpha\bigg)\right|\cdot\max_{\varphi\in[0,2\pi]}\left|R\big(e^{\beta+i\varphi}\big)\right|$$
$$\sim\beta\ln\frac{\beta}{2\pi}\cdot\max_{\varphi\in[0,2\pi]}\left|R\big(e^{\beta+i\varphi}\big)\right|\to 0\qquad\text{as}\quad\beta\to\infty$$

[24] The non-asymptotic part of this formula (that containing an infinite integral) is also known as the second Binet's expression for the logarithm of the Γ-function [12, pp. 335–336], [71, pp. 250–251], [9, vol. I, p. 22, Eq. 1.9(9)] (for more details, see also exercise no. 40 in the last section of this manuscript). As regards its asymptotic form, it was already known to Gauss [26, p. 33], and in a more simple form (for natural z), to Euler [21, part II, Chap. VI, p. 466], to Stirling and to de Moivre.

thanks to condition (30). In like manner, we can show that the third limit in (32) vanishes also. Consider now the second integral in the right part of (32). By making a change of variable $z = x + 2\pi i$ and by noticing that $R(e^z)$ is a $2\pi i$-periodic function, this integral reduces to

$$\int\limits_{\beta+2\pi i}^{-\beta+2\pi i} R\big(e^z\big)\,\Lambda\!\left(\frac{z}{2\pi i}+\alpha\right)dz = -\int\limits_{-\beta}^{+\beta} R\big(e^x\big)\,\Lambda\!\left(\frac{x}{2\pi i}+\alpha+1\right)dx.$$

Equation (32) may be therefore rewritten as

$$\begin{aligned}
&\oint\limits_{L_\infty} R\big(e^z\big)\,\Lambda\!\left(\frac{z}{2\pi i}+\alpha\right)dz \\
&\qquad= \lim_{\beta\to\infty}\int\limits_{-\beta}^{+\beta} R\big(e^x\big)\left[\Lambda\!\left(\frac{x}{2\pi i}+\alpha\right) - \Lambda\!\left(\frac{x}{2\pi i}+\alpha+1\right)\right]dx \\
&\qquad= -\int\limits_{-\infty}^{+\infty} R\big(e^x\big)\ln\!\left(\frac{x}{2\pi i}+\alpha\right)dx \\
&\qquad= -\int\limits_{-\infty}^{+\infty} R\big(e^x\big)\ln(x+2\pi i\alpha)\,dx + \left(\ln 2\pi + \frac{\pi i}{2}\right)\cdot\underbrace{\int\limits_{-\infty}^{+\infty} R\big(e^x\big)\,dx}_{J_R}.
\end{aligned} \tag{34}$$

We note, in passing, that because condition (30) holds the first integral in the second line converges, and so does the integral J_R.

On the other hand, the left part of the last equation may be computed by the residue theorem

$$\begin{aligned}
&\oint\limits_{L_\infty} R\big(e^z\big)\,\Lambda\!\left(\frac{z}{2\pi i}+\alpha\right)dz \\
&\quad= 2\pi i\left\{\sum_{l=1}^{m}\operatorname*{res}_{z=z_l}\left[R\big(e^z\big)\,\Lambda\!\left(\frac{z}{2\pi i}+\alpha\right)\right] + \frac{1}{2}\sum_{l=1}^{\widetilde{m}}\operatorname*{res}_{z=\tilde{z}_l}\left[R\big(e^z\big)\,\Lambda\!\left(\frac{z}{2\pi i}+\alpha\right)\right]\right\},
\end{aligned} \tag{35}$$

where $\{z_l\}_{l=1}^{m}$ are the isolated singularities of the integrand lying within the strip $0 < \operatorname{Im} z < 2\pi$, and $\{\tilde{z}_l\}_{l=1}^{\widetilde{m}}$ are those whose imaginary part is exactly 0 or 2π (i.e., they lie on the integration path). By equating right-hand sides of (34) and of (35), we get

$$\int_{-\infty}^{+\infty} R(e^x)\ln(x+2\pi i\alpha)\,dx = J_R\left(\ln 2\pi + \frac{\pi i}{2}\right) \qquad (36)$$
$$-2\pi i\left\{\sum_{l=1}^{m}\operatorname*{res}_{z=z_l}\left[R(e^z)\Lambda\left(\frac{z}{2\pi i}+\alpha\right)\right]+\frac{1}{2}\sum_{l=1}^{\widetilde{m}}\operatorname*{res}_{z=\tilde{z}_l}\left[R(e^z)\Lambda\left(\frac{z}{2\pi i}+\alpha\right)\right]\right\}$$

Now, on taking real parts, we obtain

$$\int_{-\infty}^{+\infty} R(e^x)\ln(x^2+4\pi^2\alpha^2)\,dx = 2J_R\ln 2\pi$$
$$+4\pi\,\mathrm{Im}\left\{\sum_{l=1}^{m}\operatorname*{res}_{z=z_l}\left[R(e^z)\Lambda\left(\frac{z}{2\pi i}+\alpha\right)\right]+\frac{1}{2}\sum_{l=1}^{\widetilde{m}}\operatorname*{res}_{z=\tilde{z}_l}\left[R(e^z)\,\Lambda\left(\frac{z}{2\pi i}+\alpha\right)\right]\right\}$$

while equating imaginary parts yields

$$\int_{-\infty}^{+\infty} R(e^x)\,\mathrm{arctg}\left(\frac{2\pi\alpha}{x}\right)dx = \frac{\pi}{2}\int_{0}^{\infty}\left\{R(e^x)-R(e^{-x})\right\}dx$$
$$-2\pi\,\mathrm{Re}\left\{\sum_{l=1}^{m}\operatorname*{res}_{z=z_l}\left[R(e^z)\Lambda\left(\frac{z}{2\pi i}+\alpha\right)\right]+\frac{1}{2}\sum_{l=1}^{\widetilde{m}}\operatorname*{res}_{z=\tilde{z}_l}\left[R(e^z)\Lambda\left(\frac{z}{2\pi i}+\alpha\right)\right]\right\}$$

Rewriting the first equation with $\alpha = \frac{a}{2\pi}$ and the second one with $\alpha = \frac{1}{2\pi a}$, and recalling that $\mathrm{arctg}\,\frac{1}{x} = \frac{\pi}{2}\,\mathrm{sgn}\,x - \mathrm{arctg}\,x$ for any real x except zero, we arrive at integral (26)

$$\int_{-\infty}^{+\infty} R(e^x)\ln(x^2+a^2)\,dx = 2J_R\ln 2\pi \qquad (37)$$
$$+4\pi\,\mathrm{Im}\left\{\sum_{l=1}^{m}\operatorname*{res}_{z=z_l}\left[R(e^z)\ln\Gamma\left(\frac{z+ai}{2\pi i}\right)\right]+\frac{1}{2}\sum_{l=1}^{\widetilde{m}}\operatorname*{res}_{z=\tilde{z}_l}\left[R(e^z)\ln\Gamma\left(\frac{z+ai}{2\pi i}\right)\right]\right\}$$

and at another integral:

$$\int_{-\infty}^{+\infty} R(e^x)\,\mathrm{arctg}(ax)\,dx \qquad (38)$$
$$= 2\pi\,\mathrm{Re}\left\{\sum_{l=1}^{m}\operatorname*{res}_{z=z_l}\left[R(e^z)\ln\Gamma\left(\frac{za+i}{2\pi ai}\right)\right]+\frac{1}{2}\sum_{l=1}^{\widetilde{m}}\operatorname*{res}_{z=\tilde{z}_l}\left[R(e^z)\ln\Gamma\left(\frac{za+i}{2\pi ai}\right)\right]\right\}.$$

For the logarithmic integral, the particular case $a=0$ may be of special interest. From (37), we easily get

$$\int_{-\infty}^{+\infty} R\big(e^x\big)\ln|x|\,dx = J_R\ln 2\pi \tag{39}$$
$$+2\pi\,\mathrm{Im}\left\{\sum_{l=1}^{m}\operatorname*{res}_{z=z_l}\left[R\big(e^z\big)\ln\Gamma\left(\frac{z}{2\pi i}\right)\right]+\frac{1}{2}\sum_{l=1}^{\widetilde{m}}\operatorname*{res}_{z=\tilde{z}_l}\left[R\big(e^z\big)\ln\Gamma\left(\frac{z}{2\pi i}\right)\right]\right\}.$$

In contrast, for the arctangent integral, the limiting case $a\to\infty$ reveals to be more interesting. The fact that $\lim_{a\to\infty}\operatorname{arctg} ax=\frac{\pi}{2}\operatorname{sgn}x$ gives the opportunity to evaluate integrals of some odd functions over interval $[0,\infty)$. Making $a\to\infty$, we obtain from (38)

$$\int_{0}^{\infty}\big\{R\big(e^x\big)-R\big(e^{-x}\big)\big\}\,dx \tag{40}$$
$$=4\,\mathrm{Re}\left\{\sum_{l=1}^{m}\operatorname*{res}_{z=z_l}\left[R\big(e^z\big)\ln\Gamma\left(\frac{z}{2\pi i}\right)\right]+\frac{1}{2}\sum_{l=1}^{\widetilde{m}}\operatorname*{res}_{z=\tilde{z}_l}\left[R\big(e^z\big)\ln\Gamma\left(\frac{z}{2\pi i}\right)\right]\right\}.$$

If $R(e^x)$ is odd then $R(e^x)-R(e^{-x})=2R(e^x)$, while if $R(e^x)$ is even the last integral vanishes identically.

Formulas (37)–(40) allow one to compute many kind of different integrals containing logarithms, inverse trigonometric functions and many others. For example, these integrals

$$\int_{-\infty}^{+\infty}\frac{R(e^x)}{(x^2+a^2)^n}\,dx,\qquad \int_{-\infty}^{+\infty}\frac{x\,R(e^x)}{(x^2+b^2)^n}\,dx,\qquad b\equiv a^{-1},\ n\in\mathbb{N},$$

may be straightforwardly obtained from derived ones by a simple differentiation with respect to the parameter a. In addition, the evenness or oddness of $R(e^x)$ may simply calculations. Moreover, if the integral on the left diverges in the classical sense, in some cases, it can be still evaluated in the sense of the Cauchy principal value with the help of the above formulas. To illustrate these matters more vividly, the next section provides several beautiful examples of applications.

As to the integral J_R, it is difficult to treat the general case, so each integral must be considered individually. For example, it may be simply an elementary integral. Otherwise, its computation may be performed by the contour integration via

$$\oint R\big(e^z\big)\,dz$$

taken around an infinitely long rectangle of breadth π or 2π by the method analogous to that in (31). Such integrals are also exhaustively treated in [59, pp. 186, 197–198], [69, p. 132], [23, pp. 276–277], [25].

At the end of the demonstration, it may be of interest to remark that some quantity of exercises considering similar methods of contour integration involving the Γ-function and its derivatives are given in the excellent Russian book [23], which, unfortunately, was never translated into other languages, and for which reason it probably remains inaccessible to most readers. It seems also quite fair to say that the above exposition is much inspired from exercises no. 31.30–31.32. In exercise no. 31.32, the reader is asked to prove a simpler variant of formula (37) provided $R(e^x) \in L^1$. Regrettably, despite the high complexity of problems (higher than in the well-known collections [59] and [69]), this book does not contain solutions, nor even hints, only answers are provided at the end of each section.

3.4 Application to Malmsten's integrals

3.4.1 The simplest Malmsten's integral

Let's calculate Malmsten's integral (1) by means of formula (37). Since the function $R = \operatorname{ch}^{-1} x$, being a rational function of e^x, it easily satisfies (30). Besides, it is a meromorphic function having two simple poles within the strip $0 \leqslant \operatorname{Im} z \leqslant 2\pi$ at points $\pi i/2$ and $3\pi i/2$. Taking additionally into account that $\operatorname{ch}^{-1} x$ is even, (37) gives

$$\begin{aligned}\int\limits_0^\infty \frac{\ln(x^2+a^2)}{\operatorname{ch} x}\,dx &= 2\pi \operatorname{Im}\left\{\operatorname*{res}_{z=\frac{1}{2}\pi i} \frac{\ln\Gamma\left(\frac{z+ai}{2\pi i}\right)}{\operatorname{ch} z} + \operatorname*{res}_{z=\frac{3}{2}\pi i} \frac{\ln\Gamma\left(\frac{z+ai}{2\pi i}\right)}{\operatorname{ch} z}\right\} + \ln(2\pi) \underbrace{\int\limits_{-\infty}^{+\infty} \frac{dx}{\operatorname{ch} x}}_{\pi} \\ &= 2\pi\left\{\ln\Gamma\left(\frac{3}{4}+\frac{a}{2\pi}\right) - \ln\Gamma\left(\frac{1}{4}+\frac{a}{2\pi}\right)\right\} + \pi\ln 2\pi. \qquad (41)\end{aligned}$$

In the last line we may easily recognize Malmsten's general formula (14). Letting $a \to 0$, we arrive at (1) as follows:

$$\begin{aligned}\frac{1}{2}\int\limits_0^\infty \frac{\ln x}{\operatorname{ch} x}\,dx &= \frac{1}{4}\lim_{a\to 0}\int\limits_0^\infty \frac{\ln(x^2+a^2)}{\operatorname{ch} x}\,dx = \frac{\pi}{2}\left\{\ln\Gamma\left(\frac{3}{4}\right) - \ln\Gamma\left(\frac{1}{4}\right)\right\} + \frac{\pi}{4}\ln 2\pi \\ &= \frac{\pi}{2}\ln\left\{\frac{\Gamma(3/4)}{\Gamma(1/4)}\sqrt{2\pi}\right\} = \frac{\pi}{2}\ln 2 + \frac{3\pi}{4}\ln\pi - \pi\ln\Gamma\left(\frac{1}{4}\right), \qquad (42)\end{aligned}$$

where the final simplification is done with the help of the reflection formula for the Γ-function. The final result is, therefore, completely expressed in terms of mathematical constants.

3.4.2 Other Malmsten's integrals

Consider now another Malmsten's integral, mentioned by Vardi and others researchers as well, namely the integral (2b). After changes of variable $y = u^{-1}$,

$u = e^x$, it can be rewritten as

$$\int_0^1 \frac{\ln\ln\frac{1}{y}}{1-y+y^2}\,dy = \int_1^\infty \frac{\ln\ln u}{1-u+u^2}\,du = \int_0^\infty \frac{\ln x}{2\operatorname{ch} x - 1}\,dx$$

Poles of the latter integrand in the strip $0 \leqslant \operatorname{Im} z \leqslant 2\pi$ are simple and occur at points $\pi i/3$ and $5\pi i/3$. Application of formula (37) yields

$$\int_0^\infty \frac{\ln(x^2+a^2)}{2\operatorname{ch} x - 1}\,dx = 2\pi \operatorname{Im}\left\{\operatorname*{res}_{z=\frac{1}{3}\pi i} \frac{\ln\Gamma(\frac{z+ai}{2\pi i})}{2\operatorname{ch} z - 1} + \operatorname*{res}_{z=\frac{5}{3}\pi i} \frac{\ln\Gamma(\frac{z+ai}{2\pi i})}{2\operatorname{ch} z - 1}\right\} \tag{43}$$

$$+\ln(2\pi)\underbrace{\int_{-\infty}^{+\infty} \frac{dx}{2\operatorname{ch} x - 1}}_{\frac{4\pi}{3\sqrt{3}}} = \frac{2\pi}{\sqrt{3}}\left\{\ln\Gamma\left(\frac{5}{6}+\frac{a}{2\pi}\right) - \ln\Gamma\left(\frac{1}{6}+\frac{a}{2\pi}\right) + \frac{2\ln 2\pi}{3}\right\}$$

Letting $a \to 0$ yields

$$\int_0^\infty \frac{\ln x}{2\operatorname{ch} x - 1}\,dx = \frac{1}{2}\lim_{a\to 0}\int_0^\infty \frac{\ln(x^2+a^2)}{2\operatorname{ch} x - 1}\,dx$$

$$= \frac{\pi}{\sqrt{3}}\left\{\ln\Gamma\left(\frac{5}{6}\right) - \ln\Gamma\left(\frac{1}{6}\right) + \frac{2\ln 2\pi}{3}\right\}$$

$$= \frac{\pi}{\sqrt{3}}\ln\left\{\frac{\Gamma(5/6)}{\Gamma(1/6)}\sqrt[3]{4\pi^2}\right\} = \frac{2\pi}{\sqrt{3}}\left\{\frac{5}{6}\ln 2\pi - \ln\Gamma\left(\frac{1}{6}\right)\right\},$$

which is identical with (2b). Again, at the last stage, the reflection formula was employed. Finally, in view of the fact that $\Gamma(1/6) = 3^{\frac{1}{2}}2^{-\frac{1}{3}}\pi^{-\frac{1}{2}}\Gamma^2(1/3)$, see e.g. [15, p. 31], the latter formula may be written in terms of $\Gamma(1/3)$, and hence

$$\int_0^1 \frac{\ln\ln\frac{1}{y}}{1-y+y^2}\,dy = \int_1^\infty \frac{\ln\ln u}{1-u+u^2}\,du = \int_0^\infty \frac{\ln x}{2\operatorname{ch} x - 1}\,dx$$

$$= \frac{\pi}{3\sqrt{3}}\left\{7\ln 2 + 8\ln\pi - 3\ln 3 - 12\ln\Gamma\left(\frac{1}{3}\right)\right\}. \tag{44}$$

Analogously, it can be shown that

$$\int_0^1 \frac{\ln\ln\frac{1}{y}}{1+y+y^2}\,dy = \int_1^\infty \frac{\ln\ln u}{1+u+u^2}\,du = \int_0^\infty \frac{\ln x}{2\operatorname{ch} x + 1}\,dx$$

$$= \frac{\pi}{6\sqrt{3}}\left\{8\ln 2\pi - 3\ln 3 - 12\ln\Gamma\left(\frac{1}{3}\right)\right\}. \tag{45}$$

The unique difference between integrals (44) and (45) is the location of poles of integrands. For the former, they occur (for the hyperbolic form) in the strip $0 \leqslant \operatorname{Im} z \leqslant 2\pi$ at $\pi i/3$ and $5\pi i/3$, while for the latter they occur at $2\pi i/3$ and $4\pi i/3$.

Another frequently encountered Malmsten's integral (however, not mentioned by Vardi) is this one:

$$\int_0^1 \frac{\ln\ln\frac{1}{y}}{(1+y)^2}\,dy = \int_1^\infty \frac{\ln\ln u}{(1+u)^2}\,du = \frac{1}{4}\int_0^\infty \frac{\ln x}{\operatorname{ch}^2(\frac{1}{2}x)}\,dx = \frac{1}{2}\int_0^\infty \frac{\ln x}{\operatorname{ch} x+1}\,dx \tag{46}$$

see e.g. [41, p. 24], [62, Table 147-7, 257-4], [61, Table 190-7], [28, no. 4.325-3]. It can be also evaluated by the proposed method. In the strip $0 \leqslant \operatorname{Im} z \leqslant 2\pi$ the integrand has one double pole at $z = \pi i$. Proceeding as above, we find that

$$\begin{aligned}\int_0^\infty \frac{\ln(x^2+a^2)}{\operatorname{ch} x+1}\,dx &= 2\pi \operatorname{Im}\left\{\operatorname*{res}_{z=\pi i} \frac{\ln\Gamma(\frac{z+ai}{2\pi i})}{\operatorname{ch} z+1}\right\} + \ln(2\pi)\underbrace{\int_{-\infty}^{+\infty}\frac{dx}{\operatorname{ch} x+1}}_{2} \\ &= 2\left\{\Psi\left(\frac{1}{2}+\frac{a}{2\pi}\right) + \ln 2\pi\right\}.\end{aligned} \tag{47}$$

When a tends to zero, we have

$$\begin{aligned}\frac{1}{2}\int_0^\infty \frac{\ln x}{\operatorname{ch} x+1}\,dx &= \frac{1}{4}\lim_{a\to 0}\int_0^\infty \frac{\ln(x^2+a^2)}{\operatorname{ch} x+1}\,dx = \frac{1}{2}\left\{\Psi\left(\frac{1}{2}\right) + \ln 2\pi\right\} \\ &= \frac{1}{2}\left\{-\gamma + \ln\frac{\pi}{2}\right\},\end{aligned}$$

which completes the evaluation of (46). Other Malmsten's integrals from [40] and [41] may be evaluated similarly.

4 New results, problems and exercises

This section is designed as a collection of original exercises to be worked out by the readers. Exercises marked with an * contain new results which were never, to our knowledge, released before (except if otherwise stated). There are also some known results which were historically obtained by other methods. For such problems original sources of the formulas are provided. The results are presented in a quite general form, which is why most of the formulas contain different parameters with respect to which they can be differentiated or integrated.

4.1 Logarithmic integrals containing hyperbolic functions

4.1.1 Main results obtained by the contour integration method

1 By using formula (37), verify that for any $a \geqslant 0$ and $\operatorname{Re} b > 0$

$$\int_0^\infty \frac{\ln(x^2+a^2)}{\operatorname{ch} bx}\,dx = \frac{2\pi}{b}\left\{\ln\Gamma\left(\frac{3}{4}+\frac{ab}{2\pi}\right) - \ln\Gamma\left(\frac{1}{4}+\frac{ab}{2\pi}\right) + \frac{1}{2}\ln\frac{2\pi}{b}\right\}$$

Hint: Make a suitable change of variable in (41).

Nota bene: This formula for $b > 0$ can be found in Gradshteyn and Ryzhik's tables [28, no. 4.373-1].

2* Prove by the contour integration method the following formula

$$\int_0^\infty \frac{\ln(x^2+a^2)}{\operatorname{ch} bx + \cos\varphi}\,dx = \frac{2\pi}{b\sin\varphi}\left\{\ln\Gamma\left(\frac{1}{2}+\frac{ab+\varphi}{2\pi}\right) - \ln\Gamma\left(\frac{1}{2}+\frac{ab-\varphi}{2\pi}\right) + \frac{\varphi}{\pi}\ln\frac{2\pi}{b}\right\}$$

$a \geqslant 0$, $\operatorname{Re} b > 0$, $|\operatorname{Re}\varphi| < \pi$, $\varphi \neq 0$; for $\varphi = 0$, see formula (47) and exercise no. 6. Note that this formula remains valid even for complex values of φ. If φ is imaginary pure, then $\varphi = it$, $t \in \mathbb{R}$, and the denominator of the integral takes the form $\operatorname{ch} x + \operatorname{ch} t$. Such integrals can be always evaluated via the above formula. Moreover, even if φ lies outside the vertical strip $|\operatorname{Re}\varphi| < \pi$, the integral can be still computed in the sense of the Cauchy principal value. Note also that in the particular case $\varphi = \pm\pi/2$, the above formula reduces to that obtained in the previous exercise.

Hint: By considering

$$\oint\limits_{0 \leqslant \operatorname{Im} z \leqslant 2\pi} \frac{e^{-i\alpha z}}{\operatorname{ch} z + \cos\varphi}\,dz, \qquad \text{prove first}$$

$$\int_{-\infty}^{+\infty} \frac{\cos\alpha x}{\operatorname{ch} x + \cos\varphi}\,dx = \frac{2\pi}{\sin\varphi}\cdot\frac{\operatorname{sh}\varphi\alpha}{\operatorname{sh}\pi\alpha}, \qquad \begin{cases} |\operatorname{Im}\alpha| < 1, \\ |\operatorname{Re}\varphi| < \pi, \ \varphi \neq 0. \end{cases}$$

Then, let $\alpha \to 0$. Note that in exercise no. 28.19-4 [23], where the last integral also appears, there is a slight inaccuracy concerning the domain of convergence: it is defined only for $\operatorname{Im}\varphi > 0$.

Nota bene: Malmsten [41, p. 24] discovered a particular case of this formula for $a = 0$, $b = 1$ and real $\varphi \in (-\pi, +\pi)$. Formula (63) on the p. 24 in [41] is actually a rewritten version of such a particular case (see also [61, Table 274-12 ⇒ Table 190-9], [62, Table 257-7 ⇒ Table 147-9], [28, no. 4.371-2]). The same particular case was independently rediscovered by Medina et al. in [44]. As regards the general formula given above, it seems to be not new.

3* By the contour integration method, prove that if p is a discrete parameter chosen so that $p = bm/n$, where $\operatorname{Re} b > 0$ and numbers m and n are positive integers such that $m < n$, then for any $a \geqslant 0$, one always has

$$\text{(a)} \quad \int_0^\infty \frac{\operatorname{sh} px \cdot \ln(x^2+a^2)}{\operatorname{sh} bx}\,dx = \frac{\pi}{b}\operatorname{tg}\frac{m\pi}{2n}\cdot\ln\frac{2\pi n}{b} + \frac{2\pi}{b}\sum_{l=1}^{2n-1}(-1)^l \sin\frac{m\pi l}{n}\cdot\ln\Gamma\left(\frac{l}{2n}+\frac{ab}{2\pi n}\right)$$

$$\text{(b)} \quad \int_0^\infty \frac{\operatorname{ch} px \cdot \ln(x^2+a^2)}{\operatorname{ch} bx}\,dx = \frac{\pi}{b}\sec\frac{m\pi}{2n}\cdot\ln\frac{2\pi n}{b} - \frac{2\pi}{b}\sum_{l=0}^{2n-1}(-1)^l \cos\frac{(2l+1)m\pi}{2n}\cdot\ln\Gamma\left(\frac{2l+1}{4n}+\frac{ab}{2\pi n}\right)$$

$$\text{(c)} \quad \int_0^\infty \frac{\operatorname{ch} px \cdot \ln(x^2+a^2)}{\operatorname{ch} bx + \cos\varphi}\,dx = \frac{2\pi}{b}\sin\frac{m\varphi}{n}\cdot\csc\varphi\cdot\csc\frac{m\pi}{n}\cdot\ln\frac{2\pi n}{b} + \frac{2\pi}{b\sin\varphi}\sum_{l=0}^{n-1}\left\{\cos\frac{(2l+1)m\pi+m\varphi}{n}\cdot\ln\Gamma\left(\frac{2l+1}{2n}+\frac{ab+\varphi}{2\pi n}\right) - \cos\frac{(2l+1)m\pi-m\varphi}{n}\cdot\ln\Gamma\left(\frac{2l+1}{2n}+\frac{ab-\varphi}{2\pi n}\right)\right\}$$

$$\text{(d)} \quad \int_0^\infty \frac{\operatorname{ch} px \cdot \ln(x^2+a^2)}{\operatorname{ch} bx + 1}\,dx = \frac{2\pi m}{bn}\csc\frac{m\pi}{n}\cdot\ln\frac{2\pi n}{b} - \frac{4\pi m}{bn}\sum_{l=0}^{n-1}\sin\frac{(2l+1)m\pi}{n}\cdot\ln\Gamma\left(\frac{2l+1}{2n}+\frac{ab}{2\pi n}\right) + \frac{2}{bn}\sum_{l=0}^{n-1}\cos\frac{(2l+1)m\pi}{n}\cdot\Psi\left(\frac{2l+1}{2n}+\frac{ab}{2\pi n}\right)$$

where in (c) $|\operatorname{Re}\varphi| < \pi$, $\varphi \neq 0$; for $\varphi = 0$, see (d). For continuous and complex values of p, see no. 63–64.

Hint: As regards exercise (a), first, put for simplicity $b = 1$ and rewrite the integral for $x = yn$ as follows:

$$\int_0^\infty \frac{\operatorname{sh}(mx/n)\cdot\ln(x^2+a^2)}{\operatorname{sh} x}\,dx = n\int_0^\infty \frac{\operatorname{sh} my\cdot\ln(y^2n^2+a^2)}{\operatorname{sh} ny}\,dy$$

$$= 2n\ln n\int_0^\infty \frac{\operatorname{sh} my}{\operatorname{sh} ny}\,dy + n\int_0^\infty \frac{\operatorname{sh} my\cdot\ln(y^2+(a/n)^2)}{\operatorname{sh} ny}\,dy.$$

Then, by noticing that the integrand of the last integral on the right is a rational function of e^y and, hence, is $2\pi i$-periodic, apply formula (38). When evaluating residues, do not forget that function $\dfrac{\operatorname{sh} mz}{\operatorname{sh} nz}$ has removable singularities at $z = 0$, $z = \pi i$, $z = 2\pi i$ and poles of the first order at $z = i\pi l/n$, $l = 1, 2, 3, \ldots, n-1, n+1, \ldots, 2n-1$. Integrals (b)–(d) are evaluated similarly.

Nota bene: Integrals (a) and (b) for $b = \pi$, as we explained in the first part of our work, were evaluated by Malmsten and colleagues, see expressions (13) and (15), respectively. Their formulas slightly differ from ours because they separately considered cases $m + n$ odd and $m + n$ even. Besides, both formulas (13) and (15) can be further simplified. Integrals (c) and (d) seem to be not evaluated previously. Formula (c) is an important general formula from which several particular cases may be derived via an appropriate limiting procedure. For example, integral d), as well as formulas in exercises no. 6b–d may be obtained in this way.

4* Prove by the contour integration method the following general formulas

$$\text{(a)}\quad \text{p.v.}\int_{-\infty}^{+\infty} \frac{\ln(x^2+a^2)}{\operatorname{sh} x \pm \operatorname{sh} t}\, dx = \pm\frac{2\pi}{\operatorname{ch} t}\left[\frac{\pi}{2} - \operatorname{arctg}\frac{a}{t}\right] \pm \frac{4t\ln 2\pi}{\operatorname{ch} t}$$
$$\pm\frac{4\pi}{\operatorname{ch} t}\operatorname{Im}\left\{\ln\Gamma\left(\frac{a}{2\pi} + \frac{it}{2\pi}\right) - \ln\Gamma\left(\frac{1}{2} + \frac{a}{2\pi} - \frac{it}{2\pi}\right)\right\}$$

$$\text{(b)}\quad \int_{-\infty}^{+\infty} \frac{\ln(x^2+1-t^2)}{\operatorname{sh} x \pm \operatorname{sh} t}\, dx = \pm\frac{2\pi}{\operatorname{ch} t}\left[\frac{\pi}{2} - \operatorname{arctg}\frac{\sqrt{1-t^2}}{t}\right] \pm \frac{4t\ln 2\pi}{\operatorname{ch} t}$$
$$\pm\frac{4\pi}{\operatorname{ch} t}\operatorname{Im}\left\{\ln\Gamma\left(\frac{\sqrt{1-t^2}}{2\pi} + \frac{it}{2\pi}\right) - \ln\Gamma\left(\frac{1}{2} + \frac{\sqrt{1-t^2}}{2\pi} - \frac{it}{2\pi}\right)\right\}$$

By the way, formula (b) may be also written in a slightly different form:

$$\text{(b}^\star\text{)}\quad \int_{0}^{\infty} \frac{\ln(x^2+1-t^2)}{\operatorname{sh}^2 t - \operatorname{sh}^2 x}\, dx = \frac{2\pi}{\operatorname{sh} 2t}\left[\frac{\pi}{2} - \operatorname{arctg}\frac{\sqrt{1-t^2}}{t}\right] + \frac{4t\ln 2\pi}{\operatorname{sh} 2t}$$
$$+\frac{4\pi}{\operatorname{sh} 2t}\operatorname{Im}\left\{\ln\Gamma\left(\frac{\sqrt{1-t^2}}{2\pi} + \frac{it}{2\pi}\right) - \ln\Gamma\left(\frac{1}{2} + \frac{\sqrt{1-t^2}}{2\pi} - \frac{it}{2\pi}\right)\right\}$$

where in (a) $a \geqslant 0,\ t > 0$, and in (b) and (b*) $0 < t \leqslant 1$.

Hint: By considering

$$\oint\limits_{0\leqslant \operatorname{Im} z\leqslant 2\pi} \frac{e^{-i\alpha z}}{\operatorname{sh} z + \operatorname{sh} t}\, dz, \qquad \text{prove first}$$

$$\text{p.v.}\int_{-\infty}^{+\infty}\frac{e^{-i\alpha x}}{\operatorname{sh}x+\operatorname{sh}t}\,dx=\frac{\pi i}{\operatorname{ch}t\cdot\operatorname{sh}\alpha\pi}\left\{e^{-i\alpha t}-e^{i\alpha t}\operatorname{ch}\alpha\pi\right\},$$

provided $|\operatorname{Im}\alpha|<1$ and $-\infty<t<\infty$. Then, let $\alpha\to 0$.[25]

Nota bene: In spite of the fact that some integrals in this exercise may be evaluated only in the sense of the Cauchy principal value, their evaluation is not as useless as it may first appear. In fact, by an appropriate choice of the numerator of the integrand, it is often possible to get rid of the p.v. sign. For instance, with the help of the last formula, we may arrive at these beautiful and quite non-trivial[26] convergent integrals

$$\text{(c)}\quad \int_{-\infty}^{+\infty}\frac{x+t}{\operatorname{sh}x+\operatorname{sh}t}\,dx=2\int_{0}^{\infty}\frac{x\operatorname{sh}x-t\operatorname{sh}t}{\operatorname{sh}^2x-\operatorname{sh}^2t}\,dx=\frac{\pi^2+4t^2}{2\operatorname{ch}t},$$

$$\text{(d)}\quad \int_{-\infty}^{+\infty}\frac{x^2-t^2}{\operatorname{sh}x+\operatorname{sh}t}\,dx=-\frac{t(\pi^2+4t^2)}{3\operatorname{ch}t},$$

$$\text{(e)}\quad \int_{0}^{\infty}\frac{x^2-t^2}{\operatorname{sh}^2x-\operatorname{sh}^2t}\,dx=\frac{t(\pi^2+4t^2)}{3\operatorname{sh}2t},$$

$$\text{(f)}\quad \int_{-\infty}^{+\infty}\frac{\sin\alpha x+\sin\alpha t}{\operatorname{sh}x+\operatorname{sh}t}\,dx=2\int_{0}^{\infty}\frac{\sin\alpha x\cdot\operatorname{sh}x-\sin\alpha t\cdot\operatorname{sh}t}{\operatorname{sh}^2x-\operatorname{sh}^2t}\,dx$$
$$=\frac{1}{\operatorname{ch}t}\left\{2t\sin\alpha t+\pi\operatorname{th}\left(\frac{\alpha\pi}{2}\right)\cos\alpha t\right\},$$

$$\text{(g)}\quad \int_{-\infty}^{+\infty}\frac{\cos\alpha x-\cos\alpha t}{\operatorname{sh}x+\operatorname{sh}t}\,dx=\frac{1}{\operatorname{ch}t}\left\{\pi\operatorname{cth}\left(\frac{\alpha\pi}{2}\right)\sin\alpha t-2t\cos\alpha t\right\},$$

$$\text{(h)}\quad \int_{0}^{\infty}\frac{\cos\alpha x-\cos\alpha t}{\operatorname{sh}^2x-\operatorname{sh}^2t}\,dx=\frac{1}{\operatorname{sh}2t}\left\{2t\cos\alpha t-\pi\operatorname{cth}\left(\frac{\alpha\pi}{2}\right)\sin\alpha t\right\}.$$

The above results hold for any $t\in(-\infty,+\infty)$ and α lying within the strip $|\operatorname{Im}\alpha|<1$. The reader is also asked to prove these formulas as an exercise.

[25] For the evaluation of the integral J_R lazy readers may directly use formula 1.4.7-15 from [53, vol. I, p. 148]. Nevertheless, it is highly recommended that readers employ the proposed method rather than the ready formula, since the procedure for the calculation of the logarithmic integral is very similar.

[26] For instance, we have not found these integrals in [28], neither in [53].

5* By using results of the previous exercise, prove that

$$\int\limits_{-\infty}^{+\infty} \frac{\ln|x| - \ln t}{\operatorname{sh} x - \operatorname{sh} t}\, dx = -\frac{2\pi}{\operatorname{ch} t} \operatorname{Im}\left\{\ln\left(\frac{it}{2\pi}\right) - \ln\Gamma\left(\frac{1}{2} - \frac{it}{2\pi}\right)\right\}$$

$$-\frac{\pi^2}{2\operatorname{ch} t} + \frac{2t}{\operatorname{ch} t} \ln\frac{t}{2\pi}$$

and

$$\int\limits_{0}^{\infty} \frac{\ln x - \ln t}{\operatorname{sh}^2 x - \operatorname{sh}^2 t}\, dx = -\frac{2\pi}{\operatorname{sh} 2t} \operatorname{Im}\left\{\ln\Gamma\left(\frac{it}{2\pi}\right) - \ln\Gamma\left(\frac{1}{2} - \frac{it}{2\pi}\right)\right\} - \frac{\pi^2}{2\operatorname{sh} 2t}$$

$$+\frac{2t}{\operatorname{sh} 2t} \ln\frac{t}{2\pi}$$

provided $t > 0$.

6* By using Cauchy's residue theorem, prove that for any $a \geqslant 0$, $\operatorname{Re} b > 0$ and $p = bm/n$, numbers m and n being positive integers, the following formulas hold

$$\text{(a)} \qquad \int\limits_{0}^{\infty} \frac{\ln(x^2 + a^2)}{\operatorname{ch}^2 bx}\, dx = \frac{2}{b}\left\{\ln\frac{\pi}{b} + \Psi\left(\frac{1}{2} + \frac{ab}{\pi}\right)\right\}.$$

If the product ab is a rational part of π or a is zero, then the above integral may be always expressed in terms of elementary functions and Euler's constant γ (in virtue of Gauss' digamma theorem). For example:

$$\int\limits_{0}^{\infty} \frac{\ln x}{\operatorname{ch}^2 x}\, dx = \int\limits_{0}^{1} \ln \operatorname{arcth} x\, dx = \ln\frac{\pi}{4} - \gamma,$$

$$\int\limits_{0}^{\infty} \frac{\ln(x^2 + \pi^2)}{\operatorname{ch}^2 x}\, dx = 2\ln\frac{\pi}{4} + 4 - 2\gamma$$

etc.

$$\text{(b)} \qquad \int\limits_{0}^{\infty} \frac{\operatorname{ch} px \cdot \ln(x^2 + a^2)}{\operatorname{ch}^2 bx} dx = \frac{\pi m}{bn} \cdot \csc\frac{m\pi}{2n} \cdot \ln\frac{2\pi n}{b}$$

$$-\frac{2\pi m}{bn} \sum_{l=0}^{2n-1} \sin\frac{(2l+1)m\pi}{2n} \cdot \ln\Gamma\left(\frac{2l+1}{4n} + \frac{ab}{2\pi n}\right)$$

$$+\frac{1}{bn} \sum_{l=0}^{2n-1} \cos\frac{(2l+1)m\pi}{2n} \cdot \Psi\left(\frac{2l+1}{4n} + \frac{ab}{2\pi n}\right)$$

$$\text{(c)}\quad \int_0^\infty \frac{\operatorname{ch}^2 px \cdot \ln(x^2+a^2)}{\operatorname{ch}^2 bx}\,dx = \frac{\pi m}{bn}\cdot\csc\frac{m\pi}{n}\cdot\ln\frac{\pi n}{b}$$

$$-\frac{2\pi m}{bn}\sum_{l=0}^{n-1}\sin\frac{(2l+1)m\pi}{n}\cdot\ln\Gamma\left(\frac{2l+1}{2n}+\frac{ab}{\pi n}\right)$$

$$+\frac{1}{bn}\sum_{l=0}^{n-1}\cos\frac{(2l+1)m\pi}{n}\cdot\Psi\left(\frac{2l+1}{2n}+\frac{ab}{\pi n}\right)+\frac{1}{b}\left\{\ln\frac{\pi}{b}+\Psi\left(\frac{1}{2}+\frac{ab}{\pi}\right)\right\}$$

$$\text{(d)}\quad \int_0^\infty \frac{\operatorname{sh}^2 px \cdot \ln(x^2+a^2)}{\operatorname{ch}^2 bx}\,dx = \frac{\pi m}{bn}\cdot\csc\frac{m\pi}{n}\cdot\ln\frac{\pi n}{b}$$

$$-\frac{2\pi m}{bn}\sum_{l=0}^{n-1}\sin\frac{(2l+1)m\pi}{n}\cdot\ln\Gamma\left(\frac{2l+1}{2n}+\frac{ab}{\pi n}\right)$$

$$+\frac{1}{bn}\sum_{l=0}^{n-1}\cos\frac{(2l+1)m\pi}{n}\cdot\Psi\left(\frac{2l+1}{2n}+\frac{ab}{\pi n}\right)-\frac{1}{b}\left\{\ln\frac{\pi}{b}+\Psi\left(\frac{1}{2}+\frac{ab}{\pi}\right)\right\}$$

where in (b) $m < 2n$, and in (c)–(d) $m < n$.

Hint: For the integral (b), see exercise no. 3. As regards last the two integrals, they may be obtained by a linear combination of integrals (a) and (b). In last two cases, at the final stage, split both sums over $l = 0, 1, 2, \ldots, 2n-1$ into two sums of equal lengths and use duplication formulas in order to simplify the result.

7* Show that for any $a \geqslant 0$ and $\operatorname{Re} b > 0$

$$\text{(a)}\quad \int_0^\infty \frac{\ln(x^2+a^2)}{2\operatorname{ch}^2 bx+1}\,dx = \frac{\pi i}{b\sqrt{3}}\ln\frac{\Gamma\left(\frac{1}{2}+\frac{ab}{\pi}+\frac{\ln(2+\sqrt{3})}{2\pi i}\right)}{\Gamma\left(\frac{1}{2}+\frac{ab}{\pi}-\frac{\ln(2+\sqrt{3})}{2\pi i}\right)}$$

$$+\frac{\ln(2+\sqrt{3})}{b\sqrt{3}}\ln\frac{\pi}{b}$$

$$\text{(b)}\quad \int_0^\infty \frac{\ln(x^2+a^2)}{2\operatorname{ch}^2 bx-1}\,dx = \frac{\pi}{b}\left\{\ln\Gamma\left(\frac{3}{4}+\frac{ab}{\pi}\right)-\ln\Gamma\left(\frac{1}{4}+\frac{ab}{\pi}\right)+\frac{1}{2}\ln\frac{\pi}{b}\right\}$$

Hint: In order to get formula (a), first use formula (37), then notice that $\operatorname{sh}\ln(2\pm\sqrt{3}) = \pm\sqrt{3}$ and $\ln(2+\sqrt{3}) = -\ln(2-\sqrt{3})$. At the final stage, use the duplication formula for the Γ-function.

Nota bene: The particular case of the integral (a) for $a = 0$ and $b = 1$ may be found in the Prudnikov et al.'s tables [53]. However, the provided expression is completely different and its numerical verification (performed with the help of Maple 12 and MATLAB 7.2) fails:

$$\underbrace{\int_0^\infty \frac{\ln x}{2\operatorname{ch}^2 x+1}\,dx}_{-0.2686306939\ldots} = \underbrace{\frac{\sqrt{\pi}}{2\sqrt{3}}\sum_{n=1}^{\infty}(-1)^n\frac{\gamma+\ln 4n}{\sqrt{n}}\sin\frac{\pi n}{3}}_{-0.2977821762\ldots}$$

[53, vol. I, p. 534, no. 2.6.29-7]. A careful study of this formula shows that the error consists in the misplaced square sign: $\operatorname{ch}^2 x$ should be replaced by $\operatorname{ch} x^2$, that is to say,

$$\int_0^\infty \frac{\ln x}{2\operatorname{ch} x^2+1}\,dx = \frac{\sqrt{\pi}}{2\sqrt{3}}\sum_{n=1}^{\infty}(-1)^n\frac{\gamma+\ln 4n}{\sqrt{n}}\sin\frac{\pi n}{3} = -0.2977821762\ldots$$

By the way, in a slightly different form the above formula appears in the so many times cited Malmsten's work [41, p. 15, Eq. (41)].

8* Prove that for any $a\geqslant 0$ and $\operatorname{Re} b>0$

$$\text{(a)}\quad \int_0^\infty \frac{\ln(x^2+a^2)}{\operatorname{ch}^2 bx+\sin^2\varphi}\,dx = \frac{\pi i}{b\sin\varphi\sqrt{\sin^2\varphi+1}}\ln\frac{\Gamma\left(\frac{1}{2}+\frac{ab}{\pi}+\frac{\ln\kappa}{2\pi i}\right)}{\Gamma\left(\frac{1}{2}+\frac{ab}{\pi}-\frac{\ln\kappa}{2\pi i}\right)} + \frac{\ln\kappa}{b\sin\varphi\sqrt{\sin^2\varphi+1}}\ln\frac{\pi}{b},$$

where $\kappa\equiv 1+2\sin^2\varphi+2\sin\varphi\sqrt{\sin^2\varphi+1}$, $\varphi\in\mathbb{C}$,

$$\text{(b)}\quad \int_0^\infty \frac{\ln(x^2+a^2)}{\operatorname{ch}^2 bx-\sin^2\varphi}\,dx = \frac{2\pi}{b\sin 2\varphi}\ln\frac{\Gamma\left(\frac{1}{2}+\frac{ab+\varphi}{\pi}\right)}{\Gamma\left(\frac{1}{2}+\frac{ab-\varphi}{\pi}\right)} + \frac{4\varphi}{b\sin 2\varphi}\ln\frac{\pi}{b},\qquad \begin{cases} |\operatorname{Re}\varphi|<\dfrac{\pi}{2}, & a\neq 1,\\[2mm] |\operatorname{Re}\varphi|\leqslant\dfrac{\pi}{2}, & a=1,\end{cases}$$

where the right-hand side should be regarded as a limit in cases $\varphi=0$ and $\varphi=\pm\pi/2$ $(a=1)$. Note also that both integrals from exercise no. 7 are actually particular cases of above ones with $\varphi=\pi/4$.

9* Prove that for any $a\geqslant 0$, $\operatorname{Re} b>0$ and $p=bm/n$, where numbers m and n are positive integers such that $m<2n$,

$$\text{(a)}\quad \int_0^\infty \frac{\operatorname{ch} px\cdot\ln(x^2+a^2)}{\operatorname{ch}^2 bx+\sin^2\varphi}\,dx = \frac{\pi\left(\kappa^{\frac{m}{2n}}-\kappa^{-\frac{m}{2n}}\right)}{2b\sin\varphi\sqrt{\sin^2\varphi+1}}\cdot\csc\frac{m\pi}{2n}\cdot\ln\frac{2\pi n}{b}$$

$$
\begin{aligned}
&-\frac{\pi}{2b\sin\varphi\sqrt{\sin^2\varphi+1}}\,\operatorname{Im}\sum_{l=0}^{2n-1}\Bigg\{\Bigg[\kappa^{\frac{m}{2n}}\ln\Gamma\left(\frac{2l+1}{4n}+\frac{ab}{2\pi n}+\frac{\ln\kappa}{4\pi i n}\right)\\
&\quad-\kappa^{-\frac{m}{2n}}\ln\Gamma\left(\frac{2l+1}{4n}+\frac{ab}{2\pi n}-\frac{\ln\kappa}{4\pi i n}\right)\Bigg]\cdot\exp\frac{(2l+1)m\pi i}{2n}\\
&\quad-\Bigg[\kappa^{\frac{m}{2n}}\ln\Gamma\left(\frac{2l+1}{4n}+\frac{ab}{2\pi n}-\frac{\ln\kappa}{4\pi i n}\right)\\
&\quad-\kappa^{-\frac{m}{2n}}\ln\Gamma\left(\frac{2l+1}{4n}+\frac{ab}{2\pi n}+\frac{\ln\kappa}{4\pi i n}\right)\Bigg]\cdot\exp\frac{-(2l+1)m\pi i}{2n}\Bigg\},
\end{aligned}
$$

where $\kappa\equiv 1+2\sin^2\varphi+2\sin\varphi\sqrt{\sin^2\varphi+1}$, $\varphi\in\mathbb{R}$,

$$
\begin{aligned}
\text{(b)}\quad &\int_0^\infty \frac{\operatorname{ch} px\cdot\ln(x^2+a^2)}{\operatorname{ch}^2 bx-\sin^2\varphi}\,dx=\frac{2\pi}{b\sin 2\varphi}\cdot\sin\frac{m\varphi}{n}\cdot\csc\frac{m\pi}{2n}\cdot\ln\frac{2\pi n}{b}\\
&+\frac{2\pi}{b\sin 2\varphi}\sum_{l=0}^{2n-1}\Bigg\{\cos\frac{(2l+1)m\pi+2m\varphi}{2n}\cdot\ln\Gamma\left(\frac{2l+1}{4n}+\frac{ab+\varphi}{2\pi n}\right)\\
&\qquad-\cos\frac{(2l+1)m\pi-2m\varphi}{2n}\cdot\ln\Gamma\left(\frac{2l+1}{4n}+\frac{ab-\varphi}{2\pi n}\right)\Bigg\}
\end{aligned}
$$

with $|\operatorname{Re}\varphi|\leqslant\pi/2$, and where the right-hand side should be regarded as a limit in cases $\varphi=0$ and $\varphi=\pm\pi/2$ $(a=1)$.

Hint: For exercise (a), first, evaluate the auxiliary integral J_R with the help of

$$
\oint\limits_{0\leqslant\operatorname{Im}z\leqslant\pi}\frac{e^{\alpha z}}{\operatorname{ch}^2 z+\sin^2\varphi}\,dz.\qquad\text{This will give}
$$

$$
\int_0^\infty\frac{\operatorname{ch}\alpha x}{\operatorname{ch}^2 x+\sin^2\varphi}dx=\frac{\pi\left(\kappa^{\alpha/2}-\kappa^{-\alpha/2}\right)}{4\sin\varphi\sqrt{\sin^2\varphi+1}}\cdot\csc\frac{\alpha\pi}{2},\qquad|\operatorname{Re}\alpha|<2.
$$

Then, use a similar procedure as described in no. 3. Analogously, for exercise (b), show first that

$$
\int_0^\infty\frac{\operatorname{ch}\alpha x}{\operatorname{ch}^2 x-\sin^2\varphi}\,dx=\frac{\pi\sin\alpha\varphi}{\sin 2\varphi}\cdot\csc\frac{\alpha\pi}{2},\qquad|\operatorname{Re}\alpha|<2.
$$

10* Prove by the contour integration method that for any $a\geqslant 0$ and $\operatorname{Re}b>0$

$$
\text{(a)}\quad\int_0^\infty\frac{\ln(x^2+1)}{\operatorname{sh}^2 bx}\,dx=\frac{2}{b}\left\{\ln\frac{b}{\pi}-\frac{\pi}{2b}-\Psi\left(\frac{b}{\pi}\right)\right\}
$$

(b) $$\int_0^\infty \frac{\operatorname{ch} bx \cdot \ln(x^2+1)}{\operatorname{sh}^2 bx}\, dx = \frac{1}{b}\left\{\Psi\left(\frac{1}{2}+\frac{b}{2\pi}\right)-\Psi\left(\frac{b}{2\pi}\right)-\frac{\pi}{b}\right\},$$

(c) $$\int_0^\infty \frac{(1-\operatorname{ch} bx)\ln(x^2+a^2)}{\operatorname{sh}^2 bx}\, dx = -\frac{2}{b}\left\{\ln\frac{2\pi}{b}+\Psi\left(\frac{1}{2}+\frac{ab}{2\pi}\right)\right\},$$

(d) $$\int_0^\infty \frac{\ln(x^2+a^2)}{\operatorname{sh}^2 bx+\cos^2\varphi}\, dx = \frac{2\pi}{b\sin 2\varphi}\ln\frac{\Gamma\left(\frac{1}{2}+\frac{ab+\varphi}{\pi}\right)}{\Gamma\left(\frac{1}{2}+\frac{ab-\varphi}{\pi}\right)}+\frac{4\varphi}{b\sin 2\varphi}\ln\frac{\pi}{b}$$

$$\begin{cases} |\operatorname{Re}\varphi|<\dfrac{\pi}{2}, & \text{if } a\neq 1,\\ |\operatorname{Re}\varphi|\leqslant\dfrac{\pi}{2}, & \text{if } a=1,\end{cases}$$

where in the last expression the right part must be considered as a limit for $\varphi=0$ and $\varphi=\pm\pi/2$ $(a=1)$.

Nota bene: Formula (a), in the unsimplified form, can be found in Bierens de Haan tables [62, Table 258-5] and in [28, no. 4.373-4]. Formulas (b)–(d) seem to be new.

11* By using the contour integration method prove that if p is a rational part of b, i.e. $p=bm/n$, where b is some positive parameter and numbers m and n are positive integers, then for any $a>0$,

(a) $$\int_0^\infty \frac{\operatorname{sh}^2 px\cdot\ln(x^2+a^2)}{\operatorname{sh}^2 bx}\, dx = \frac{1}{b}\left(1-\frac{\pi m}{n}\operatorname{ctg}\frac{\pi m}{n}\right)\cdot\ln\frac{\pi n}{b}$$
$$+\frac{2m\pi}{bn}\sum_{l=1}^{n-1}\sin\frac{2\pi ml}{n}\cdot\ln\Gamma\left(\frac{l}{n}+\frac{ab}{\pi n}\right)-\frac{1}{bn}\sum_{l=1}^{n-1}\cos\frac{2\pi ml}{n}\cdot\Psi\left(\frac{l}{n}+\frac{ab}{\pi n}\right)$$
$$+\frac{1}{b}\Psi\left(\frac{ab}{\pi}\right)-\frac{1}{bn}\Psi\left(\frac{ab}{\pi n}\right)-\frac{1}{b}\ln n,$$

(b) $$\int_0^\infty \frac{\operatorname{sh}^2 px\cdot\ln x}{\operatorname{sh}^2 bx}\, dx = -\frac{\pi m}{2bn}\operatorname{ctg}\frac{\pi m}{n}\cdot\ln\frac{\pi n}{b}$$
$$+\frac{m\pi}{bn}\sum_{l=1}^{n-1}\sin\frac{2\pi ml}{n}\cdot\ln\Gamma\left(\frac{l}{n}\right)-\frac{1}{2b}\ln\left(\frac{2b}{\pi}\sin\frac{m\pi}{n}\right)-\frac{\gamma}{2b}$$

(c) $$\int_0^\infty \frac{\operatorname{ch} px\cdot\ln(x^2+1)}{\operatorname{sh}^2 bx}\, dx = -\frac{\pi}{b^2}-\frac{\pi m}{bn}\operatorname{ctg}\frac{\pi m}{2n}\cdot\ln\frac{2\pi n}{b}$$

$$+\frac{2m\pi}{bn}\sum_{l=1}^{2n-1}\sin\frac{\pi ml}{n}\cdot\ln\Gamma\left(\frac{l}{2n}+\frac{b}{2\pi n}\right)$$

$$-\frac{1}{bn}\sum_{l=1}^{2n-1}\cos\frac{\pi ml}{n}\cdot\Psi\left(\frac{l}{2n}+\frac{b}{2\pi n}\right)-\frac{1}{bn}\Psi\left(\frac{b}{2\pi n}\right)$$

where $m<n$ in exercises (a) an (b), and $m<2n$ in exercise (c). For continuous and complex values of p, see no. 67.

Hint: For exercise (a), after having proved that

$$\int_0^\infty\frac{\operatorname{sh}^2\alpha x}{\operatorname{sh}^2 x}\,dx\;=\;\frac{1}{2}-\frac{\alpha\pi}{2}\operatorname{ctg}\alpha\pi,\qquad |\operatorname{Re}\alpha|<1,$$

use a similar procedure as described in exercise no. 3. When evaluating residues, do not forget that function $\dfrac{\operatorname{sh}^2 mz}{\operatorname{sh}^2 nz}$ has removable singularities at $z=0$, $z=\pi i$, $z=2\pi i$ and poles of the second order at $z=i\pi l/n$, $l=1,2,3,\ldots,n-1,n+1,\ldots,2n-1$. At the final stage, split the sum over $l=1,2,\ldots,2n-1$ in two sums of equal lengths (over $l=0,1,2,\ldots,n-1$ and over $l=n,n+1,n+2,\ldots,2n-1$), then, use duplication formulas for both Γ- and Ψ-functions, and finally, employ the Gauss' multiplication theorem for the Ψ-function

$$\Psi(nz)=\ln n+\frac{1}{n}\sum_{l=0}^{n-1}\Psi\left(z+\frac{l}{n}\right),\qquad z\in\mathbb{C},\;n\in\mathbb{N}.$$

Result (b) is obtained from (a) by an appropriate limiting procedure. In its final form it appears after this elegant simplification

$$\sum_{l=1}^{n-1}\cos\frac{2\pi ml}{n}\cdot\Psi\left(\frac{l}{n}\right)=n\ln\left(2\sin\frac{m\pi}{n}\right)+\gamma,\qquad m=1,2,\ldots,n-1.\tag{48}$$

Formula (c) may be got from an intermediate formula obtained when deriving formula (a) and with the help of no. 10a.

12* In precedent exercises we saw that the evaluation of certain logarithmic integrals by the contour integration method may lead to the Γ-function of a complex argument. By supposing that parameters α, β and ϑ are real and $a\geqslant 0$, show that integrals

$$\text{(a)}\quad\int_0^\infty\frac{\ln(x^2+a^2)}{\operatorname{ch}x+\vartheta}\,dx,\qquad\text{(b)}\quad\int_0^\infty\frac{\ln(x^2+a^2)}{\operatorname{ch}^2 x+\alpha}\,dx,$$

$$\text{(c)}\quad\int_0^\infty\frac{\ln(x^2+a^2)}{\operatorname{sh}^2 x+\beta}\,dx$$

will lead to the Γ-function of a real argument (if using the proposed contour integration method) only if

$$\text{(a)}\quad \begin{cases} -1 < \vartheta \leqslant 1, & a \neq 1, \\ -1 \leqslant \vartheta \leqslant 1, & a = 1, \end{cases} \qquad \text{(b)}\quad \begin{cases} -1 < \alpha \leqslant 0, & a \neq 1, \\ -1 \leqslant \alpha \leqslant 0, & a = 1, \end{cases}$$

$$\text{(c)}\quad \begin{cases} 0 < \beta \leqslant 1, & a \neq 1, \\ 0 \leqslant \beta \leqslant 1, & a = 1 \end{cases}$$

respectively. Note that such integrals can be also calculated by using expansions in geometric series (we employed such a method in exercises no. 18–21). By the way, Malmsten et al. in [40] and [41] evaluated only such a kind of integral.

13* By making use of the contour integration method show that for any $a \geqslant 0$, $\operatorname{Re} b > 0$,

$$\text{(a)}\quad \int_0^\infty \frac{\ln(x^2+a^2)}{\operatorname{ch}^3 bx}\,dx = \frac{\pi}{b}\left\{\ln\Gamma\left(\frac{3}{4}+\frac{ab}{2\pi}\right) - \ln\Gamma\left(\frac{1}{4}+\frac{ab}{2\pi}\right)\right\} + \frac{\pi}{2b}\ln\frac{2\pi}{b}$$
$$+\frac{1}{4\pi b}\left\{\Psi_1\left(\frac{3}{4}+\frac{ab}{2\pi}\right) - \Psi_1\left(\frac{1}{4}+\frac{ab}{2\pi}\right)\right\},$$

$$\text{(b)}\quad \int_0^\infty \frac{\operatorname{ch} px\cdot\ln(x^2+a^2)}{\operatorname{ch}^3 bx}\,dx = \frac{\pi(n^2-m^2)}{2bn^2}\sec\frac{m\pi}{2n}\cdot\ln\frac{2\pi n}{b}$$
$$-\frac{(n^2-m^2)\pi}{bn^2}\sum_{l=0}^{2n-1}(-1)^l\cos\frac{(2l+1)m\pi}{2n}\cdot\ln\Gamma\left(\frac{2l+1}{4n}+\frac{ab}{2\pi n}\right)$$
$$+\frac{m}{bn^2}\sum_{l=0}^{2n-1}(-1)^l\sin\frac{(2l+1)m\pi}{2n}\cdot\Psi\left(\frac{2l+1}{4n}+\frac{ab}{2\pi n}\right)$$
$$-\frac{1}{4b\pi n^2}\sum_{l=0}^{2n-1}(-1)^l\cos\frac{(2l+1)m\pi}{2n}\cdot\Psi_1\left(\frac{2l+1}{4n}+\frac{ab}{2\pi n}\right)$$

$$\text{(c)}\quad \int_0^\infty \frac{\operatorname{sh}^3 px\cdot\ln(x^2+a^2)}{\operatorname{sh}^3 bx}\,dx = \frac{\pi}{b}\cdot\frac{\dfrac{3m^2}{n^2}-\operatorname{tg}^2\dfrac{m\pi}{2n}}{1-3\operatorname{tg}^2\dfrac{m\pi}{2n}}\cdot\operatorname{tg}\frac{m\pi}{2n}\cdot\ln\frac{2\pi n}{b}$$
$$+\frac{\pi}{b}\sum_{l=1}^{2n-1}(-1)^l\sin^3\frac{m\pi l}{n}\cdot\ln\Gamma\left(\frac{l}{2n}+\frac{ab}{2\pi n}\right)$$
$$+\frac{3\pi m^2}{bn^2}\sum_{l=1}^{2n-1}(-1)^l\sin\frac{m\pi l}{n}\cdot\left(3\cos^2\frac{m\pi l}{n}-1\right)\cdot\ln\Gamma\left(\frac{l}{2n}+\frac{ab}{2\pi n}\right)$$

$$+\frac{3m}{bn^2}\sum_{l=1}^{2n-1}(-1)^l\sin^2\frac{m\pi l}{n}\cdot\cos\frac{m\pi l}{n}\cdot\Psi\left(\frac{l}{2n}+\frac{ab}{2\pi n}\right)$$

$$+\frac{1}{4b\pi n^2}\sum_{l=1}^{2n-1}(-1)^l\sin^3\frac{m\pi l}{n}\cdot\Psi_1\left(\frac{l}{2n}+\frac{ab}{2\pi n}\right)$$

where $p = bm/n$, numbers m and n being positive integers such that $m < 3n$ in (b) and $m < n$ in (c).

Hint: For exercise (b): use the procedure described in the hint of exercise no. 3. For the evaluation of the auxiliary integral J_R, consider

$$\oint\limits_{0\leqslant\operatorname{Im}z\leqslant\pi}\frac{e^{\alpha z}}{\operatorname{ch}^3 z}\,dz, \qquad \text{and then, show that}$$

$$\int_0^\infty\frac{\operatorname{ch}\alpha x}{\operatorname{ch}^3 x}\,dx = \frac{\pi(1-\alpha^2)}{4}\cdot\sec\frac{\alpha\pi}{2}, \qquad |\operatorname{Re}\alpha| < 3.$$

As regards exercise (c), the procedure is very similar. The evaluation of the integral J_R may be done with the help of

$$\text{p.v.}\oint\limits_{0\leqslant\operatorname{Im}z\leqslant\pi}\frac{e^{az}}{\operatorname{sh}^3 z}\,dz, \quad |\operatorname{Re}a| < 3, \qquad \text{which yields}$$

$$\int_0^\infty\frac{\operatorname{sh}^3\alpha x}{\operatorname{sh}^3 x}\,dx = \frac{\pi}{2}\cdot\frac{3\alpha^2-\operatorname{tg}^2\frac{\alpha\pi}{2}}{1-3\operatorname{tg}^2\frac{\alpha\pi}{2}}\cdot\operatorname{tg}\frac{\alpha\pi}{2}, \qquad |\operatorname{Re}\alpha| < 1.$$

14* Prove by the contour integration method that for any $a \geqslant 0$ and $\operatorname{Re}b > 0$,

(a) $$\int_0^\infty\frac{\ln(x^2+a^2)}{\operatorname{ch}^4 bx}\,dx = \frac{4}{3b}\left\{\ln\frac{\pi}{b}+\Psi\left(\frac{1}{2}+\frac{ab}{\pi}\right)\right\}+\frac{1}{3\pi^2 b}\Psi_2\left(\frac{1}{2}+\frac{ab}{\pi}\right),$$

(b) $$\int_0^\infty\frac{(1-\operatorname{ch}bx)^2\ln(x^2+a^2)}{\operatorname{sh}^4 bx}\,dx = \frac{2}{3b}\left\{\ln\frac{2\pi}{b}+\Psi\left(\frac{1}{2}+\frac{ab}{2\pi}\right)\right\}$$
$$+\frac{1}{6\pi^2 b}\Psi_2\left(\frac{1}{2}+\frac{ab}{2\pi}\right),$$

(c) $$\int_0^\infty\frac{\operatorname{ch}px\cdot\ln(x^2+a^2)}{\operatorname{ch}^4 bx}\,dx = \frac{(4n^2-m^2)m\pi}{6bn^3}\csc\frac{m\pi}{2n}\cdot\ln\frac{2\pi n}{b}$$

$$
\begin{aligned}
&-\frac{(4n^2-m^2)m\pi}{3bn^3}\sum_{l=0}^{2n-1}\sin\frac{(2l+1)m\pi}{2n}\cdot\ln\Gamma\left(\frac{2l+1}{4n}+\frac{ab}{2\pi n}\right)\\
&+\frac{4n^2-3m^2}{6bn^3}\sum_{l=0}^{2n-1}\cos\frac{(2l+1)m\pi}{2n}\cdot\Psi\left(\frac{2l+1}{4n}+\frac{ab}{2\pi n}\right)\\
&-\frac{m}{4b\pi n^3}\sum_{l=0}^{2n-1}\sin\frac{(2l+1)m\pi}{2n}\cdot\Psi_1\left(\frac{2l+1}{4n}+\frac{ab}{2\pi n}\right)\\
&+\frac{1}{24b\pi^2n^3}\sum_{l=0}^{2n-1}\cos\frac{(2l+1)m\pi}{2n}\cdot\Psi_2\left(\frac{2l+1}{4n}+\frac{ab}{2\pi n}\right)
\end{aligned}
$$

where $p = bm/n$, numbers m and n being positive integers such that $m < 4n$.

Hint: For exercise (c): proceed similarly to exercise no. 13c. As regards the integral J_R, consider

$$
\oint\limits_{0\leqslant \operatorname{Im} z\leqslant \pi}\frac{e^{\alpha z}}{\operatorname{ch}^4 z}\,dz \qquad \text{in order to prove that}
$$

$$
\int_0^\infty\frac{\operatorname{ch}\alpha x}{\operatorname{ch}^4 x}\,dx=\frac{\alpha\pi(4-\alpha^2)}{12}\cdot\csc\frac{\alpha\pi}{2},\qquad |\operatorname{Re}\alpha|<4.
$$

15* Show by the contour integration technique that

$$
\begin{aligned}
\int_0^\infty\frac{\ln(x^2+a^2)}{\operatorname{ch}^5 bx}\,dx=&\frac{3\pi}{4b}\left\{\ln\Gamma\left(\frac{3}{4}+\frac{ab}{2\pi}\right)-\ln\Gamma\left(\frac{1}{4}+\frac{ab}{2\pi}\right)\right\}+\frac{3\pi}{8b}\ln\frac{2\pi}{b}\\
&+\frac{5}{24\pi b}\left\{\Psi_1\left(\frac{3}{4}+\frac{ab}{2\pi}\right)-\Psi_1\left(\frac{1}{4}+\frac{ab}{2\pi}\right)\right\}\\
&+\frac{1}{192\pi^3 b}\left\{\Psi_3\left(\frac{3}{4}+\frac{ab}{2\pi}\right)-\Psi_3\left(\frac{1}{4}+\frac{ab}{2\pi}\right)\right\},
\end{aligned}
$$

provided that $a \geqslant 0$ and $\operatorname{Re} b > 0$.

16* Prove that

$$
\begin{aligned}
\int_0^\infty\frac{\ln(x^2+a^2)}{\operatorname{ch}^6 bx}\,dx=&\frac{16}{15b}\left\{\ln\frac{\pi}{b}+\Psi\left(\frac{1}{2}+\frac{ab}{\pi}\right)\right\}+\frac{1}{3\pi^2 b}\Psi_2\left(\frac{1}{2}+\frac{ab}{\pi}\right)\\
&+\frac{1}{60\pi^4 b}\Psi_4\left(\frac{1}{2}+\frac{ab}{\pi}\right),
\end{aligned}
$$

provided that $a \geqslant 0$ and $\operatorname{Re} b > 0$.

17* Show that for $a \geqslant 0$ and $\operatorname{Re} b > 0$

$$\text{(a)} \quad \int_0^\infty \frac{\ln(x^2+a^2)}{\operatorname{ch}^n bx}\,dx = \frac{\pi A_n}{b}\left\{\ln\Gamma\left(\frac{3}{4}+\frac{ab}{2\pi}\right) - \ln\Gamma\left(\frac{1}{4}+\frac{ab}{2\pi}\right) + \frac{1}{2}\ln\frac{2\pi}{b}\right\}$$

$$+\frac{1}{b}\sum_{l=1}^{\frac{1}{2}(n-1)}\frac{B_{n,l}}{\pi^{2l-1}}\left\{\Psi_{2l-1}\left(\frac{3}{4}+\frac{ab}{2\pi}\right) - \Psi_{2l-1}\left(\frac{1}{4}+\frac{ab}{2\pi}\right)\right\}$$

for odd n, and

$$\text{(b)} \quad \int_0^\infty \frac{\ln(x^2+a^2)}{\operatorname{ch}^n bx}\,dx = \frac{A_n}{b}\left\{\ln\frac{\pi}{b} + \Psi\left(\frac{1}{2}+\frac{ab}{\pi}\right)\right\} + \frac{1}{b}\sum_{l=1}^{\frac{1}{2}n-1}\frac{B_{n,l}}{\pi^{2l}}\Psi_{2l}\left(\frac{1}{2}+\frac{ab}{\pi}\right)$$

for even n, A_n and $B_{n,l}$ being some rational coefficients. After some effort, one can obtain their numeric values for any positive integer n. Table 1 gives these coefficients for n up to 20. Curiously enough, A_n for odd n are equal to the coefficients in the Maclaurin expansion of $2(1-x)^{-\frac{1}{2}}$, while A_n^{-1} for even n are equal to the coefficients in the Maclaurin expansion of $\frac{1}{2}(1-x)^{-\frac{3}{2}}$.

4.1.2 Further results obtained by a combination of various methods

18* By using geometric series expansions and term-by-term integration, prove

$$\text{(a)} \quad \int_0^\infty \frac{x^a \ln x}{e^{bx}-1}\,dx = \frac{\Gamma(a+1)}{b^{a+1}}\left\{\Psi(a+1)\zeta(a+1) + \zeta'(a+1) - \zeta(a+1)\ln b\right\},$$

$$\operatorname{Re} a > 0,$$

$$\text{(b)} \quad \int_0^\infty \frac{x^a \ln x}{e^{bx}+1}\,dx = \frac{(1-2^{-a})\,\Gamma(a+1)}{b^{a+1}}\left\{\Psi(a+1)\zeta(a+1) + \frac{\ln 2}{2^a-1}\zeta(a+1)\right.$$

$$\left. + \zeta'(a+1) - \zeta(a+1)\ln b\right\}, \qquad \operatorname{Re} a > -1,\ a \neq 0,$$

$$\text{(c)} \quad \int_0^\infty \frac{\ln x}{e^{bx}+1}\,dx = -\frac{(\ln 2 + 2\ln b)\ln 2}{2b},$$

$$\text{(d)} \quad \int_0^\infty \frac{\ln^2 x}{e^{bx}+1}\,dx = \frac{\ln 2}{b}\left\{-2\gamma_1 - \gamma^2 + \frac{1}{3}\ln^2 2 + \ln 2 \ln b + \ln^2 b + \frac{\pi^2}{6}\right\},$$

n	A_n	$B_{n,1}$	$B_{n,2}$	$B_{n,3}$	$B_{n,4}$	$B_{n,5}$	$B_{n,6}$	$B_{n,7}$	$B_{n,8}$	$B_{n,9}$
1	2	–	–	–	–	–	–	–	–	–
3	1	$\frac{1}{4}$	–	–	–	–	–	–	–	–
5	$\frac{3}{4}$	$\frac{5}{24}$	$\frac{1}{192}$	–	–	–	–	–	–	–
7	$\frac{5}{8}$	$\frac{259}{1440}$	$\frac{7}{1152}$	$\frac{1}{23040}$	–	–	–	–	–	–
9	$\frac{35}{64}$	$\frac{3229}{20160}$	$\frac{47}{7680}$	$\frac{1}{15360}$	$\frac{1}{5160960}$	–	–	–	–	–
11	$\frac{63}{128}$	$\frac{117469}{806400}$	$\frac{17281}{2903040}$	$\frac{209}{2764800}$	$\frac{11}{30965760}$	$\frac{1}{1857945600}$	–	–	–	–
13	$\frac{231}{512}$	$\frac{7156487}{53222400}$	$\frac{1997021}{348364800}$	$\frac{28067}{348364800}$	$\frac{871}{1857945600}$	$\frac{13}{11147673600}$	$\frac{1}{980995276800}$	–	–	–
15	$\frac{429}{1024}$	$\frac{2430898831}{19372953600}$	$\frac{1206053}{218972160}$	$\frac{230443}{2786918400}$	$\frac{8521}{15606743040}$	$\frac{11}{6370099200}$	$\frac{1}{392398110720}$	$\frac{1}{714164561510400}$	–	–
17	$\frac{6435}{16384}$	$\frac{60997921}{516612096}$	$\frac{24615717239}{4649508864000}$	$\frac{15313957}{183936614400}$	$\frac{5599613}{9364045824000}$	$\frac{4097}{1872809164800}$	$\frac{493}{117719433216000}$	$\frac{17}{4284987369062400}$	$\frac{1}{685597979049984000}$	–
19	$\frac{12155}{32768}$	$\frac{141433003757}{1264666411008}$	$\frac{426540447313}{83691159552000}$	$\frac{249938765093}{3012881743872000}$	$\frac{391080857}{618027024384000}$	$\frac{574123}{224737099776000}$	$\frac{85177}{14832648585216000}$	$\frac{1843}{257099242143744000}$	$\frac{19}{411358787429990400}$	$\frac{1}{83917192635718041600}$

n	A_n	$B_{n,1}$	$B_{n,2}$	$B_{n,3}$	$B_{n,4}$	$B_{n,5}$	$B_{n,6}$	$B_{n,7}$	$B_{n,8}$	$B_{n,9}$
2	2	–	–	–	–	–	–	–	–	–
4	$\frac{4}{3}$	$\frac{1}{3}$	–	–	–	–	–	–	–	–
6	$\frac{16}{15}$	$\frac{1}{3}$	$\frac{1}{60}$	–	–	–	–	–	–	–
8	$\frac{32}{35}$	$\frac{14}{45}$	$\frac{1}{45}$	$\frac{1}{2520}$	–	–	–	–	–	–
10	$\frac{256}{315}$	$\frac{164}{567}$	$\frac{13}{540}$	$\frac{1}{1512}$	$\frac{1}{181440}$	–	–	–	–	–
12	$\frac{512}{693}$	$\frac{3832}{14175}$	$\frac{139}{5670}$	$\frac{31}{37800}$	$\frac{1}{90720}$	$\frac{1}{19958400}$	–	–	–	–
14	$\frac{2048}{3003}$	$\frac{1888}{7425}$	$\frac{148}{6075}$	$\frac{311}{340200}$	$\frac{1}{64800}$	$\frac{1}{8553600}$	$\frac{1}{3113510400}$	–	–	–
16	$\frac{4096}{6435}$	$\frac{17067584}{70945875}$	$\frac{33608}{1403325}$	$\frac{2473}{2551500}$	$\frac{67}{3572100}$	$\frac{41}{224532000}$	$\frac{1}{1167566400}$	$\frac{1}{653837184000}$	–	–
18	$\frac{65536}{109395}$	$\frac{16229632}{70945875}$	$\frac{14956868}{638512875}$	$\frac{4679}{4677750}$	$\frac{2021}{95256000}$	$\frac{757}{3143448000}$	$\frac{23}{15567552000}$	$\frac{1}{217945728000}$	$\frac{1}{177843714048000}$	–
20	$\frac{131072}{230945}$	$\frac{263495168}{1206079875}$	$\frac{2919352}{127702575}$	$\frac{5839219}{5746615875}$	$\frac{21713}{943034400}$	$\frac{5473}{18860688000}$	$\frac{619}{294226732800}$	$\frac{17}{19615115520000}$	$\frac{1}{53353114214400}$	$\frac{1}{60822550204416000}$

Table 1 Coefficients A_n and $B_{n,l}$ from exercise no. 17 (top table corresponds to exercise 17-a, bottom table to 17-b)

$$\text{(e)}\quad \int\limits_0^\infty \frac{x^a \ln x}{\operatorname{sh} bx}\, dx = \frac{(2-2^{-a})\Gamma(a+1)}{b^{a+1}}\left\{\Psi(a+1)\zeta(a+1) + \frac{\ln 2}{2^{a+1}-1}\zeta(a+1)\right.$$
$$\left. + \zeta'(a+1) - \zeta(a+1)\ln b\right\}, \qquad \operatorname{Re} a > 0,$$

$$\text{(f)}\quad \int\limits_0^\infty \frac{x^a \ln x}{\operatorname{ch} bx}\, dx = \frac{\Gamma(a+1)}{2^{2a} b^{a+1}}\left\{\left[\Psi(a+1) - \ln 4b\right]\zeta\left(a+1, \frac{1}{4}\right) + \zeta'\left(a+1, \frac{1}{4}\right)\right.$$
$$- 2^a\left(2^{a+2}-1\right)\ln 2 \cdot \zeta(a+1) - 2^a\left(2^{a+1}-1\right)\left\{\zeta'(a+1)\right.$$
$$\left.\left. + \left[\Psi(a+1) - \ln 4b\right]\zeta(a+1)\right\}\right\}, \quad \operatorname{Re} a > -1,\ a \neq 0,$$

$$\text{(g)}\quad \int\limits_0^\infty \frac{\ln x}{\operatorname{ch} bx}\, dx = \frac{1}{b}\left\{\gamma_1 - \gamma_1\left(\frac{1}{4}\right) - 3\gamma\ln 2 - \frac{7}{2}\ln^2 2 - \frac{\pi\gamma}{2} - \frac{\pi}{2}\ln 4b\right\},$$

$$\text{(h)}\quad \int\limits_0^\infty \frac{\ln^2 x}{\operatorname{ch} bx}\, dx$$
$$= \frac{1}{b}\left\{-\gamma_2 + \gamma_2\left(\frac{1}{4}\right) + 2\gamma_1\ln 2 + 12\gamma\ln^2 2 + 9\ln^3 2 + 2\gamma\gamma_1\left(\frac{1}{4}\right)\right.$$
$$- 2\gamma\gamma_1 + 6\gamma^2\ln 2 + \frac{\pi^3}{12} + 2\gamma_1\left(\frac{1}{4}\right)\ln 4b - 2\gamma_1\ln b + 6\gamma\ln 2\ln b$$
$$\left. + 7\ln^2 2\ln b + \frac{\pi\gamma^2}{2} + 2\pi\ln^2 2 + \frac{\pi}{2}\ln^2 b + \pi\gamma\ln 4b + 2\pi\ln 2\ln b\right\},$$

which hold for any $b > 0$. Integrals containing higher powers of the logarithm in the numerator[27] may be easily evaluated by calculating the derivative with respect to a of the corresponding right parts [e.g., results (d) and (h) were obtained in this way]. However, the derivation is usually long and quite tedious. As regards non-integers powers of logarithm in the numerator, calculation of such integrals may be sometimes carried out by other methods; for example, Malmsten [41] treated similar integrals by a simple change of variable.

Nota bene: We have not found formulas (a), (b), (d)–(h) in Gradshteyn and Ryzhik's tables [28], neither in Prudnikov et al.'s [53] tables. However, these results are not very complicated to derive, so, probably, they were presented elsewhere before.
Solution: all demonstrations are based on the summation of certain geometric series which can be integrated term-by-term. For instance, in exercise d), which is perhaps the most complicated, one should first represent $\operatorname{ch}^{-1} x$ in the following form:

$$\frac{1}{\operatorname{ch} x} = 2\sum_{n=0}^{\infty}(-1)^n e^{-(2n+1)x}, \qquad \operatorname{Re} x > 0.$$

[27] Integer powers only.

The latter series, being uniformly convergent, can be integrated term-by-term

$$\begin{aligned}\int\limits_0^\infty \frac{x^a}{\operatorname{ch} bx}\,dx &= 2\sum_{n=0}^\infty (-1)^n \int\limits_0^\infty x^a e^{-(2n+1)bx}\,dx\\&= \frac{2\Gamma(a+1)}{b^{a+1}}\sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)^{a+1}} = \frac{\Gamma(a+1)}{2^a b^{a+1}}\eta\left(a+1,\frac{1}{2}\right),\end{aligned}$$

$\operatorname{Re} a > -1,\ b > 0$. By recalling that $\eta(s, v)$ can be reduced to $\zeta(s, v)$, see (4), and in view of the fact that $\zeta\left(s, \frac{1}{2}\right) = (2^s - 1)\zeta(s)$, the above formula reduces to

$$\int\limits_0^\infty \frac{x^a}{\operatorname{ch} bx}\,dx = \frac{\Gamma(a+1)}{2^{2a} b^{a+1}}\left\{\zeta\left(a+1,\frac{1}{4}\right) - 2^a\left(2^{a+1}-1\right)\zeta(a+1)\right\},$$

$\operatorname{Re} a > -1,\ b > 0$. Differentiating once/twice the latter expression with respect to a, and then letting $a \to 0$, yields formulas (g)/(h) respectively. For more information about the relationship between the Hurwitz ζ-function and the Stieltjes constants, see exercises no. 63–64.

19* By using results of the previous exercise, show that

(a) $$\sum_{n=1}^\infty \frac{(-1)^n \ln bn}{n} = (\gamma - \ln b)\ln 2 - \frac{1}{2}\ln^2 2, \qquad b > 0,$$

(b) $$\sum_{n=1}^\infty \frac{(-1)^{n-1}\ln(2n-1)}{2n-1} = \pi \ln \Gamma\left(\frac{1}{4}\right) - \frac{\pi}{2}\ln 2 - \frac{3\pi}{4}\ln \pi - \frac{\pi\gamma}{4}.$$

Hint: For (a): consider no. 18-c and set $b = 1$. Expand $\operatorname{ch}^{-1} x$ as indicated in the hint of no. 18, and then interchange the integration and summation. Proceed similarly for series (b) and recall that integral no. 18-g is also the simplest Malsmten's integral (1).

Nota bene: The logarithmic series from exercise a) is quite well known and can be found in many sources, e.g. in [28, no. 4.325-1] or in [53, vol. I, p. 746, no. 5.5.1-3]. Series (b) was derived by Malmsten in [41, unnumbered equation on the pp. 20 and 26] and also appears in [53, vol. I, p. 747, no. 5.5.1-6] in the form containing Malmsten's integral (1).

20* Proceeding as above, prove that for any $b > 0$

(a) $$\int\limits_0^\infty \frac{x^a \ln x}{\operatorname{ch} bx + \cos\varphi}\,dx = \frac{2\Gamma(a+1)}{b^{a+1}\sin\varphi}\sum_{n=1}^\infty (-1)^{n-1}\frac{\Psi(a+1) - \ln bn}{n^{a+1}}\sin\varphi n,$$

$$a > -1,\ -\pi < \varphi < \pi,$$

(b) $$\sum_{n=1}^{\infty}(-1)^n\frac{\ln bn\cdot\sin\varphi n}{n}=\pi\ln\Gamma\left(\frac{1}{2}+\frac{\varphi}{2\pi}\right)+\frac{\pi}{2}\ln\cos\frac{\varphi}{2}-\frac{\pi}{2}\ln\pi$$
$$+\frac{\varphi}{2}\left[\gamma+\ln\frac{2\pi}{b}\right],\qquad -\pi<\varphi<\pi,$$

(c) $$\sum_{n=1}^{\infty}\frac{\ln bn\cdot\sin\varphi n}{n}=\pi\ln\Gamma\left(\frac{\varphi}{2\pi}\right)+\frac{\pi}{2}\ln\sin\frac{\varphi}{2}-\frac{\pi}{2}\ln\pi$$
$$+\frac{\varphi-\pi}{2}\left[\gamma+\ln\frac{2\pi}{b}\right],\qquad 0<\varphi<2\pi,$$

(d) $$\sum_{n=1}^{\infty}\frac{\ln(2n-1)\cdot\sin[(2n-1)\varphi]}{2n-1}=\frac{\pi}{2}\left[\ln\Gamma\left(\frac{\varphi}{2\pi}\right)-\ln\Gamma\left(\frac{1}{2}+\frac{\varphi}{2\pi}\right)\right]$$
$$-\frac{\pi}{4}\left[\ln 2\pi+\gamma-\ln\operatorname{tg}\frac{\varphi}{2}\right],\qquad 0<\varphi<\pi,$$

(e) $$\sum_{n=1}^{\infty}(-1)^n\frac{\ln(2n-1)\cdot\sin[(2n-1)\varphi]}{2n-1}=\frac{1}{4}\left\{\Phi\left(\varphi-\frac{\pi}{2}\right)-\Phi\left(\varphi+\frac{\pi}{2}\right)\right\}$$
$$+\frac{\gamma}{2}\ln\operatorname{tg}\left[\frac{\varphi}{2}+\frac{\pi}{4}\right]-\frac{\pi\operatorname{tg}\varphi}{4}\left\{2\ln\Gamma\left(\frac{\varphi}{2\pi}+\frac{1}{4}\right)-2\ln\Gamma\left(\frac{\varphi}{2\pi}+\frac{3}{4}\right)\right.$$
$$\left.+\ln\operatorname{tg}\left[\frac{\varphi}{2}+\frac{\pi}{4}\right]-\ln 2\pi\right\},\qquad -\frac{\pi}{2}<\varphi<\frac{\pi}{2},$$

where $\Phi(\varphi)$ is defined in exercise no. 21.

Hint: Formula (b) may be obtained by comparing result (a) with that obtained previously in exercise no. 2. Expansion (c) may be obtained from (b) by a shifting φ. The difference between (c) and (b) with parameter $b=1$ yields (d). Analogously, formula (e) may be deduced from no. 21-c.

Nota bene: Formulas (b) and (c), in a slightly different form, were previously derived by Malmsten et al. [40, p. 62] and [41, p. 25, eqs. (64)–(65)]. They also appear in [53, vol. I, p. 748] as no. 5.5.1-25 and 5.5.1-24 respectively; however, formula (b) appears incorrectly in no. 5.5.1-25, and (c) in no. 5.5.1-24 contains an additional modulus that must be removed. Once again, we regret that Prudnikov et al. [53] do not specify sources. As regards expansion (c), which actually represents the Fourier series expansion for the logarithm of the Γ-function, this important result is attributed to Ernst Eduard Kummer [39, p. 86], [71, p. 250], [9, vol. I, p. 23], albeit he obtained this expression only in 1847 [35, p. 4], i.e. 5 years later after the publication of the Malmsten et al.'s dissertation [40], see Fig. 2.[28] Moreover, taking into account that

[28] Strictly speaking, Kummer's result [35, p. 4] is obtained from the Malmsten et al.'s one [40, formula (74), p. 62] by putting $a=\pi(2x-1)$.

expansion no. 21-e was also derived by Malmsten et al. [40, p. 74], the authorship of such a kind of series should without doubt be, attributed to Malmsten and not to Kummer. As regards formulas (a), (d), and (e), they seem to be unpublished yet.

21* Analogous to the previous exercise, prove the following results related to sums containing cosines and logarithms:

$$\text{(a)}\quad \sum_{n=1}^{\infty}(-1)^n\frac{\ln bn\cdot\cos\varphi n}{n} = (\gamma-\ln b)\ln\left(2\cos\frac{\varphi}{2}\right)+\frac{\Phi(\varphi)}{2}$$
$$+\frac{\pi\operatorname{ctg}\varphi}{2}\left\{2\ln\Gamma\left(\frac{1}{2}+\frac{\varphi}{2\pi}\right)+\ln\cos\frac{\varphi}{2}+\frac{\varphi}{\pi}\ln 2\pi-\ln\pi\right\}$$
$$-\pi<\varphi<\pi$$

$$\text{(b)}\quad \sum_{n=1}^{\infty}\frac{\ln bn\cdot\cos\varphi n}{n} = (\gamma-\ln b)\ln\left(2\sin\frac{\varphi}{2}\right)+\frac{\Phi(\varphi-\pi)}{2}$$
$$+\frac{\pi\operatorname{ctg}\varphi}{2}\left\{2\ln\Gamma\left(\frac{\varphi}{2\pi}\right)+\ln\sin\frac{\varphi}{2}+\frac{\varphi-\pi}{\pi}\ln 2\pi-\ln\pi\right\}$$
$$0<\varphi<2\pi,$$

$$\text{(c)}\quad \sum_{n=1}^{\infty}\frac{\ln(2n-1)\cdot\cos[(2n-1)\varphi]}{2n-1} = \frac{\gamma}{2}\ln\operatorname{tg}\frac{\varphi}{2}+\frac{1}{4}\{\Phi(\varphi-\pi)-\Phi(\varphi)\}$$
$$+\frac{\pi\operatorname{ctg}\varphi}{4}\left\{2\ln\Gamma\left(\frac{\varphi}{2\pi}\right)-2\ln\Gamma\left(\frac{1}{2}+\frac{\varphi}{2\pi}\right)+\ln\operatorname{tg}\frac{\varphi}{2}-\ln 2\pi\right\}$$
$$0<\varphi<\pi,$$

$$\text{(d)}\quad \sum_{n=1}^{\infty}(-1)^n\frac{\ln(2n-1)\cdot\cos[(2n-1)\varphi]}{2n-1} = \frac{\pi}{4}(\gamma+3\ln\pi+2\ln 2-\ln\cos\varphi)$$
$$-\frac{\pi}{2}\ln\left[\Gamma\left(\frac{1}{4}+\frac{\varphi}{2\pi}\right)\Gamma\left(\frac{1}{4}-\frac{\varphi}{2\pi}\right)\right],\qquad -\frac{\pi}{2}<\varphi<\frac{\pi}{2}$$

$b>0$, and where we denoted by $\Phi(\varphi)$ the following improper integral:

$$\Phi(\varphi)\equiv\int_0^{\infty}\frac{e^{-x}\ln x}{\operatorname{ch}x+\cos\varphi}\,dx = 2\int_1^{\infty}\frac{\ln\ln x}{x(x^2+2x\cos\varphi+1)}\,dx,\qquad -\pi<\varphi<\pi.$$

Nota bene: Formula (d) was previously derived by Malmsten et al. [40, p. 74] and [41, p. 39][29] and does not appear, to our knowledge, in other sources. As regards the other sums, they do not appear in Gradshteyn and Ryzhik's tables [28], neither in

[29] Malmsten originally wrote $a/2$ instead of φ.

Prudnikov et al.'s tables [53] nor in other sources that we could found. The reader may also remark that the integral $\Phi(\varphi)$ may be trivially expressed in terms of the polylogarithm's derivative. Although much less trivially, but it can be also expressed in terms of the second-order derivatives of the Hurwitz ζ-function:

$$\begin{aligned}\Phi(\varphi) = 2\ln 2\pi\cdot\ln\biggl(2\cos\frac{\varphi}{2}\biggr) - \pi\operatorname{ctg}\varphi\biggl\{2\ln\Gamma\biggl(\frac{1}{2}+\frac{\varphi}{2\pi}\biggr) + \ln\cos\frac{\varphi}{2}\\ + \frac{\varphi}{\pi}\ln 2\pi - \ln\pi\biggr\} + \zeta''\biggl(0,\frac{1}{2}+\frac{\varphi}{2\pi}\biggr) + \zeta''\biggl(0,\frac{1}{2}-\frac{\varphi}{2\pi}\biggr).\end{aligned} \tag{49}$$

This formula straightforwardly follows from the comparison of no. 21-a to no. 22-a.

22* Results in exercises no. 18–21 are obtained by means of geometric series. Another way to treat such problems consists is to use the Mittag–Leffler theorem and similar expansions. By proceeding in this manner, prove

(a) $$\begin{aligned}&\sum_{n=1}^{\infty}(-1)^n\frac{\ln n\cdot\cos\varphi n}{n}\\ &= \ln\pi\cdot\ln\cos\frac{\varphi}{2} - \frac{1}{2}\ln^2 2 + \gamma\ln 2 + \underbrace{\int_0^{\infty}\frac{\operatorname{ch}(\varphi x/\pi)-1}{x\operatorname{sh}x}\ln x\,dx}_{\Upsilon(\varphi)}\\ &= (\gamma+\ln 2\pi)\cdot\ln\biggl(2\cos\frac{\varphi}{2}\biggr) + \frac{1}{2}\biggl\{\zeta''\biggl(0,\frac{1}{2}+\frac{\varphi}{2\pi}\biggr) + \zeta''\biggl(0,\frac{1}{2}-\frac{\varphi}{2\pi}\biggr)\biggr\}\\ &= -\frac{1}{2}\ln^2 2 + \gamma\ln 2 + (\gamma+\ln 2\pi)\ln\cos\frac{\varphi}{2} - 2\Gamma_1\biggl(\frac{1}{2}\biggr) + \Gamma_1\biggl(\frac{1}{2}+\frac{\varphi}{2\pi}\biggr)\\ &\quad + \Gamma_1\biggl(\frac{1}{2}-\frac{\varphi}{2\pi}\biggr), \qquad -\pi<\varphi<\pi,\end{aligned}$$

(b) $$\begin{aligned}&\sum_{n=1}^{\infty}\frac{\ln n\cdot\cos\varphi n}{n} = \ln\pi\cdot\ln\sin\frac{\varphi}{2} - \frac{1}{2}\ln^2 2 + \gamma\ln 2 + \Upsilon(\varphi-\pi)\\ &= (\gamma+\ln 2\pi)\cdot\ln\biggl(2\sin\frac{\varphi}{2}\biggr) + \frac{1}{2}\biggl\{\zeta''\biggl(0,\frac{\varphi}{2\pi}\biggr) + \zeta''\biggl(0,1-\frac{\varphi}{2\pi}\biggr)\biggr\},\\ &\qquad\qquad 0<\varphi<2\pi,\end{aligned}$$

(c) $$\begin{aligned}&\sum_{n=1}^{\infty}\frac{\ln(2n-1)\cdot\cos[(2n-1)\varphi]}{2n-1} = \frac{1}{2}(\gamma+\ln 2\pi)\cdot\ln\operatorname{tg}\frac{\varphi}{2}\\ &+\frac{1}{4}\biggl\{\zeta''\biggl(0,\frac{\varphi}{2\pi}\biggr) + \zeta''\biggl(0,1-\frac{\varphi}{2\pi}\biggr) - \zeta''\biggl(0,\frac{1}{2}+\frac{\varphi}{2\pi}\biggr) - \zeta''\biggl(0,\frac{1}{2}-\frac{\varphi}{2\pi}\biggr)\biggr\},\\ &\qquad\qquad 0<\varphi<\pi,\end{aligned}$$

(d) $$\sum_{n=0}^{\infty}(-1)^n\frac{\ln(2n+1)\cdot\sin[(2n+1)\varphi]}{2n+1}$$

$$=\frac{1}{2}\ln\frac{\pi}{2}\cdot\ln\operatorname{tg}\left[\frac{\pi}{4}-\frac{\varphi}{2}\right]+\frac{1}{2}\int\limits_0^{\infty}\frac{\operatorname{sh}(2\varphi x/\pi)}{x\operatorname{ch}x}\ln x\,dx,\qquad -\frac{\pi}{2}<\varphi<\frac{\pi}{2}$$

where $\zeta''(s,v)$ stands for the second derivative of the Hurwitz ζ-function with respect to s, and $\Gamma_1(x)$ stands for the antiderivative of the first generalized Stieltjes constant $\gamma_1(x)$. Note that formulas (a)–(c) actually represent Fourier series expansions for the sum of two second-order derivative of the Hurwitz ζ-function at $s=0$ and have a comparatively simple form. At the same time, they can be regarded as the reflection formula for the second-order derivative of the Hurwitz ζ-function at $s=0$.

Hint: For (a): by using a theorem from [23, no. 27.09], one can show that the following expansion holds (see also [23, no. 27.10.1, p. 265] and [29, no. 641, p. 73])

$$\frac{\operatorname{sh}\alpha z}{\operatorname{sh}z}=-2\pi\sum_{n=1}^{\infty}(-1)^n\frac{n\cdot\sin(\alpha\pi n)}{z^2+\pi^2n^2},\qquad -1<\alpha<1 \tag{50}$$

and is valid in the entire complex plane except at points $z=\pi i n$, $n\in\mathbb{Z}$. Computing the antiderivative of the above expansion with respect to α, and then bearing in mind that

$$\int\limits_0^{\infty}\frac{\ln x}{x^2+\pi^2n^2}\,dx=\frac{\ln\pi n}{2n},\qquad n>0, \tag{51}$$

as well as using the series from no. 19-c, yields the first part of the formula (that containing the integral $\Upsilon(\varphi)$). The second part of the formula (that containing Hurwitz ζ-functions) is derived as follows. First, as in no. 18, write $\operatorname{sh}^{-1}x$ as a sum of a geometric series. Then, apply term-by-term integration[30] and utilize the integral definition of the Hurwitz ζ-function, see e.g. [9, vol. I, p. 25, Eq. 1.10(3)], [5, p. 251, §12.3]. Proceeding in this manner yields

$$\int\limits_0^{\infty}\frac{x^a(\operatorname{ch}bx-1)}{\operatorname{sh}x}\,dx=\frac{\Gamma(a+1)}{2^{a+1}}\left\{\zeta\left(a+1,\frac{1}{2}+\frac{b}{2}\right)+\zeta\left(a+1,\frac{1}{2}-\frac{b}{2}\right)\right.$$
$$\left.-2\left(2^{a+1}-1\right)\zeta(a+1)\right\} \tag{52}$$

[30] The series being uniformly convergent.

$a > -2$, $|\operatorname{Re} b| < 1$. Differentiating the latter expression with respect to a, and then, using an appropriate limiting procedure, we obtain

$$\int_0^\infty \frac{\operatorname{ch} bx - 1}{x \operatorname{sh} x} \ln x \, dx = \frac{1}{2}\left\{\zeta''\left(0, \frac{1}{2} + \frac{b}{2}\right) + \zeta''\left(0, \frac{1}{2} - \frac{b}{2}\right)\right\} + \frac{3}{2}\ln^2 2$$
$$+ \ln 2 \cdot \ln \pi + (\gamma + \ln 2) \ln \cos \frac{\pi b}{2}$$

$|\operatorname{Re} b| < 1$. The second part of the formula in its final form is now straightforward. In order to obtain the third variant of the formula (that containing two antiderivatives of the first Stieltjes constants) consider again $\Upsilon(\varphi)$, which is also the antiderivative of no. 63-a with respect to p at $p = \varphi/\pi$. The constant of integration is easily determined by putting $\varphi = 0$, which yields for the latter the value of $-2\Gamma_1(1/2)$. In order to get result (b), write $\varphi - \pi$ instead of φ in (a). For (d): it is sufficient to ascertain that

$$\frac{\operatorname{ch} \alpha z}{\operatorname{ch} z} = \pi \sum_{n=0}^\infty (-1)^n \frac{(2n+1)\cos\left[(n+\frac{1}{2})\alpha\pi\right]}{z^2 + (n+\frac{1}{2})^2\pi^2}, \qquad -1 < \alpha < 1,$$

which holds in the whole complex plane except at $z = \left(n + \frac{1}{2}\right)\pi i$, $n \in \mathbb{Z}$. If necessary, results (b)–(d) may be also written in terms of the antiderivatives of the first Stieltjes constants.

23* By using a similar method, prove that for any $a \geqslant 0$ and $-\pi < \varphi < \pi$, we have

(a) $$\sum_{n=1}^\infty (-1)^n \frac{\ln(n+a) \cdot [\cos \varphi n - 1]}{n}$$
$$= \ln \pi \cdot \ln \cos \frac{\varphi}{2} + \frac{1}{2} \int_0^\infty \frac{\operatorname{ch}(\varphi x/\pi) - 1}{x \operatorname{sh} x} \ln\left(x^2 + \pi^2 a^2\right) dx,$$

(b) $$\sum_{n=0}^\infty (-1)^n \frac{\ln(n+a) \cdot \sin\left[(n+\frac{1}{2})\varphi\right]}{2n+1}$$
$$= \frac{1}{2} \ln \pi \cdot \ln \operatorname{tg}\left[\frac{\pi}{4} - \frac{\varphi}{4}\right] + \frac{1}{4} \int_0^\infty \frac{\operatorname{sh}(2\varphi x/\pi)}{x \operatorname{ch} x} \ln\left[x^2 + \pi^2\left(a - \tfrac{1}{2}\right)^2\right] dx.$$

Prove that if φ is a rational part of π, i.e. $\varphi \equiv \pi m/n$ where m is integer and n is positive integer, then for any $a \in \mathbb{C}$ (except points specified below) and $k = 2, 3, 4, \ldots$

(c) $$\sum_{l=1}^\infty \frac{\sin \varphi l}{l + a} = -\frac{1}{2n} \sum_{l=1}^{2n-1} \sin \varphi l \cdot \Psi\left(\frac{l+a}{2n}\right)$$

$$\text{(d)}\quad \sum_{l=1}^{\infty} \frac{\sin \varphi l}{(l+a)^k} = \frac{(-1)^k}{(2n)^k (k-1)!} \sum_{l=1}^{2n-1} \sin \varphi l \cdot \Psi_{k-1}\!\left(\frac{l+a}{2n}\right)$$

$$\text{(e)}\quad \sum_{l=0}^{\infty} \frac{\cos \varphi l}{l+a} = -\frac{1}{2n} \sum_{l=0}^{2n-1} \cos \varphi l \cdot \Psi\!\left(\frac{l+a}{2n}\right)$$

$$\text{(f)}\quad \sum_{l=0}^{\infty} \frac{\cos \varphi l}{(l+a)^k} = \frac{(-1)^k}{(2n)^k (k-1)!} \sum_{l=0}^{2n-1} \cos \varphi l \cdot \Psi_{k-1}\!\left(\frac{l+a}{2n}\right)$$

where in (c)–(d) $a \neq -1, -2, -3, \ldots$, and in (e)–(f) $a \neq 0, -1, -2, \ldots$
Nota bene: Trigonometric series have been the subject of numerous studies since the 18th century; it is therefore very difficult to know if formulas (c)–(f) represent some new contribution or not. We, however, remark that no closed-form expressions for these series are given in Prudnikov et al.'s tables [53], nor in Gradshteyn and Ryzhik's tables [28].[31]

Hint: For (a) and (b), demonstrations are similar to no. 22, except that integral (51) is replaced with

$$\int_0^{\infty} \frac{\ln(x^2+a^2)}{x^2+\varepsilon^2}\, dx = \frac{\pi}{\varepsilon} \ln(a+\varepsilon), \qquad a > 0,\ \varepsilon > 0,$$

see, e.g., the solution for [59, no. 22, p. 187]. Result (c) is obtained by applying $\frac{\partial^2}{\partial\varphi\partial a}$ to (a). After the differentiation, if φ is a rational part of π, then the integral in the right part of (a) coincide with the derivative of Malmsten's integral no. 3-(a) with respect to a; this leads to the Ψ-function in the right part. Shifting φ by π yields the series (c) in its final form. Result (d) is obtained from (c) by calculating its $(k-1)$th derivative with respect to a. In like manner, we obtain (e) from (b), but result (c) should be also used. Finally, extensions to $a \in \mathbb{C}$ follow from the principle of analytic continuation.

24 From the results obtained in no. 22 and no. 19, prove that

$$\text{(a)}\quad \zeta''\!\left(0, \frac{1}{2}\right) = -\frac{3}{2}\ln^2 2 - \ln\pi \ln 2,$$

$$\text{(b)}\quad \int_0^{\infty} \frac{\ln(x^2+1/4)\cdot \operatorname{arctg}(2x)}{e^{2\pi x}-1}\, dx = \frac{3}{4}\ln^2 2 + \frac{(\ln\pi - 1)\ln 2}{2} - \frac{1}{2}.$$

Nota bene: Mark Coffey [18] suggested that it may be possible to obtain integral (b) by contour integration. Regrettably, he did not indicate how to circumvent the prob-

[31] Prudnikov et al.' tables provides, however, several formulas for the series (c) and (e) when a is rational [53, vol. I, § 5.4.3].

lem of branch points that have both logarithm and arctangent (we pointed out the importance of this problem in Sect. 3.1).

Hint: For (a): put $\varphi = 0$ in no. 22-a and compare to no. 19-a. Another way to prove the same result is to recall that $\zeta\left(s, \frac{1}{2}\right) = (2^s - 1)\zeta(s)$, and then, to use these well-known results $\zeta(0) = -\frac{1}{2}$ and $\zeta'(0) = -\frac{1}{2}\ln 2\pi$. For (b): compute the second derivative of the Hermite representation for the Hurwitz ζ-function with respect to s at $s = 0$ and $v = \frac{1}{2}$, see (3).

25* By combining various methods, prove that if p is a rational part of b, i.e. $p = bm/n$, where b is some parameter with positive real part, and numbers m and n are positive integers such that $m < n$, then

$$\text{(a)}\quad \int_0^\infty \frac{\operatorname{sh} px \cdot \ln x}{e^{bx} - 1}\, dx = -\frac{\pi}{2b}\operatorname{ctg}\frac{m\pi}{n} \cdot \ln\frac{2\pi n}{b} + \frac{\pi}{b}\sum_{l=1}^{n-1}\sin\frac{2m\pi l}{n} \cdot \ln\Gamma\left(\frac{l}{n}\right) - \frac{n}{2bm}\left(\gamma + \ln\frac{bm}{n}\right),$$

$$\text{(b)}\quad \int_0^\infty \frac{\operatorname{sh} px \cdot \ln x}{e^{bx} + 1}\, dx = \frac{\pi}{2b}\csc\frac{m\pi}{n} \cdot \ln\frac{2\pi n}{b} - \frac{\pi}{b}\sum_{l=0}^{n-1}\sin\frac{(2l+1)m\pi}{n} \cdot \ln\Gamma\left(\frac{2l+1}{2n}\right) + \frac{n}{2bm}\left(\gamma + \ln\frac{bm}{n}\right).$$

Hint: From these two well-known (at the time) integrals

$$\int_0^\infty \frac{\operatorname{ch} ax \cdot \sin rx}{\operatorname{sh}\pi x}\, dx = \frac{\operatorname{sh} r}{2(\operatorname{ch} r + \cos a)} \quad \text{and} \quad \int_0^\infty e^{-\omega x}\sin rx\, dx = \frac{r}{r^2 + \omega^2},$$

Legendre [64, vol. II, p. 189] derived, by using elementary transformations the value of the following integral[32]

$$\int_0^\infty \frac{\sin rx}{e^{2\pi x} - 1}\, dx = \frac{1}{4}\operatorname{cth}\frac{r}{2} - \frac{1}{2r}, \qquad |\operatorname{Re} r| < 2\pi,\ r \neq 0.$$

[32]This integral was first evaluated by Poisson in 1813 in the famous work [51, pp. 214–220], see also [12, p. 240]. The work [51] was later republished in several volumes of the *Journal de l'École Polytechnique*, see namely [52].

By the same line of reasoning, one can show that

$$\frac{\operatorname{sh} ax}{\operatorname{sh} bx} = -\frac{2\operatorname{sh}\omega x}{e^{2bx}-1} + e^{-\omega x}, \qquad \frac{\operatorname{ch} ax}{\operatorname{ch} bx} = \frac{2\operatorname{sh}\omega x}{e^{2bx}+1} + e^{-\omega x},$$
$$\frac{\operatorname{ch} ax}{\operatorname{sh} bx} = \frac{2\operatorname{ch}\omega x}{e^{2bx}-1} + e^{-\omega x}, \qquad \frac{\operatorname{sh} ax}{\operatorname{ch} bx} = -\frac{2\operatorname{ch}\omega x}{e^{2bx}+1} + e^{-\omega x},$$

where $a \equiv b - \omega$. Accounting for these elementary formulas and using previously obtained integrals in exercise no. 3, as well as bearing in mind that

$$\int_0^\infty e^{-\omega x} \ln x \, dx = -\frac{\gamma + \ln\omega}{\omega}, \qquad \operatorname{Re}\omega > 0,$$

we obtain both integrals (a) and (b). The final formulas are obtained by further simplification with the help of the duplication formula for the Γ-function and with the help of these results known from elementary analysis:

$$\sum_{l=1}^{n-1} l \cdot \sin\frac{(2l+1)m\pi}{n} = -\frac{n}{2}\csc\frac{m\pi}{n}, \qquad \sum_{l=1}^{n-1} l \cdot \sin\frac{2m\pi l}{n} = -\frac{n}{2}\operatorname{ctg}\frac{m\pi}{n}$$

for $m = 1, 2, \ldots, n-1$.

Nota bene: The evaluation of

$$\int_0^\infty \frac{\operatorname{ch} px \cdot \ln x}{e^{bx}+1}\, dx$$

requires the knowledge of the following integral

$$\int_0^\infty \frac{\operatorname{sh} px \cdot \ln x}{\operatorname{ch} bx}\, dx.$$

The latter may be expressed by means of the first generalized Stieltjes constants, see exercise no. 65.

26 By employing results obtained in exercise no. 25, prove that

$$\int_0^\infty e^{-bx} \cdot \operatorname{th} bx \cdot \ln x \, dx = -\frac{2\pi}{b}\ln\Gamma\left(\frac{1}{4}\right) + \frac{\pi}{2b}\ln\frac{4\pi^3}{b} + \frac{\gamma + \ln b}{b}, \qquad \operatorname{Re} b > 0.$$

Nota bene: alternatively, the same result may be derived by the series expansion method. By recalling that $\operatorname{ch}^{-1} x$ is the sum of a geometric progression, one may easily show that

$$\operatorname{th} x = 1 + 2\sum_{n=1}^\infty (-1)^n e^{-2nx}, \qquad \operatorname{Re} x > 0.$$

This expansion and the use of the result obtained in no. 19-d yield the above formula.

27 Show that

$$\text{(a)}\quad \int_0^\infty \frac{e^{-x}\ln x}{e^x+1}\,dx = \int_1^\infty \frac{\ln\ln x}{x^2(1+x)}\,dx = \frac{1}{2}\ln^2 2-\gamma,$$

$$\text{(b)}\quad \int_0^\infty \frac{(\operatorname{ch} x-1)\ln x}{e^{2x}-1}\,dx = \frac{1}{2}\int_0^\infty e^{-x}\cdot \operatorname{th}\frac{x}{2}\cdot \ln x\,dx$$

$$= \frac{1}{2}\int_1^\infty \frac{(x-1)\ln\ln x}{x^2(x+1)}\,dx = \frac{1}{2}\left(\gamma-\ln^2 2\right).$$

4.2 Logarithmic integrals of ln ln-type in combination with polynomials

An appropriate change of variable applied to integrals treated in the preceding exercises allows one to evaluate many beautiful ln ln-integrals. Below we give several examples of such integrals, but the given list is far from exhaustive.

28* Prove that

$$\int_0^1 \frac{x^2-3x+1}{1+x^2+x^4}\ln\ln\frac{1}{x}\,dx = \int_1^\infty \frac{x^2-3x+1}{1+x^2+x^4}\ln\ln x\,dx = \frac{\pi\ln 2}{3\sqrt{3}}.$$

Hint: Use formulas (44) and (45).

29 By using formula (37) show that

$$\text{(a)}\quad \int_0^1 \frac{x\ln\ln\frac{1}{x}}{1+x^4}\,dx = \int_1^\infty \frac{x\ln\ln x}{1+x^4}\,dx = \frac{\pi}{8}\left\{\ln 2+3\ln\pi-4\ln\Gamma\left(\frac{1}{4}\right)\right\},$$

$$\text{(b)}\quad \int_0^1 \frac{x^{n-1}\ln\ln\frac{1}{x}}{(1+x^n)^2}\,dx = \int_1^\infty \frac{x^{n-1}\ln\ln x}{(1+x^n)^2}\,dx = \frac{1}{2n}\left(-\gamma+\ln\frac{\pi}{2n}\right),$$

$$\text{(c)}\quad \int_0^1 \frac{x\ln\ln\frac{1}{x}}{1+x^2+x^4}\,dx = \int_1^\infty \frac{x\ln\ln x}{1+x^2+x^4}\,dx$$

$$= \frac{\pi}{12\sqrt{3}}\left\{6\ln 2-3\ln 3+8\ln\pi-12\ln\Gamma\left(\frac{1}{3}\right)\right\},$$

$$\text{(d)}\quad \int_0^1 \frac{\ln\ln\frac{1}{x}}{1+\sqrt{2}x+x^2}\,dx = \int_1^\infty \frac{\ln\ln x}{1+\sqrt{2}x+x^2}\,dx$$

$$= \frac{\pi}{4\sqrt{2}}\left\{5\ln 2\pi - 2\ln(2+\sqrt{2}) - 8\ln\Gamma\left(\frac{3}{8}\right)\right\},$$

$$\text{(e)}\quad \int_0^1 \frac{\ln\ln\frac{1}{x}}{1-\sqrt{2}x+x^2}\,dx = \int_1^\infty \frac{\ln\ln x}{1-\sqrt{2}x+x^2}\,dx$$

$$= \frac{\pi}{4\sqrt{2}}\left\{7\ln 2\pi - 2\ln(2-\sqrt{2}) - 8\ln\Gamma\left(\frac{1}{8}\right)\right\},$$

$$\text{(f)}\quad \int_0^1 \frac{x\ln\ln\frac{1}{x}}{1+\sqrt{2}x^2+x^4}\,dx = \int_1^\infty \frac{x\ln\ln x}{1+\sqrt{2}x^2+x^4}\,dx$$

$$= \frac{\pi}{8\sqrt{2}}\left\{4\ln 2 - 2\ln(2+\sqrt{2}) + 5\ln\pi - 8\ln\Gamma\left(\frac{3}{8}\right)\right\},$$

$$\text{(g)}\quad \int_0^1 \frac{x\ln\ln\frac{1}{x}}{1-\sqrt{2}x^2+x^4}\,dx = \int_1^\infty \frac{x\ln\ln x}{1-\sqrt{2}x^2+x^4}\,dx$$

$$= \frac{\pi}{8\sqrt{2}}\left\{4\ln 2 - 2\ln(2-\sqrt{2}) + 7\ln\pi - 8\ln\Gamma\left(\frac{1}{8}\right)\right\},$$

$$\text{(h)}\quad \int_0^1 \frac{\ln\ln\frac{1}{x}}{1+2x\cos\varphi+x^2}\,dx = \int_1^\infty \frac{\ln\ln x}{1+2x\cos\varphi+x^2}\,dx$$

$$= \frac{\pi}{2\sin\varphi}\ln\left\{\frac{(2\pi)^{\frac{\varphi}{\pi}}\,\Gamma\left(\frac{1}{2}+\frac{\varphi}{2\pi}\right)}{\Gamma\left(\frac{1}{2}-\frac{\varphi}{2\pi}\right)}\right\}, \qquad |\operatorname{Re}\varphi| < \pi.$$

Show that the function $Q(x)$ in each of these integrals satisfies the functional relationship $Q(x^{-1}) = x^2 Q(x)$.

Nota bene: Many of the above integrals for bounds [0, 1] were already treated by different authors. However, none of them noticed that they can be equally taken from 1 to ∞ and that they all obey $Q(x^{-1}) = x^2 Q(x)$. In particular, result (a), in a slightly different form, as well as result (b) were presented by Adamchik [2] (it should be noted however that both integrals may be easily obtained from Malmsten's integrals (1) and (46), respectively, by means of a simple change of variable). Result (c) is a particular case of Malmsten's integral (17) for $n = 3$ (see also exercise no. 32 below), and it was also independently evaluated in [44]. Integral (h) for real φ was evaluated by Malmsten [41, p. 24] and also appears in [61, Table 190-9], in [62, Table 147-9] and in [28, no. 4.325-7] (see also exercise no. 2 above). Integrals (d) and (e) are particular cases of (h) with $\varphi = \pi/4$ and $\varphi = 3\pi/4$ respectively. Integrals (f) and (g) may be deduced from integrals (d) and (e) respectively by making a suitable change of variable; integral (g) was also independently evaluated in [44] (however the authors did not simplify their result). By the way, formula (h) may provide many other useful results. For instance, the result given in the Proposition 7.5 [44] may be obtained directly from h) by differentiating it with respect to φ; formulas obtained in examples 7.8–7.10 [44] follows immediately from such a derivative.

30* In [67, p. 314] Vardi, after having considered three basic Malmsten's formulas (1) and (2a), (2b), supposed that the multiplicative inverse of the argument of the Γ-function is the degree in which the poles of the integrand are the roots of unity.[33] Prove that such an assumption is not generally true for the integrals of the form

$$\int_0^1 \frac{\ln\ln\frac{1}{x}}{ax^2+bx+c}\,dx \tag{53}$$

where a, b, c are arbitrary real coefficients. Determine exact conditions under which this assumption is true.

Proof From condition (27), which implies that $Q(x^{-1}) = x^2 Q(x)$ [see Sect. 3.2], it follows that the above integral may be expressed in terms of the Γ-function if $a = c$. Consider the case $b \in (-2a, +2a)$. As shown in no. 12-a, in such a case integral (53) may be written by means of the Γ-function of a real argument. Since $-1 \leqslant \cos\varphi \leqslant +1$ for any $0 \leqslant \varphi \leqslant 2\pi$, the integral in question may be always written in the form analogous to no. 29-h, namely:

$$\int_0^1 \frac{\ln\ln\frac{1}{x}}{1-2x\cos\varphi+x^2}\,dx = \int_1^\infty \frac{\ln\ln x}{1-2x\cos\varphi+x^2}\,dx$$
$$= -\frac{\pi}{2\sin\varphi}\left\{\frac{\varphi-\pi}{\pi}\ln 2\pi - \ln\pi + \ln\sin\frac{\varphi}{2} + 2\ln\Gamma\left(\frac{\varphi}{2\pi}\right)\right\}$$

where $\varphi \in (0, 2\pi)$. Accordingly to Vardi's statement, the zeros of the denominator $x_{1,2}$ must be the $(2\pi/\varphi)$th roots of 1, i.e., $x_{1,2}^{\frac{2\pi}{\varphi}} = 1$. Computing the roots of the quadratic polynomial in the denominator yields $x_{1,2} = e^{\pm i\varphi}$. Hence $x_{1,2}^{\frac{2\pi}{\varphi}} = e^{\pm 2\pi i} = 1$, and thus, Vardi's hypothesis is true. Consider now the case $b \notin (-2a, +2a)$. This case, in virtue of what was established in no. 12-a, leads to the Γ-function of a complex argument (integrals no. 4, 5, 7-a, 8-a and 9-a are typical

[33] In fact, Vardi was not very clear in defining his idea of the relationship between the poles of the integrand and the argument of the Γ-function with the help of which Malmsten's integrals are expressed. The statement "in Eq. (2a) the number 3 plays the 'key role' and in Eq. (2b) 6 is the 'magic number'" [67, p. 313] may be also interpreted in the sense that the least possible integer in the denominator of the argument of the Γ-function (and not the inverse multiplicative of the argument of the Γ-function) should be equal to the degree in which the poles of the integrand are the roots of unity. However, the fallacy of this statement is also evident from the proof given above.

examples of such a case). More precisely, integral (53) reduces to[34]

$$\int_0^1 \frac{\ln\ln\frac{1}{x}}{1+2x\operatorname{ch}t+x^2}\,dx = \int_1^\infty \frac{\ln\ln x}{1+2x\operatorname{ch}t+x^2}\,dx$$

$$= -\frac{\pi i}{2\operatorname{sh}t}\left\{\frac{it}{\pi}\ln 2\pi - \ln\pi + \ln\operatorname{ch}\frac{t}{2} + 2\ln\Gamma\left(\frac{1}{2}+\frac{it}{2\pi}\right)\right\}$$

with $t\in(0,\infty)$. By Vieta's formulas, we easily find the roots of the quadratic polynomial in the denominator: $x_{1,2}=-e^{\mp t}$. Are these values the pth roots of 1? where

$$p \equiv \frac{1}{\frac{1}{2}+\frac{it}{2\pi}} = \frac{2\pi^2}{\pi^2+t^2} - i\frac{2\pi t}{\pi^2+t^2}.$$

The straightforward verification shows that only $x_1^p=1$, while $x_2^p\neq 1$, and thus, Vardi's assumption is false. Now, consider the case $b=2a$. In this case, the quadratic polynomial ax^2+bx+c takes the form $a(x+1)^2$, and hence (53) reduces to Malmsten's integral (46), which is given in terms of the Euler's constant γ and not of the logarithm of the Γ-function. However, Vardi remarked that his assumption remains applicable only if ax^2+bx+c is irreducible, which is obviously not the case if $b=2a$. Finally, when $b=-2a$ integral (53) does not converge.

As regards the case $a\neq c$, the general procedure for the evaluation of (53) is not yet well known, but in many cases such integrals have higher transcendence than the Γ-function (see, e.g. the last paragraph in Sect. 3.2). Consequently, Vardi's assumption for integral (53) remains true only if coefficients a,b,c are chosen so that $a=c$ and $-2a<b<+2a$, where we may suppose, without loss of generality, that $a>0$. □

31* With the help of (37) show that

$$\int_0^1 \frac{x^{\alpha-1}\ln\ln\frac{1}{x}}{1+x^{2\alpha}}\,dx = \int_1^\infty \frac{x^{\alpha-1}\ln\ln x}{1+x^{2\alpha}}\,dx = \frac{\pi}{4\alpha}\left\{\ln\frac{4\pi^3}{\alpha} - 4\ln\Gamma\left(\frac{1}{4}\right)\right\}, \qquad \alpha>0.$$

Note that case $\alpha=1$ corresponds to the simplest Malmsten's integral (1); case $\alpha=2$ gives the result obtained by Adamchik (see the previous exercise). Cases for other α (not necessarily integer) seem to be new (at least, in this form). The above result may be therefore regarded as a generalization of Malmsten's integral (1).

32* Show that if

$$\int_0^1 \frac{x^{\delta}\ln\ln\frac{1}{x}}{1\pm x^{\alpha}+x^{\beta}}\,dx = \int_1^\infty \frac{x^{\delta}\ln\ln x}{1\pm x^{\alpha}+x^{\beta}}\,dx,$$

[34] To assure the convergence, we should take in the denominator $1+2x\operatorname{ch}t+x^2$ rather than $1-2x\operatorname{ch}t+x^2$.

then, coefficients α, β and δ should be chosen so that $\beta = 2\alpha$ and $\delta = \alpha - 1$. Prove then that

$$\int_0^1 \frac{x^{\alpha-1}\ln\ln\frac{1}{x}}{1+x^\alpha+x^{2\alpha}}\,dx = \int_1^\infty \frac{x^{\alpha-1}\ln\ln x}{1+x^\alpha+x^{2\alpha}}\,dx = \frac{2\pi}{3\alpha\sqrt{3}}\left\{\ln\frac{4\pi^2}{\sqrt[4]{27\alpha^2}} - 3\ln\Gamma\left(\frac{1}{3}\right)\right\}$$

$$\int_0^1 \frac{x^{\alpha-1}\ln\ln\frac{1}{x}}{1-x^\alpha+x^{2\alpha}}\,dx = \int_1^\infty \frac{x^{\alpha-1}\ln\ln x}{1-x^\alpha+x^{2\alpha}}\,dx = \frac{4\pi}{3\alpha\sqrt{3}}\left\{\ln\frac{4\pi^2}{\sqrt[4]{54\alpha^2}} - 3\ln\Gamma\left(\frac{1}{3}\right)\right\}$$

where $\alpha > 0$. Note that these integrals for $\alpha = 1$ give Malmsten's integrals (2a), (2b) [or (44) and (45)], respectively. Hence, the above formulas may be seen as a generalization of (2a), (2b). Moreover, in the case $\alpha = 2$, these integrals become Malmsten's integrals (17) and (18) with $n = 3$, respectively.

Hint: Establish the condition under which $Q(x^{-1}) = x^2 Q(x)$. Then, make a change of variable $x = e^{t/\alpha}$.

33 Show that the function $Q(x)$ of Malmsten's integral (17) satisfies the functional relationship $Q(x^{-1}) = x^2 Q(x)$. Show that so do functions

$$\frac{x^{m-1} - x^{-m-1}}{x^n - x^{-n}} \quad \text{and} \quad \frac{x^{m-1} + x^{-m-1}}{x^n + x^{-n}}$$

where n and m are natural numbers.

Nota bene: Integrals containing such functions $Q(x)$ were evaluated by Malmsten in [41, pp. 7, 29].

34 Prove that for any positive α

$$\text{(a)} \quad \int_0^1 \frac{x^{\frac{\alpha n}{2}-1}\ln\ln\frac{1}{x}}{1+x^\alpha+x^{2\alpha}+\cdots+x^{n\alpha}}\,dx = \int_1^\infty \frac{x^{\frac{\alpha n}{2}-1}\ln\ln x}{1+x^\alpha+x^{2\alpha}+\cdots+x^{n\alpha}}\,dx$$

$$= \frac{2}{\alpha}\left\{Y_{n+1} - \frac{\pi}{2n+2}\operatorname{tg}\frac{\pi}{2n+2}\cdot\ln\frac{\alpha}{2}\right\}, \qquad n = 1, 2, 3, \ldots,$$

$$\text{(b)} \quad \int_0^1 \frac{x^{\frac{\alpha n}{2}-1}\ln\ln\frac{1}{x}}{1-x^\alpha+x^{2\alpha}-\cdots+x^{n\alpha}}\,dx = \int_1^\infty \frac{x^{\frac{\alpha n}{2}-1}\ln\ln x}{1-x^\alpha+x^{2\alpha}-\cdots+x^{n\alpha}}\,dx$$

$$= \frac{2}{\alpha}\left\{X_{n+1} - \frac{\pi}{2n+2}\sec\frac{\pi}{2n+2}\cdot\ln\frac{\alpha}{2}\right\}, \qquad n = 2, 4, 6, \ldots,$$

where Y_n and X_n are Malmsten's integrals (17) and (18), respectively.

Hint: For exercise (a), show first that

$$\int\limits_0^1 \frac{x^{n-2}}{1+x^2+x^4+\cdots+x^{2n-2}}\,dx = \int\limits_1^\infty \frac{x^{n-2}}{1+x^2+x^4+\cdots+x^{2n-2}}\,dx = \frac{\pi}{2n}\operatorname{tg}\frac{\pi}{2n}$$

As regards exercise (b), follow the same line of reasoning.

35* A family of logarithmic integrals I_n

$$I_n = \int\limits_0^1 \frac{x^{n-1}\ln\ln\frac{1}{x}}{(1+x^2)^n}\,dx = \int\limits_1^\infty \frac{x^{n-1}\ln\ln x}{(1+x^2)^n}\,dx = \frac{1}{2^n}\int\limits_0^\infty \frac{\ln x}{\operatorname{ch}^n x}\,dx,$$

$n = 1, 2, 3, \ldots$, generates special mathematical constants in the following way:

$$I_1 = \frac{\pi}{2}\ln 2 + \frac{3\pi}{4}\ln\pi - \pi\ln\Gamma\left(\frac{1}{4}\right),$$

$$I_2 = -\frac{1}{2}\ln 2 + \frac{1}{4}\ln\pi - \frac{\gamma}{4},$$

$$I_3 = \frac{\pi}{16}\ln 2 + \frac{3\pi}{32}\ln\pi - \frac{\mathrm{G}}{4\pi} - \frac{\pi}{8}\ln\Gamma\left(\frac{1}{4}\right),$$

$$I_4 = \begin{cases} -\dfrac{1}{12}\ln 2 + \dfrac{1}{24}\ln\pi - \dfrac{\gamma}{24} - \dfrac{7\,\zeta(3)}{48\pi^2}, & [\zeta\text{-form}], \\ -\dfrac{1}{12}\ln 2 + \dfrac{1}{24}\ln\pi - \dfrac{\gamma}{24} + \dfrac{1}{96\pi^2}\Psi_2\left(\dfrac{1}{2}\right), & [\Psi\text{-form}], \end{cases}$$

$$I_5 = \frac{3\pi}{256}\ln 2 + \frac{9\pi}{512}\ln\pi + \frac{\pi}{768} - \frac{5\,\mathrm{G}}{96\pi} - \frac{3\pi}{128}\ln\Gamma\left(\frac{1}{4}\right) - \frac{1}{6144\pi^3}\Psi_3\left(\frac{1}{4}\right),$$

$$I_6 = \begin{cases} -\dfrac{1}{60}\ln 2 + \dfrac{1}{120}\ln\pi - \dfrac{\gamma}{120} - \dfrac{7\,\zeta(3)}{192\pi^2} - \dfrac{31\,\zeta(5)}{320\pi^4}, & [\zeta\text{-form}], \\ -\dfrac{1}{60}\ln 2 + \dfrac{1}{120}\ln\pi - \dfrac{\gamma}{120} + \dfrac{1}{384\pi^2}\Psi_2\left(\dfrac{1}{2}\right) + \dfrac{1}{7680\pi^4}\Psi_4\left(\dfrac{1}{2}\right), & [\Psi\text{-form}], \end{cases}$$

$$\begin{aligned} I_7 = {} & \frac{5\pi}{2048}\ln 2 + \frac{15\pi}{4096}\ln\pi + \frac{43\pi}{92160} \\ & - \frac{259\,\mathrm{G}}{23040\pi} - \frac{5\pi}{1024}\ln\Gamma\left(\frac{1}{4}\right) - \frac{7}{147456\pi^3}\Psi_3\left(\frac{1}{4}\right) - \frac{1}{2949120\pi^5}\Psi_5\left(\frac{1}{4}\right), \end{aligned}$$

$$I_8 = \begin{cases} -\dfrac{1}{280}\ln 2 + \dfrac{1}{560}\ln\pi - \dfrac{\gamma}{560} - \dfrac{49\,\zeta(3)}{5760\pi^2} - \dfrac{31\,\zeta(5)}{960\pi^4} - \dfrac{127\,\zeta(7)}{1792\pi^6}, \\ \quad [\zeta\text{-form}], \\ -\dfrac{1}{280}\ln 2 + \dfrac{1}{560}\ln\pi - \dfrac{\gamma}{560} + \dfrac{7}{11520\pi^2}\Psi_2\left(\dfrac{1}{2}\right) + \dfrac{1}{23040\pi^4}\Psi_4\left(\dfrac{1}{2}\right) \\ \quad + \dfrac{1}{1290240\pi^6}\Psi_6\left(\dfrac{1}{2}\right), \quad [\Psi\text{-form}], \end{cases}$$

Prove the results above by the contour integration technique.

Nota bene: The particular case $n=1$ is the simplest Malmsten's integral (1); case $n=2$ is also known, see e.g., [62, Table 257-4], [28, no. 4.371-3], [44]. The result for the case $n=3$ can be found in [44]. Integral I_4 was also treated in [44], but the presented formula differs from the above ones and contains the derivative of the Riemann ζ-function. As regards integrals with higher n, they seem never to have been evaluated before in the literature.

36* By using results of the previous exercise, show that following mathematical constants may be defined by ln ln-integrals:

$$\text{(a)} \quad \mathrm{G} = \frac{\pi}{2}\int_0^1 \frac{(x^4-6x^2+1)\ln\ln\frac{1}{x}}{(1+x^2)^3}\,dx = \frac{\pi}{2}\int_1^\infty \frac{(x^4-6x^2+1)\ln\ln x}{(1+x^2)^3}\,dx$$

$$\text{(b)} \quad \zeta(3) = \frac{8\pi^2}{7}\int_0^1 \frac{x(x^4-4x^2+1)\ln\ln\frac{1}{x}}{(1+x^2)^4}\,dx = \frac{8\pi^2}{7}\int_1^\infty \frac{x(x^4-4x^2+1)\ln\ln x}{(1+x^2)^4}\,dx$$

The first result was already presented by Adamchik [2], while the second one seems to be new. Similar integral definitions for the Euler's constant γ and for $\ln\Gamma(1/4)$ are straightforward from the previous exercise.

37* By using results of exercise no. 4, show that

$$\frac{1}{2\,\mathrm{sh}\,1}\int_{-\infty}^{+\infty} \frac{\ln|x|}{\mathrm{sh}\,x \pm \mathrm{sh}\,1}\,dx = \mp\int_0^\infty \frac{\ln x}{\mathrm{sh}^2 x - \mathrm{sh}^2 1}\,dx$$

$$= \mp 4\int_1^\infty \frac{x\ln\ln x}{x^4 - 2x^2\,\mathrm{ch}\,2 + 1}\,dx = \mp 4\int_0^1 \frac{x\ln\ln\frac{1}{x}}{x^4 - 2x^2\,\mathrm{ch}\,2 + 1}\,dx$$

$$= \pm\left[\frac{2\pi}{\mathrm{sh}\,2}\,\mathrm{Im}\left\{\ln\Gamma\left(\frac{i}{2\pi}\right) - \ln\Gamma\left(\frac{1}{2} - \frac{i}{2\pi}\right)\right\} + \frac{\pi^2}{2\,\mathrm{sh}\,2} + \frac{2\ln 2\pi}{\mathrm{sh}\,2}\right].$$

38* Prove that for any $t > 0$

$$\int_1^\infty \frac{x(\ln\ln x - \ln t)}{x^4 - 2x^2\operatorname{ch}2t + 1}\,dx = \int_0^1 \frac{x\bigl(\ln\ln\frac{1}{x} - \ln t\bigr)}{x^4 - 2x^2\operatorname{ch}2t + 1}\,dx$$

$$= -\frac{\pi}{2\operatorname{sh}2t}\operatorname{Im}\left\{\ln\Gamma\left(\frac{it}{2\pi}\right) - \ln\Gamma\left(\frac{1}{2} - \frac{it}{2\pi}\right)\right\} - \frac{\pi^2}{8\operatorname{sh}2t} + \frac{t}{2\operatorname{sh}2t}\ln\frac{t}{2\pi}.$$

4.3 Arctangent integrals containing hyperbolic functions

Integrals of the arctangent function in combination with hyperbolic functions are not presented at all in [28], and there are few of them in [53, vol. I, § 2.7]. For example, integral no. 39-c is given in [53, vol. I] as no. 2.7.5-10, but the provided formula is incorrect.[35]

39* By using Cauchy's residue theorem, prove that for any $\operatorname{Re}a > 0$ and $\operatorname{Re}b > 0$

(a) $$\int_0^\infty \frac{\operatorname{arctg}x}{\operatorname{sh}x}\,dx = \int_0^\infty \frac{\operatorname{arctg}\ln x}{x^2-1}\,dx = 2\int_1^\infty \frac{\operatorname{arctg}\ln x}{x^2-1}\,dx = 2\int_0^1 \frac{\operatorname{arctg}\ln x}{x^2-1}\,dx$$

$$= \int_0^1 \frac{\operatorname{arctg}(2\operatorname{arcth}x)}{x}\,dx = \int_0^\infty \operatorname{arctg}\bigl(2\operatorname{arcth}e^{-x}\bigr)\,dx = \int_0^{\pi/2} \frac{x}{\operatorname{sh}\operatorname{tg}x\cdot\cos^2 x}\,dx$$

$$= \pi\left\{\ln\Gamma\left(\frac{1}{2\pi}\right) - \ln\Gamma\left(\frac{1}{2} + \frac{1}{2\pi}\right) - \frac{1}{2}\ln 2\pi\right\}$$

$$= \pi\left\{2\ln\Gamma\left(\frac{1}{2\pi}\right) - \ln\Gamma\left(\frac{1}{\pi}\right) - \ln\pi\sqrt{8}\right\} + \ln 2$$

(b) $$\int_0^1 \frac{\operatorname{arctg}\operatorname{arcth}x}{x}\,dx = \pi\left\{\ln\Gamma\left(\frac{1}{\pi}\right) - \ln\Gamma\left(\frac{1}{2} + \frac{1}{\pi}\right) - \frac{1}{2}\ln\pi\right\}$$

(c) $$\int_0^1 \frac{\operatorname{arcth}\operatorname{arcth}x}{x}\,dx = -i\int_0^i \frac{\operatorname{arctg}\operatorname{arctg}x}{x}\,dx$$

$$= -\pi i\left\{\ln\Gamma\left(-\frac{i}{\pi}\right) - \ln\Gamma\left(\frac{1}{2} - \frac{i}{\pi}\right) - \frac{1}{2}\ln\pi - \frac{\pi i}{4}\right\}$$

[35] More precisely, the last term in the right-hand side of [53, vol. I, no. 2.7.5-10] is incorrect: the argument of the square root should be multiplied by 2. Curiously, in the original Russian edition of [53], this integral is neither correctly evaluated, but the error is not the same: the coefficient 2 in the argument of the logarithm must be placed under the square root sign.

(d) $$\int\limits_0^\infty \frac{\operatorname{arctg} x}{\operatorname{sh} \pi x}\, dx = \frac{1}{2}\ln\frac{\pi}{2}$$

(e) $$\int\limits_0^\infty \frac{\operatorname{arctg} ax}{\operatorname{sh} bx}\, dx = \frac{\pi}{b}\left\{\ln\Gamma\left(\frac{b}{2\pi a}\right) - \ln\Gamma\left(\frac{1}{2}+\frac{b}{2\pi a}\right) - \frac{1}{2}\ln\frac{2\pi a}{b}\right\}$$

Nota bene: Although these formulas do not appear correctly in modern mathematical literature, a particular case of one of them may be found in the Malmsten et al.'s dissertation [40, p. 52, Eq. (63)] and it is correct. As usual, Malmsten et al. used the series expansion technique in order to get the result.

40* By using results of the previous exercise, prove that for any $\operatorname{Re} a > 0$ and $\operatorname{Re} b > 0$
(a) this analog of Binet's formula:

$$\int\limits_0^\infty \frac{\operatorname{arctg} ax}{e^{bx}+1}\, dx = -\frac{\pi}{b}\ln\Gamma\left(\frac{1}{2}+\frac{b}{2\pi a}\right) - \frac{1}{2a}\left(1+\ln\frac{2\pi a}{b}\right) + \frac{\pi}{2b}\ln 2\pi,$$

(b) this analog of Legendre's formula:

$$\int\limits_0^\infty \frac{x\,dx}{(e^{bx}+1)(x^2+a^2)} = \int\limits_0^\infty \frac{x\,dx}{(e^{ax}+1)(x^2+b^2)} = \frac{1}{2}\left\{\Psi\left(\frac{1}{2}+\frac{ab}{2\pi}\right) - \ln\frac{ab}{2\pi}\right\}$$

Hint: Make use of Binet's formula for the logarithm of the Γ-function

$$\ln\Gamma(z) = \left(z-\frac{1}{2}\right)\ln z - z + \frac{1}{2}\ln 2\pi + 2\int\limits_0^\infty \frac{\operatorname{arctg}(x/z)}{e^{2\pi x}-1}\, dx, \tag{54}$$

see [12, pp. 335–336] and [71, pp. 250–251], [9, vol. I, p. 22, Eq. 1.9(9)]. It is interesting that Legendre was very close to this expression and Binet also remarked this. On p. 190 [64, vol. II], we find this expression

$$\int\limits_0^\infty \frac{x\,dx}{(e^{2\pi x}-1)(m^2+x^2)} = -\frac{1}{4m} + \frac{1}{2}\ln m - \frac{1}{2}\Psi(m), \tag{55}$$

which is exactly the derivative of the above Binet's formula with respect to z at $z = m$. In order to arrive at Binet's formula (54), it was just sufficient to integrate (55) over m and to find the constant of integration, but Legendre left this honour to Binet. It seems also almost incredible that it took 23 years to get formula (54) from (55). Finally, a more general version of Binet's formula containing $e^{bx}-1$ in the denominator of the integrand appears also in Prudnikov et al.'s tables [53, vol. I] as no. 2.7.5-6 and is correct.

By the way, Lerch in [37, p. 19, Eq. (30)] (written in Czech) gives a more general case of formula (a)

$$\int\limits_0^\infty \frac{e^{2\pi x}\cos\varphi - 1}{e^{4\pi x} - 2e^{2\pi x}\cos\varphi + 1}\operatorname{arctg}\frac{u}{x}\,dx = -\frac{1}{4}\left\{\ln\Gamma\left(u - \frac{\varphi}{2\pi}\right) + \ln\Gamma\left(u + \frac{\varphi}{2\pi}\right)\right.$$
$$\left. + \ln\left(u - \frac{\varphi}{2\pi}\right) + 2u(1 - \ln u) + \ln\sin\frac{\varphi}{2} - \ln\pi\right\}$$

in which parameters u and φ are such that $0 < u \leqslant 1$ and $0 < \varphi < 2\pi u$. This formula reduces to exercise a) when setting $u = 1$ and $\varphi = \pi/2$.

41* By using the contour integration method, prove that if p is a rational part of b, i.e. $p = bm/n$, where $a \geqslant 0$, $\operatorname{Re} b > 0$ and numbers m and n are positive integers such that $m < n$, then

(a) $$\int\limits_0^\infty \frac{\operatorname{sh} px \cdot \operatorname{arctg} ax}{\operatorname{ch} bx}\,dx$$
$$= \frac{\pi}{b}\sum_{l=0}^{2n-1}(-1)^l \sin\frac{(2l+1)m\pi}{2n}\cdot\ln\Gamma\left(\frac{2l+1}{4n} + \frac{b}{2\pi an}\right)$$

(b) $$\int\limits_0^\infty \frac{\operatorname{ch} px \cdot \operatorname{arctg} ax}{\operatorname{sh} bx}\,dx = \frac{\pi}{b}\sum_{l=1}^{2n}(-1)^l \cos\frac{m\pi l}{n}\cdot\ln\Gamma\left(\frac{l}{2n} + \frac{b}{2\pi an}\right)$$
$$+ \frac{\pi}{2b}\ln\frac{2\pi an}{b}$$

(c) $$\int\limits_0^\infty \frac{\operatorname{sh} px \cdot \operatorname{arctg} ax}{\operatorname{ch} bx + \cos\varphi}\,dx = -\frac{\pi}{b\sin\varphi}$$
$$\times\sum_{l=0}^{n-1}\left\{\sin\frac{(2l+1)m\pi + m\varphi}{n}\cdot\ln\Gamma\left(\frac{2l+1}{2n} + \frac{b+a\varphi}{2\pi an}\right)\right.$$
$$\left. - \sin\frac{(2l+1)m\pi - m\varphi}{n}\cdot\ln\Gamma\left(\frac{2l+1}{2n} + \frac{b-a\varphi}{2\pi an}\right)\right\}$$

(d) $$\int\limits_0^\infty \frac{\operatorname{sh} px \cdot \operatorname{arctg} ax}{\operatorname{ch} bx + 1}\,dx$$
$$= -\frac{2\pi m}{bn}\sum_{l=0}^{n-1}\cos\frac{(2l+1)m\pi}{n}\cdot\ln\Gamma\left(\frac{2l+1}{2n} + \frac{b}{2\pi an}\right)$$

$$-\frac{1}{bn}\sum_{l=0}^{n-1}\sin\frac{(2l+1)m\pi}{n}\cdot\Psi\left(\frac{2l+1}{2n}+\frac{b}{2\pi an}\right)$$

where in (c) $|\operatorname{Re}\varphi|<\pi,\ \varphi\neq 0$; see (d) for $\varphi=0$.

Nota bene: A particular case of formula (a) for $b=\pi$ was already derived by Malmsten et al. [40, p. 70, Eq. (83)]. He separately treated cases $(m+n)$ odd and $(m+n)$ even and simplified the result in both cases (simplification is not the same, see e.g. exercise no. 48 where we also treat separately these two cases and performed such a simplification). Other formulas obtained above seem to be never released before.[36]

42* By letting $a\to\infty$ in the previous exercise, prove formula (23).

43* Prove by the contour integration method the following formulas:

$$\text{(a)}\quad \text{p.v.}\int_{-\infty}^{+\infty}\frac{\operatorname{arctg}x}{\operatorname{sh}x\pm\operatorname{sh}t}\,dx=\frac{\pi}{2\operatorname{ch}t}\left\{\ln\left(1+t^2\right)-2\ln 2\pi\right\}$$
$$+\frac{2\pi}{\operatorname{ch}t}\operatorname{Re}\left\{\ln\Gamma\left(\frac{1}{2\pi}+\frac{it}{2\pi}\right)-\ln\Gamma\left(\frac{1}{2}+\frac{1}{2\pi}-\frac{it}{2\pi}\right)\right\},$$

$$\text{(b)}\quad \text{p.v.}\int_{-\infty}^{+\infty}\frac{\operatorname{arctg}ax}{\operatorname{sh}bx\pm\operatorname{sh}bt}\,dx=\frac{\pi}{2b\operatorname{ch}bt}\left\{\ln\left(1+a^2t^2\right)-2\ln\frac{2\pi a}{b}\right\}$$
$$+\frac{2\pi}{b\operatorname{ch}bt}\operatorname{Re}\left\{\ln\Gamma\left(\frac{b}{2\pi a}+\frac{ibt}{2\pi}\right)-\ln\Gamma\left(\frac{1}{2}+\frac{b}{2\pi a}-\frac{ibt}{2\pi}\right)\right\},$$

$$\text{(c)}\quad \int_{-\infty}^{+\infty}\frac{\operatorname{arctg}ax\pm\operatorname{arctg}at}{\operatorname{sh}bx\pm\operatorname{sh}bt}\,dx=\frac{\pi}{2b\operatorname{ch}bt}\left\{\ln\left(1+a^2t^2\right)-2\ln\frac{2\pi a}{b}\right\}$$
$$+\frac{2t\operatorname{arctg}at}{\operatorname{ch}bt}+\frac{2\pi}{b\operatorname{ch}bt}\operatorname{Re}\left\{\ln\Gamma\left(\frac{b}{2\pi a}+\frac{ibt}{2\pi}\right)-\ln\Gamma\left(\frac{1}{2}+\frac{b}{2\pi a}-\frac{ibt}{2\pi}\right)\right\}$$

where $a>0,\ b>0,\ t>0$.

44* Show that

$$\text{(a)}\quad \int_0^{\infty}\frac{1-x\operatorname{cth}x}{\operatorname{sh}x}\operatorname{arctg}\left(\frac{2x}{\pi}\right)dx=\frac{\pi}{2}\left(1-\frac{\pi}{2}\right),$$

$$\text{(b)}\quad \int_0^{\infty}\frac{(1-x\operatorname{cth}x)\operatorname{arctg}ax}{\operatorname{sh}x}\,dx=\frac{1}{2a}\left\{\Psi\left(\frac{1}{2\pi a}\right)-\Psi\left(\frac{1}{2}+\frac{1}{2\pi a}\right)+\pi a\right\},$$

[36] However, it seems fair to remark that an integral quite similar to (b) was also evaluated by Malmsten et al. [40, p. 55, Eq. (67)].

where $\operatorname{Re} a > 0$.

Hint: Use results of no. 39.

45* Prove the following results:

$$\text{(a)}\quad \int_0^\infty \frac{\operatorname{sh} x}{\operatorname{ch}^2 x} \operatorname{arctg} x\, dx = \int_0^\infty \frac{(x^2-1)\operatorname{arctg}\ln x}{(x^2+1)^2}\, dx$$

$$= 2\int_1^\infty \frac{(x^2-1)\operatorname{arctg}\ln x}{(x^2+1)^2}\, dx = 2\int_0^1 \frac{(x^2-1)\operatorname{arctg}\ln x}{(x^2+1)^2}\, dx$$

$$= \int_1^\infty \frac{\operatorname{arctg}\operatorname{arcch} x}{x^2}\, dx = \int_1^e \operatorname{arctg}\left(\operatorname{arcch}\frac{1}{\ln x}\right)\frac{dx}{x}$$

$$= \int_0^1 \operatorname{arctg}\left(\operatorname{arcch}\frac{1}{x}\right) dx = \int_0^{\operatorname{tg} 1} \operatorname{arctg}\left(\operatorname{arcch}\frac{1}{\operatorname{arctg} x}\right)\frac{dx}{1+x^2}$$

$$= \int_0^{\operatorname{th} 1} \operatorname{arctg}\left(\operatorname{arcch}\frac{1}{\operatorname{arcth} x}\right)\frac{dx}{1-x^2} = \int_0^{\pi/2} \frac{x\cdot\operatorname{th}\operatorname{tg} x}{\operatorname{ch}\operatorname{tg} x\cdot\cos^2 x}\, dx$$

$$= \frac{1}{2}\left\{\Psi\left(\frac{3}{4}+\frac{1}{2\pi}\right) - \Psi\left(\frac{1}{4}+\frac{1}{2\pi}\right)\right\},$$

$$\text{(b)}\quad \int_0^\infty \frac{\operatorname{sh} x}{\operatorname{ch}^2 x} \operatorname{arctg}\left(\frac{2x}{\pi}\right) dx = \ln 2,$$

$$\text{(c)}\quad \int_0^\infty \frac{\operatorname{sh} bx}{\operatorname{ch}^2 bx} \operatorname{arctg} ax\, dx = \frac{1}{2b}\left\{\Psi\left(\frac{3}{4}+\frac{b}{2\pi a}\right) - \Psi\left(\frac{1}{4}+\frac{b}{2\pi a}\right)\right\},$$

where $a > 0$, $\operatorname{Re} b > 0$.

46* Show that for any $a \geqslant 0$ and $\operatorname{Re} b > 0$

$$\text{(a)}\quad \int_0^\infty \frac{\operatorname{sh} px\cdot\operatorname{arctg} ax}{\operatorname{sh}^2 bx}\, dx = \frac{\pi m}{bn}\left\{\ln\Gamma\left(\frac{b}{2\pi an}\right) - \frac{1}{2}\ln\frac{2\pi an}{b}\right\}$$

$$+ \frac{\pi m}{bn}\sum_{l=1}^{2n-1} \cos\frac{m\pi l}{n}\cdot\ln\Gamma\left(\frac{l}{2n}+\frac{b}{2\pi an}\right)$$

$$+\frac{1}{2bn}\sum_{l=1}^{2n-1}\sin\frac{m\pi l}{n}\cdot\Psi\left(\frac{l}{2n}+\frac{b}{2\pi an}\right)$$

(b) $$\int_0^\infty \frac{\operatorname{sh} px\cdot\operatorname{arctg} ax}{\operatorname{ch}^2 bx}\,dx=-\frac{\pi m}{bn}\sum_{l=0}^{2n-1}\cos\frac{(2l+1)m\pi}{2n}\cdot\ln\Gamma\left(\frac{2l+1}{4n}+\frac{b}{2\pi an}\right)$$

$$-\frac{1}{2bn}\sum_{l=0}^{2n-1}\sin\frac{(2l+1)m\pi}{2n}\cdot\Psi\left(\frac{2l+1}{4n}+\frac{b}{2\pi an}\right),$$

where $p=bm/n$ and numbers m and n are positive integers such that $m<2n$.

47* Show that for any $a\geqslant 0$ and $\operatorname{Re} b>0$

(a) $$\int_0^\infty \frac{\operatorname{sh} bx}{\operatorname{ch}^3 bx}\operatorname{arctg} ax\,dx=\frac{1}{2\pi b}\Psi_1\left(\frac{1}{2}+\frac{b}{\pi a}\right),$$

(b) $$\int_0^\infty \frac{(1-\operatorname{ch} bx)\operatorname{arctg} ax}{\operatorname{sh}^3 bx}\,dx=\frac{\pi}{2b}\left\{\ln\Gamma\left(\frac{1}{2}+\frac{b}{2\pi a}\right)-\ln\Gamma\left(\frac{b}{2\pi a}\right)\right\}$$

$$+\frac{\pi}{4b}\ln\frac{2\pi a}{b}+\frac{1}{4\pi b}\Psi_1\left(\frac{1}{2}+\frac{b}{2\pi a}\right),$$

(c) $$\int_0^\infty \frac{(1-\operatorname{ch} bx)^2\operatorname{arctg} ax}{\operatorname{sh}^5 bx}\,dx=\frac{\pi}{4b}\left\{\ln\Gamma\left(\frac{b}{2\pi a}\right)-\ln\Gamma\left(\frac{1}{2}+\frac{b}{2\pi a}\right)\right\}$$

$$-\frac{\pi}{8b}\ln\frac{2\pi a}{b}-\frac{1}{6\pi b}\Psi_1\left(\frac{1}{2}+\frac{b}{2\pi a}\right)-\frac{1}{96\pi^3 b}\Psi_3\left(\frac{1}{2}+\frac{b}{2\pi a}\right).$$

48* (a) By the contour integration method, prove that if p is a rational part of b, i.e. $p=bm/n$, where b is some positive parameter and numbers m and n are positive integers such that $m<3n$, then

$$\int_0^\infty \frac{\operatorname{sh} px\cdot\operatorname{arctg} ax}{\operatorname{ch}^3 bx}\,dx$$

$$=\frac{\pi(n^2-m^2)}{2bn^2}\sum_{l=0}^{2n-1}(-1)^l\sin\frac{(2l+1)m\pi}{2n}\cdot\ln\Gamma\left(\frac{2l+1}{4n}+\frac{b}{2\pi an}\right)$$

$$+\frac{m}{2bn^2}\sum_{l=0}^{2n-1}(-1)^l\cos\frac{(2l+1)m\pi}{2n}\cdot\Psi\left(\frac{2l+1}{4n}+\frac{b}{2\pi an}\right)$$

$$+\frac{1}{8b\pi n^2}\sum_{l=0}^{2n-1}(-1)^l\sin\frac{(2l+1)m\pi}{2n}\cdot\Psi_1\left(\frac{2l+1}{4n}+\frac{b}{2\pi an}\right)$$

or, if considering separately cases $(m+n)$ odd and $(m+n)$ even,

$$\int\limits_0^\infty\frac{\operatorname{sh}px\cdot\operatorname{arctg}ax}{\operatorname{ch}^3bx}\,dx$$

$$=\frac{1}{b}\cdot\begin{cases}\dfrac{\pi(n^2-m^2)}{2n^2}\displaystyle\sum_{l=0}^{n-1}(-1)^{l+1}\sin\frac{(2l+1)m\pi}{2n}\cdot\ln\left\{\frac{\Gamma\left(\frac{1}{2}+\frac{2l+1}{4n}+\frac{b}{2\pi an}\right)}{\Gamma\left(\frac{2l+1}{4n}+\frac{b}{2\pi an}\right)}\right\}\\ \quad+\dfrac{m}{2n^2}\displaystyle\sum_{l=0}^{n-1}(-1)^l\cos\frac{(2l+1)m\pi}{2n}\cdot\left\{\Psi\left(\frac{2l+1}{4n}+\frac{b}{2\pi an}\right)\right.\\ \qquad\qquad\left.-\Psi\left(\frac{1}{2}+\frac{2l+1}{4n}+\frac{b}{2\pi an}\right)\right\}\\ \quad+\dfrac{1}{8\pi n^2}\displaystyle\sum_{l=0}^{n-1}(-1)^l\sin\frac{(2l+1)m\pi}{2n}\cdot\left\{\Psi_1\left(\frac{2l+1}{4n}+\frac{b}{2\pi an}\right)\right.\\ \qquad\qquad\left.-\Psi_1\left(\frac{1}{2}+\frac{2l+1}{4n}+\frac{b}{2\pi an}\right)\right\},\quad\text{if } m+n \text{ is odd;}\\ \dfrac{\pi(n^2-m^2)}{2n^2}\displaystyle\sum_{l=0}^{n-1}(-1)^l\sin\frac{(2l+1)m\pi}{2n}\cdot\ln\Gamma\left(\frac{2l+1}{2n}+\frac{b}{\pi an}\right)\\ \quad+\dfrac{m}{n^2}\displaystyle\sum_{l=0}^{n-1}(-1)^l\cos\frac{(2l+1)m\pi}{2n}\cdot\Psi\left(\frac{2l+1}{2n}+\frac{b}{\pi an}\right)\\ \quad+\dfrac{1}{2\pi n^2}\displaystyle\sum_{l=0}^{n-1}(-1)^l\sin\frac{(2l+1)m\pi}{2n}\cdot\Psi_1\left(\frac{2l+1}{2n}+\frac{b}{\pi an}\right),\\ \quad\text{if } m+n \text{ is even.}\end{cases}$$

(b) Following Malmsten's idea of establishing relationships between the Γ-function and its logarithmic derivative, see (23), prove this more general formula implying the trigamma function:

$$\frac{n^2-m^2}{4n^2}\left\{\Psi\left(\frac{1}{4}+\frac{m}{4n}\right)-\Psi\left(\frac{1}{4}-\frac{m}{4n}\right)-\pi\operatorname{tg}\frac{m\pi}{2n}\right\}$$
$$=\frac{n^2-m^2}{n^2}\sum_{l=0}^{2n-1}(-1)^l\sin\frac{(2l+1)m\pi}{2n}\cdot\ln\Gamma\left(\frac{2l+1}{4n}\right)$$
$$+\frac{m}{\pi n^2}\sum_{l=0}^{2n-1}(-1)^l\cos\frac{(2l+1)m\pi}{2n}\cdot\Psi\left(\frac{2l+1}{4n}\right)$$

$$+\frac{1}{4\pi^2 n^2}\sum_{l=0}^{2n-1}(-1)^l\sin\frac{(2l+1)m\pi}{2n}\cdot\Psi_1\left(\frac{2l+1}{4n}\right)+\frac{m}{2n}$$

which holds for any positive integers m and n such that $m < 3n$. Show that the right part, analogous to (24), may be transformed into elementary functions and thus does not contain any additional information about the trigamma function.

Hint: First, put for simplicity $b = 1$ and write the integral (a) in the following form:

$$\int_0^\infty \frac{\operatorname{sh}\left(\frac{m}{n}x\right)}{\operatorname{ch}^3 x}\operatorname{arctg}(ax)\,dx$$

$$=\begin{cases} n\displaystyle\int_0^\infty \frac{\operatorname{sh} my}{\operatorname{ch}^3 ny}\operatorname{arctg}(any)\,dy, & y\equiv\dfrac{x}{n},\ \text{if } m+n \text{ is odd},\\ \dfrac{n}{2}\displaystyle\int_0^\infty \frac{\operatorname{sh}(\frac{1}{2}my)}{\operatorname{ch}^3(\frac{1}{2}ny)}\operatorname{arctg}(\tfrac{1}{2}any)\,dy, & y\equiv\dfrac{2x}{n},\ \text{if } m+n \text{ is even}.\end{cases}$$

Then, by taking into account that both integrands are rational functions of e^y and, hence, are $2\pi i$-periodic, apply formula (38) to each of these integrals.[37] At the final stage, put $b = 1$, make $a \to \infty$, and then compare the answer with the integral

$$\int_0^\infty \frac{\operatorname{sh}\alpha x}{\operatorname{ch}^3 x}\,dx = -\frac{\alpha}{2}+\frac{1-\alpha^2}{4}\left\{\Psi\left(\frac{1}{4}+\frac{\alpha}{4}\right)-\Psi\left(\frac{1}{4}-\frac{\alpha}{4}\right)-\pi\operatorname{tg}\frac{\pi\alpha}{2}\right\},$$

$|\operatorname{Re}\alpha| < 3$. The latter result may be obtained by various methods. For instance, it may be obtained by making use of the following integral

$$\int_0^\infty \frac{e^{-\alpha x}}{\operatorname{sh}^\beta x}\,dx = 2^{\beta-1}\cdot B\left(\frac{\alpha+\beta}{2}, 1-\beta\right),\qquad \operatorname{Re}\beta<1,\ \operatorname{Re}(\alpha+\beta)>0$$

which is, in turn, easily derived from the definition of the Euler's B-function by a simple change of variable, see, e.g., exercise no. 31.14.3 from [23]. By the way, in this book, we found several errors related to the latter kind of integrals. In exercise no. 31.11.6, in the right part the coefficient $2^{\beta-1}$ should be replaced by $2^{\beta-2}$. The answer given in exercise no. 31.11.5 is wrong: the integral in the left part cannot be expressed via the Euler's B-function. Unfortunately, these errors were not corrected in the second and the last edition of this book. Finally, as regards the simplification of the right part of (b) in terms of elementary functions, it is sufficient to show that the

[37] Alternatively, it is also possible to consider only the upper integral and then study two cases: $(m+n)$ is odd and $(m+n)$ is even. This method will lead to the same formula, albeit the calculation might seem more tedious.

pairwise summation of terms in each sum, i.e. the summation of the kind $a_l + a_{2n-1-l}$, where, for example, as regards the third sum

$$a_l \equiv (-1)^l \sin\frac{(2l+1)m\pi}{2n}\cdot\Psi_1\left(\frac{2l+1}{4n}\right),$$

does not contain the Γ-function, nor polygamma functions.

49* Analogously to the previous exercise, show that for any $a \geqslant 0$ and $\operatorname{Re} b > 0$

$$\begin{aligned}
&\int_0^\infty \frac{\operatorname{sh} px\cdot\operatorname{arctg} ax}{\operatorname{ch}^4 bx}\,dx\\
&\quad= -\frac{(4n^2-m^2)\pi m}{6bn^3}\sum_{l=0}^{2n-1}\cos\frac{(2l+1)m\pi}{2n}\cdot\ln\Gamma\left(\frac{2l+1}{4n}+\frac{b}{2\pi an}\right)\\
&\qquad-\frac{4n^2-3m^2}{12bn^3}\sum_{l=0}^{2n-1}\sin\frac{(2l+1)m\pi}{2n}\cdot\Psi\left(\frac{2l+1}{4n}+\frac{b}{2\pi an}\right)\\
&\qquad-\frac{m}{8b\pi n^3}\sum_{l=0}^{2n-1}\cos\frac{(2l+1)m\pi}{2n}\cdot\Psi_1\left(\frac{2l+1}{4n}+\frac{b}{2\pi an}\right)\\
&\qquad-\frac{1}{48b\pi^2 n^3}\sum_{l=0}^{2n-1}\sin\frac{(2l+1)m\pi}{2n}\cdot\Psi_2\left(\frac{2l+1}{4n}+\frac{b}{2\pi an}\right)
\end{aligned}$$

where $p = bm/n$ and numbers m and n are positive integers such that $m < 4n$.

50* Prove that Catalan's constant is the following limit:

$$\mathrm{G} = 1 + \lim_{\alpha\to 1}\left\{\int_0^\alpha \frac{(1+6x^2+x^4)\operatorname{arctg} x}{x(1-x^2)^2}\,dx + 2\operatorname{arcth}\alpha - \frac{\pi\alpha}{1-\alpha^2}\right\}.$$

Hint: Show first that

$$\iint\limits_{\mathbb{R}^2}\frac{x\sin(2xy/\pi)}{(x^2+\pi^2)\operatorname{ch} x\operatorname{sh} y}\,dx\,dy = 8(1-\mathrm{G}).$$

4.4 Integrals containing logarithm of the Γ-function or polygamma functions in combination with hyperbolic functions

Such integrals are not presented at all in Gradshteyn and Ryzhik's tables [28], nor in the second and third volumes of Prudnikov et al.'s tables [53]. Moreover, to our knowledge, all results given below are new.

51* Prove that for any $a \geqslant 0$

$$\int_{-\infty}^{+\infty} \frac{\ln \Gamma\left(\frac{x}{\pi i}+a\right)}{\operatorname{ch} x}\, dx = \pi \left\{ (a-1)\ln 2 - \frac{1}{2}\ln \pi + 2\ln \Gamma\left(\frac{1}{4}+\frac{a}{2}\right)\right\}.$$

Hint: Consider $\displaystyle\oint\limits_{0\leqslant \operatorname{Im} z\leqslant \pi} \frac{\ln \Gamma\left(\frac{z}{\pi i}+a\right)}{\operatorname{ch} z}\, dz.$

52* Prove that

$$\int_{-\infty}^{+\infty} \frac{\Psi\left(\frac{x}{\pi i}\pm\frac{1}{2}\right)}{\operatorname{ch} x}\, dx = -\pi(\gamma \pm \ln 2)$$

Hint: Consider $\displaystyle\oint\limits_{0\leqslant \operatorname{Im} z\leqslant \pi} \frac{\Psi\left(\frac{z}{\pi i}-\frac{1}{2}\right)}{\operatorname{ch} z}\, dz.$

53* Prove that for any $a > 0$

$$\int_{-\infty}^{+\infty} \frac{\Psi\left(\frac{x}{\pi i}+a\right)}{\operatorname{ch} x}\, dx = \pi \left\{ \ln 2 + \Psi\left(\frac{1}{4}+\frac{a}{2}\right)\right\}$$

Hint: Consider $\displaystyle\oint\limits_{0\leqslant \operatorname{Im} z\leqslant \pi} \frac{\Psi\left(\frac{z}{\pi i}+a\right)}{\operatorname{ch} z}\, dz.$

54* The formula in exercise no. 51 is valid for $a \geqslant 0$, while that in no. 53 is valid only for $a > 0$. A simple way to shown that it is not valid for $a = 0$ is to put $a = 0$ into no. 53, which yields for the corresponding integral $\pi\left(-\frac{1}{2}\pi - \gamma - 2\ln 2\right)$. However, numeric computation[38] shows that this result is incorrect; the correct one is

$$\text{p.v.} \int_{-\infty}^{+\infty} \frac{\Psi\left(\frac{x}{\pi i}\right)}{\operatorname{ch} x}\, dx = \int_{0}^{\infty} \frac{\Psi\left(\frac{x}{\pi i}\right)+\Psi\left(-\frac{x}{\pi i}\right)}{\operatorname{ch} x}\, dx = \pi\left(+\frac{1}{2}\pi - \gamma - 2\ln 2\right).$$

Explain this paradox and prove the above result.

55* Prove that for any $a > 0$

$$\int_{-\infty}^{+\infty} \frac{\Psi_n\left(\frac{x}{\pi i}+a\right)}{\operatorname{ch} x}\, dx = \frac{\pi}{2^n}\Psi_n\left(\frac{1}{4}+\frac{a}{2}\right), \qquad n = 1, 2, 3, \ldots$$

[38] With the help of Maple 12.

where Ψ_n denotes the nth polygamma function.

56* Prove that

$$\int\limits_{-\infty}^{+\infty} \frac{\Psi\left(\frac{x}{2\pi i}-\frac{1}{2}\right)}{(x^2+\pi^2)\operatorname{ch} x}\,dx = \int\limits_{-\infty}^{+\infty} \frac{\Psi\left(\frac{3}{2}-\frac{x}{2\pi i}\right)}{(x^2+\pi^2)\operatorname{ch} x}\,dx = \frac{12-4\gamma-12\ln 2}{\pi}+\gamma-1.$$

Hint: Consider $\displaystyle\oint\limits_{0\leqslant \operatorname{Im} z\leqslant 2\pi} \frac{\Psi\left(\frac{z}{2\pi i}-\frac{1}{2}\right)}{(z-\pi i)\operatorname{ch} z}\,dz.$

57* Prove that

(a) $$\int\limits_{-\infty}^{+\infty} \frac{\Psi\left(\pm\frac{x}{2\pi i}\right)}{(x^2+\pi^2)\operatorname{ch} x}\,dx = \int\limits_{-\infty}^{+\infty} \frac{\Psi\left(1\pm\frac{x}{2\pi i}\right)}{(x^2+\pi^2)\operatorname{ch} x}\,dx = \frac{4-4\gamma-12\ln 2}{\pi}+\gamma+2\ln 2,$$

(b) $$\int\limits_{-\infty}^{+\infty} \frac{\Psi\left(\frac{1}{2}\pm\frac{x}{2\pi i}\right)}{(x^2+\pi^2)\operatorname{ch} x}\,dx = \gamma-\frac{4(\gamma+3\ln 2-2\,\mathrm{G})}{\pi},$$

(c) $$\int\limits_{-\infty}^{+\infty} \frac{x\,\Psi\left(\frac{1}{2}\pm\frac{x}{2\pi i}\right)}{(x^2+\pi^2)\operatorname{ch} x}\,dx = \pm\pi i(2-3\ln 2),$$

(d) $$\int\limits_{-\infty}^{+\infty} \frac{x\,\Psi\left(\frac{x}{2\pi i}-\frac{1}{2}\right)}{(x^2+\pi^2)\operatorname{ch} x}\,dx = i\,(4+3\pi-3\pi\ln 2-8\,\mathrm{G}),$$

(e) $$\int\limits_{-\infty}^{+\infty} \frac{x\,\Psi\left(\frac{x}{\pi i}\right)}{(4x^2+\pi^2)\operatorname{ch} x}\,dx = \frac{i}{8}\big(2\pi-\pi^2-4\ln 2\big).$$

4.5 Exercises concerning the Γ-function at rational arguments and the Stieltjes constants

58* Prove that the Γ-function of any rational argument may be always expressed via a finite combination of Malmsten's integrals $T_a(0)$ from (10), $a=m\pi/n$, and elementary functions:

$$\ln\Gamma\left(\frac{k}{n}\right) = \frac{(n-2k)\ln 2\pi n}{2n}+\frac{1}{2}\left\{\ln\pi-\ln\sin\frac{\pi k}{n}\right\}+\frac{1}{\pi n}\sum_{m=1}^{n-1} I_{m,n}\cdot\sin\frac{2\pi mk}{n},$$

$k = 1, 2, \ldots, n-1,\ n \in \mathbb{N}_{\geqslant 2}$, where, for brevity, we designated by $I_{m,n}$ the following quantity:

$$I_{m,n} \equiv \int_0^\infty \frac{\operatorname{sh}(\frac{m}{n}x)}{\operatorname{sh} x} \ln x \, dx = \frac{\pi}{2} \operatorname{tg} \frac{\pi m}{2n} \cdot \ln n + n \int_0^1 \frac{x^{m-1} - x^{-m-1}}{x^n - x^{-n}} \ln \ln \frac{1}{x} \, dx$$

$$= \frac{\pi}{2} \operatorname{tg} \frac{\pi m}{2n} \cdot \ln n + n \int_1^\infty \frac{x^{m-1} - x^{-m-1}}{x^n - x^{-n}} \ln \ln x \, dx \; = \; \frac{\pi}{2} \left\{ \operatorname{tg} \frac{\pi m}{2n} \cdot \ln \pi \, + T_a(0) \right\}.$$

Hint: First, take the second equation from (13) and put: $x = 0$, $2m$ instead of m and $2n$ instead of n. This trick makes the equation valid for any integer values of m and n because $2m + 2n$ is always even. Then, notice that the obtained expression represents a kind of the discrete sine transform for finite-length sequences. Hence, use the orthogonality property of $\sin(\pi ml/n)$ over the discrete interval $[1, n-1]$

$$\sum_{m=1}^{n-1} \sin\left(\frac{\pi m l}{n}\right) \sin\left(\frac{\pi m k}{n}\right) = \frac{n}{2}\, \delta_{l,k},$$

$l = 1, 2, \ldots, n-1,\ k = 1, 2, \ldots, n-1$. Perform a simplification with the help of this formula

$$\sum_{m=1}^{n-1} \operatorname{tg} \frac{\pi m}{2n} \cdot \sin \frac{\pi m k}{n} = (-1)^{k+1} \cdot (n-k).$$

which is valid for positive integrers k and n such that $k \leqslant 2n - 1$. At the final stage, rewrite the result for $2k$ instead of k.

There is also another way to prove this formula. Remark first that

$$\sum_{m=1}^{n-1} \operatorname{sh}\left(\frac{mx}{n}\right) \sin\left(\frac{2\pi m k}{n}\right) = \frac{\sin \frac{2\pi k}{n} \cdot \operatorname{sh} x}{2\left(\cos \frac{2\pi k}{n} - \operatorname{ch} \frac{x}{n}\right)}$$

$k = 1, 2, \ldots, n-1,\ x \in \mathbb{C}$. Then, apply the main formula from exercise no. 2.

59* Analogously to the previous exercise, prove that for any $k = 0, 1, \ldots, 2n-1$, where n is a positive integer,

$$\ln \Gamma\left(\frac{2k+1}{4n}\right) = \frac{(-1)^k}{4n} \ln \frac{2\pi^2}{n} + \frac{\ln 2\pi n}{2n}\left(n - 2\left\lfloor \frac{k+1}{2} \right\rfloor\right) - \frac{(-1)^k}{n} \ln \Gamma\left(\frac{1}{4}\right)$$

$$+ \frac{1}{2}\left\{ \ln \pi - \ln \sin \frac{\pi(2k+1)}{4n} \right\} - \frac{(-1)^k}{\pi n} \sum_{m=0}^{n-1} I_{m,n} \cdot \cos \frac{(2k+1)m\pi}{2n},$$

where

$$I_{m,n} \equiv \int_0^\infty \frac{\operatorname{ch}\left(\frac{m}{n}x\right)}{\operatorname{ch} x} \ln x \, dx = \frac{\pi}{2} \sec \frac{\pi m}{2n} \cdot \ln n + n \int_1^\infty \frac{x^{m-1} + x^{-m-1}}{x^n + x^{-n}} \ln \ln x \, dx.$$

Hint: First, consider formula (b) obtained in exercise no. 3 and put $a = 0$ and $b = 1$. Then, make use of the following semi-orthogonality property:

$$\sum_{m=0}^{n-1} \cos \frac{(2l+1)m\pi}{2n} \cdot \cos \frac{(2k+1)m\pi}{2n} = \frac{n}{2}\delta_{l,k} + \frac{n}{2}\delta_{l,2n-1-k} + \frac{1}{2},$$

$l = 0, 1, \ldots, 2n-1$, $k = 0, 1, \ldots, 2n-1$. At the final stage, use the reflection formula for the Γ-function, the Gauss' multiplication theorem and the fact that

$$\sum_{m=0}^{n-1} \sec \frac{\pi m}{2n} \cdot \cos \frac{(2k+1)m\pi}{2n} = (-1)^k \left(n - 2\left\lfloor \frac{k+1}{2} \right\rfloor \right),$$

for any $k = 0, 1, \ldots, 2n-1$.

60* Similarly to the previous exercises, show that

$$\ln \Gamma\left(\frac{k}{n}\right) = \frac{(n-2k)\ln 2\pi n}{2n} + \frac{1}{2}\left\{\ln \pi - \ln \sin \frac{\pi k}{n}\right\}$$
$$+ \frac{1}{2\pi} \sum_{m=1}^{n-1} \frac{\gamma + \ln(m/n)}{m} \cdot \sin \frac{2\pi mk}{n} + \frac{1}{\pi n} \sum_{m=1}^{n-1} I_{m,n} \cdot \sin \frac{2\pi mk}{n}$$

$k = 1, 2, \ldots, n-1$, $n \in \mathbb{N}_{\geqslant 2}$, where

$$I_{m,n} \equiv \int_0^\infty \frac{\operatorname{sh}(\frac{m}{n}x)}{e^x - 1} \ln x \, dx = \left(\frac{n}{2m} - \frac{\pi}{2} \operatorname{ctg} \frac{\pi m}{n} \right) \ln n + \frac{n}{2} \int_1^\infty \frac{x^{m-1} - x^{-m-1}}{x^n - 1} \ln \ln x \, dx.$$

Hint: First, consider formula (a) obtained in exercise no. 25. Then, use the semi-orthogonality of $\sin(2\pi ml/n)$ over $[1, n-1]$

$$\sum_{m=1}^{n-1} \sin\left(\frac{2\pi ml}{n}\right) \sin\left(\frac{2\pi mk}{n}\right) = \frac{n}{2}\delta_{l,k} - \frac{n}{2}\delta_{l,n-k}$$

$l = 1, 2, \ldots, n-1$, $k = 1, 2, \ldots, n-1$. Finally, use the reflection formula for the Γ-function and recall that

$$\sum_{m=1}^{n-1} \operatorname{ctg} \frac{\pi m}{n} \cdot \sin \frac{2\pi mk}{n} = n - 2k$$

where k and n are positive integers such that $k < n$.

61* Prove that for $k = 1, 2, \ldots, n-1,\ n \in \mathbb{N}_{\geqslant 2}$,

$$\ln \Gamma\left(\frac{k}{n}\right) = \frac{(n-2k)\ln \pi n}{2n} + \frac{1}{2}\left\{\ln \pi - \ln \sin \frac{\pi k}{n}\right\} + \frac{2}{\pi n}\sum_{m=1}^{n-1} I_{m,n} \cdot \sin \frac{2\pi mk}{n},$$

where

$$I_{m,n} \equiv \int_0^\infty \frac{\operatorname{sh}^2\left(\frac{m}{n}x\right)}{\operatorname{sh}^2 x} \ln x \, dx.$$

Hint: Take formula (b) from no. 11 and set $b = 1$. Multiply both sides by $\sin \frac{2\pi mk}{n}$ and then sum over $m = 1, 2, \ldots, n-1$. Remarking that

$$\sum_{m=1}^{n-1} m \cdot \sin \frac{2\pi ml}{n} \cdot \sin \frac{2\pi mk}{n} = \frac{n^2}{4}(\delta_{l,k} - \delta_{l,n-k}),$$

$l = 1, 2, \ldots, n-1,\ k = 1, 2, \ldots, n-1$, and that

$$\sum_{m=1}^{n-1} m \cdot \operatorname{ctg} \frac{\pi m}{n} \cdot \sin \frac{2\pi mk}{n} = \frac{n}{2}(n-2k),$$

for $k = 1, 2, \ldots, n-1$, we obtain the above formula. Another way to prove the same result is to show that

$$\sum_{m=1}^{n-1} I_{m,n} \cdot \sin \frac{2\pi mk}{n} = \frac{1}{4}\sin \frac{2\pi k}{n} \int_0^\infty \frac{\ln x}{\cos^2 \frac{\pi k}{n} - \operatorname{ch}^2 \frac{x}{n}}\, dx,$$

$k = 1, 2, \ldots, n-1;\ x \in \mathbb{C}$, and then, to apply formula (37) to the latter integral.

62* Let $\varphi_{l,n}$ be some known function of discrete arguments l and n, which is defined at least for $l = 1, 2, \ldots, n-1$ and $n \in \mathbb{N}_{\geqslant 2}$. It is obvious that any countable combination of functions $\varphi_{l,n}$ with other known functions will be a finite and known quantity. Such a quantity is, for example,

$$\Phi_{k,n} = \sum_{l=1}^{n-1} \varphi_{l,n} \sin \frac{\pi lk}{n}, \qquad k = 1, 2, \ldots, n-1.$$

One may easily note that the above formula is actually the expansion of the function $\Phi_{k,n}$ terms of orthogonal sines with coefficients $\varphi_{l,n}$; in other words, $\Phi_{k,n}$ is the discrete sine transform of the sequence $\varphi_{1,n}, \varphi_{2,n}, \ldots, \varphi_{n-1,n}$. By using intermediate results of exercise no. 58, prove the following functional relationship on $\Gamma(\frac{1}{2n}), \Gamma(\frac{2}{2n}), \Gamma(\frac{3}{2n}), \ldots, \Gamma(\frac{1}{2} - \frac{1}{2n})$:

$$\sum_{k=1}^{n-1}(-1)^k \Phi_{k,n} \ln\Gamma\left(\frac{k}{2n}\right) = \frac{1}{2\pi}\sum_{m=1}^{n-1} I_{m,n}\,\varphi_{m,n} - \frac{1}{2}\sum_{l=1}^{n-1}\varphi_{l,n}\,\sigma_{l,n}$$
$$- \frac{\ln(2\pi^2 n)}{4}\sum_{l=1}^{n-1}\varphi_{l,n}\operatorname{tg}\frac{\pi l}{2n} + \frac{\ln\pi}{4}\sum_{l=1}^{n-1}(-1)^{l+n}\varphi_{l,n}\operatorname{tg}\frac{\pi l}{2n},$$

where

$$\sigma_{l,n} \equiv \sum_{k=1}^{n-1}(-1)^k \sin\frac{\pi l k}{n}\ln\sin\frac{\pi k}{2n},$$

the integral $I_{l,n}$ was defined previously in no. 58, and the above formula is valid for any $n \in \mathbb{N}_{\geqslant 2}$.

(b) Suppose that the discrete function $\Upsilon_{k,n}$ may be represented by the finite Fourier series[39]

$$\Upsilon_{k,n} = \beta_{0,n} + \sum_{l=1}^{n-1}\left(\alpha_{l,n}\sin\frac{2\pi l k}{n} + \beta_{l,n}\cos\frac{2\pi l k}{n}\right), \quad k = 1, 2, \ldots, n-1, \qquad (56)$$

where coefficients $\alpha_{l,n}$ and $\beta_{l,n}$ are known and finite (alternatively, $\Upsilon_{k,n}$ can be regarded as a discrete Fourier transform). Prove then that the following functional relationship for the logarithm of the Γ-function holds:

$$\sum_{k=1}^{n-1}\Upsilon_{k,n}\ln\Gamma\left(\frac{k}{n}\right) = \frac{1}{\pi}\sum_{m=1}^{n-1} I_{2m-n,n}\alpha_{m,n} + \frac{\ln\pi n}{2}\sum_{l=1}^{n-1}\alpha_{l,n}\operatorname{ctg}\frac{\pi l}{n} - \frac{1}{2}\sum_{l=1}^{n-1}\alpha_{l,n}\varsigma_{l,n}$$
$$- \frac{\ln\pi}{2}\sum_{l=1}^{n-1}\beta_{l,n} + \frac{(n-1)\ln\pi}{2}\beta_{0,n} - \frac{1}{2}\sum_{l=0}^{n-1}\beta_{l,n}\,\omega_{l,n},$$

$n \in \mathbb{N}_{\geqslant 2}$, where we designated for brevity

$$\varsigma_{l,n} \equiv \sum_{k=1}^{n-1}\sin\frac{2\pi l k}{n}\ln\sin\frac{\pi k}{n}, \qquad \omega_{l,n} \equiv \sum_{k=1}^{n-1}\cos\frac{2\pi l k}{n}\ln\sin\frac{\pi k}{n}$$

and the integral $I_{l,n}$ was defined previously in no. 58. Note that coefficients $\beta_{l,n}$ do not affect the "computability" of the integral $\sum I_{2m-n,n}\,\alpha_{m,n}$. Moreover, if all coefficients $\alpha_{l,n} = 0$, then the right part contains elementary functions only.

Nota bene: A close study of the above-derived formulas reveals several interesting things. First of all, physically, all equations represent a kind of Parseval's equations of closure (i.e., the law of conservation of energy). Moreover, from exercise (b), we can straightforwardly establish Parseval's theorem containing the sum of $\ln^2\Gamma(k/n)$ in the left part, but the right part will contain products of integrals $I_{m,n}$ with different indices whose evaluation seems to be quite difficult. Second, if one could find such a

[39] For more details of the finite Fourier series, see e.g. [30, Chap. 6].

discrete functions $\Phi_{k,n}$ or $\Upsilon_{k,n}$ that coefficients $\varphi_{m,n}$ or $\alpha_{m,n}$ being summed with the integrand from $I_{m,n}$ or $I_{2m-n,n}$ respectively, gave a new and "computable" integral, then, it should be possible to obtain a new functional relationship for the logarithm of the Γ-function.

63* (a) Show that Malmsten's integral from exercise no. 3-a for $a=0$ may be also computed by means of the first generalized Stieltjes constant $\gamma_1(v)$

$$\int_0^\infty \frac{\operatorname{sh} px \cdot \ln x}{\operatorname{sh} x}\, dx = -\frac{1}{2}\left\{\pi(\gamma+\ln 2)\operatorname{tg}\frac{\pi p}{2} + \gamma_1\left(\frac{1}{2}-\frac{p}{2}\right) - \gamma_1\left(\frac{1}{2}+\frac{p}{2}\right)\right\},$$

and this result holds for continuous and complex values of p such that $|\operatorname{Re} p| < 1$.
(b) By making use of the previous result, prove following functional relationships for the derivatives of the Hurwitz ζ-functions and for the first generalized Stieltjes constants:

$$\begin{aligned}
(1)\quad &\lim_{s\to 1}\left\{\zeta'\left(s, \frac{1}{2}-\frac{m}{2n}\right) - \zeta'\left(s, \frac{1}{2}+\frac{m}{2n}\right)\right\} = \gamma_1\left(\frac{1}{2}+\frac{m}{2n}\right) - \gamma_1\left(\frac{1}{2}-\frac{m}{2n}\right) \\
&\quad = 2\pi\sum_{l=1}^{2n-1}(-1)^l \sin\frac{m\pi l}{n}\cdot\ln\Gamma\left(\frac{l}{2n}\right) + \pi(\gamma+\ln 4\pi n)\operatorname{tg}\frac{m\pi}{2n}, \\
(2)\quad &\lim_{s\to 1}\left\{\zeta'\left(s, 1-\frac{m}{n}\right) - \zeta'\left(s, \frac{m}{n}\right)\right\} = \gamma_1\left(\frac{m}{n}\right) - \gamma_1\left(1-\frac{m}{n}\right) \\
&\quad = 2\pi\sum_{l=1}^{n-1}\sin\frac{2\pi m l}{n}\cdot\ln\Gamma\left(\frac{l}{n}\right) - \pi(\gamma+\ln 2\pi n)\operatorname{ctg}\frac{m\pi}{n}
\end{aligned}$$

where $m = 1, 2, \ldots, n-1$, $n \in \mathbb{N}_{\geqslant 2}$, and where derivatives are taken with respect to the first argument of $\zeta(s, v)$.

Nota bene: Formulas (b.1) and (b.2) are not new[40] and may be directly obtained by differentiating the functional equation for the Hurwitz ζ-function, see e.g. [5, p. 261, §12.9], [45, Eq. (6)], and by taking into account that $\zeta'(0, v) = \ln\Gamma(v) + \zeta'(0) = \ln\Gamma(v) - \frac{1}{2}\ln 2\pi$, see e.g. [9, vol. I, p. 26, Eq. 1.10(10)], [11, Eq. (3)] or [2, Eq. (3)]. Though these relationships do not appear to be completely novel, the derivation from Malmsten's results (dated 1842!) and from the Mittag–Leffler theorem seems to be original.

Hint: On the one hand, it appears from exercise no. 22 that the integral $\Upsilon(\varphi)$ may be calculated by means of the Hurwitz ζ-function. On the other hand, one may remark that the derivative $d\Upsilon/d\varphi$ at $\varphi = \pi m/n$, where m and n are positive integers such that $m < n$, coincide with Malmsten's integral from exercise no. 3. By taking into account that

$$\frac{\partial}{\partial x}\zeta''\big(s, f(x)\big) = -f'_x \cdot \left\{2\zeta'\big(s+1, f(x)\big) + s\zeta''\big(s+1, f(x)\big)\right\},$$

[40] According to [2] formula (c) was first proved by Almkvist and Meurman in a private communication.

where derivatives ζ' and ζ'' are taken with respect to s, one easily arrives at the first part of formula (b.1) [the part without the Stieltjes constants]. Now, it is well known that the Hurwitz ζ-function $\zeta(s,v)$ is a meromorphic function on the entire complex s-plane and that its only pole is a simple pole at $s=1$ with residue 1. It can be, therefore, expanded as a Laurent series in a neighborhood of $s=1$ in the following way

$$\zeta(s,v) = \frac{1}{s-1} - \Psi(v) + \sum_{n=1}^{\infty} \frac{(-1)^n (s-1)^n}{n!} \gamma_n(v), \qquad s \neq 1.$$

The coefficients $\gamma_n(v)$ appearing in the regular part of this expansion are called generalized Stieltjes constants.[41] From this formula, it follows that in a neighborhood of $s=1$ $\zeta'(s,v) = -(s-1)^{-2} - \gamma_1(v) + O(s-1)$ and $\zeta''(s,v) = 2(s-1)^{-3} + \gamma_2(v) + O(s-1)$, and thus, the limit in (b.1) reduces to the difference between two Stieltjes constants. This yields formula (b.1) in its final form, as well as explaining how formula (a) was obtained. Now, formula (b.1) may be rewritten in a slightly different way. Putting $2m-n$ instead of m, and then, using the duplication formula for the Γ-function, as well as bearing in mind that

$$\sum_{l=1}^{2n-1} l \cdot \sin \frac{2\pi m l}{n} = -n \operatorname{ctg} \frac{m\pi}{n}, \qquad m = 1, 2, \ldots, n-1,$$

one arrives at formula (b.2).

64* (a) Show that Malmsten's integral from exercise no. 3-b for $a=0$ may be also evaluated by means of the first generalized Stieltjes constant $\gamma_1(v)$

$$\begin{aligned}
\int_0^\infty \frac{\operatorname{ch} px \cdot \ln x}{\operatorname{ch} x}\, dx = \frac{1}{2} \biggl\{ &\gamma_1\left(\frac{1}{2} + \frac{p}{2}\right) + \gamma_1\left(\frac{1}{2} - \frac{p}{2}\right) \\
&- \gamma_1\left(\frac{1}{4} + \frac{p}{4}\right) - \gamma_1\left(\frac{1}{4} - \frac{p}{4}\right)\biggr\} \\
&- \frac{1}{2}\ln^2 2 + \ln 2 \cdot \Psi\left(\frac{1}{2} + \frac{p}{2}\right) + \frac{\pi}{2}(\gamma + \ln 2) \operatorname{tg} \frac{\pi p}{2} \\
&- \frac{\pi}{2}(\gamma + 2\ln 2) \operatorname{ctg}\left(\frac{\pi}{4} - \frac{\pi p}{4}\right)
\end{aligned}$$

where parameter p is assumed to be continuous and complex lying within the strip $|\operatorname{Re} p| < 1$.

[41] This expansion for the Riemann ζ-function was first given by Stieltjes, and therefore, was written in terms of constants $\gamma_n \equiv \gamma_n(1)$, which were later called the *Stieltjes constants*. Constants $\gamma_n(v)$ with arbitrary v represent a more general case and occur when expanding the Hurwitz ζ-function instead of the Riemann ζ-function; such constants are called *generalized Stieltjes constants*. For more information, see [9, vol. I, p. 26, Eq. 1.10(9), and vol. III, §17.7, p. 189], [14], [66, p. 16], [11, 17, 18, 38, 48].

(b) Prove that the following functional relationship between first derivatives of the Hurwitz ζ-function, first generalized Stieltjes constants and the logarithm of the Γ-function takes place:

$$\begin{aligned}
&\lim_{s\to 1}\left\{\zeta'\left(s,\frac{m}{n}\right)+\zeta'\left(s,1-\frac{m}{n}\right)-\zeta'\left(s,\frac{m}{2n}\right)-\zeta'\left(s,\frac{1}{2}-\frac{m}{2n}\right)\right\}\\
&\quad=\lim_{s\to 1}\left\{\zeta'\left(s,\frac{m}{n}\right)-\zeta'\left(s,\frac{m}{2n}\right)\right\}\\
&\qquad+\lim_{s\to 1}\left\{\zeta'\left(s,1-\frac{m}{n}\right)-\zeta'\left(s,\frac{1}{2}-\frac{m}{2n}\right)\right\}\\
&\quad=-\left\{\gamma_1\left(\frac{m}{n}\right)+\gamma_1\left(1-\frac{m}{n}\right)-\gamma_1\left(\frac{m}{2n}\right)-\gamma_1\left(\frac{1}{2}-\frac{m}{2n}\right)\right\}\\
&\quad=2\pi\sum_{l=0}^{n-1}\sin\frac{(2l+1)m\pi}{n}\cdot\ln\Gamma\left(\frac{2l+1}{2n}\right)-\pi\csc\frac{m\pi}{n}\cdot\ln\pi n-\ln^2 2\\
&\qquad+2\ln 2\cdot\Psi\left(\frac{m}{n}\right)-\pi(\gamma+\ln 2)\operatorname{ctg}\frac{m\pi}{n}-\pi(\gamma+2\ln 2)\operatorname{tg}\frac{m\pi}{2n},
\end{aligned}\tag{57}$$

$m=1,2,\ldots,n-1,\ n\in\mathbb{N}_{\geqslant 2}$.
Nota bene: This formula, as we come to see later, permits one to evaluate the first generalized Stieltjes constant at some rational arguments. First of all, the combination of the above formula with that from exercise no. 63 yields another elegant result

$$\begin{aligned}
&2\gamma_1\left(\frac{m}{n}\right)-\gamma_1\left(\frac{m}{2n}\right)-\gamma_1\left(\frac{1}{2}-\frac{m}{2n}\right)\\
&\quad=2\pi\sum_{l=1}^{n-1}\sin\frac{2\pi ml}{n}\cdot\ln\Gamma\left(\frac{l}{n}\right)-2\pi\sum_{l=0}^{n-1}\sin\frac{(2l+1)m\pi}{n}\cdot\ln\Gamma\left(\frac{2l+1}{2n}\right)\\
&\qquad+\ln^2 2-2\ln 2\cdot\Psi\left(\frac{m}{n}\right)+\pi(\gamma+\ln 4\pi n)\operatorname{tg}\frac{m\pi}{2n},
\end{aligned}\tag{58}$$

where $m=1,2,\ldots,n-1$, $n\in\mathbb{N}_{\geqslant 2}$. Several important particular cases follow immediately from this expression. Thus, putting $m=1$ and $n=2$ yields

$$\gamma_1\left(\frac{1}{2}\right)-\gamma_1\left(\frac{1}{4}\right)=-2\pi\ln\Gamma\left(\frac{1}{4}\right)+\frac{3\pi}{2}\ln\pi+2\pi\ln 2+\frac{5}{2}\ln^2 2+\frac{\gamma}{2}(\pi+2\ln 2).$$

Setting $m=1$ and $n=3$ gives

$$\begin{aligned}
\gamma_1\left(\frac{1}{3}\right)-\gamma_1\left(\frac{1}{6}\right)&=-2\pi\sqrt{3}\ln\Gamma\left(\frac{1}{3}\right)+\ln^2 2+(3\ln 3+2\gamma)\ln 2\\
&\quad+\frac{\pi}{\sqrt{3}}\left(5\ln 2+4\ln\pi-\frac{1}{2}\ln 3+\gamma\right),
\end{aligned}\tag{59}$$

and so on. However, other particular cases look less beautiful.

Now, we proceed with the evaluation of the generalized Stieltjes constants at some rational arguments. Let us first calculate $\gamma_1(1/4)$. By chance, we have $\zeta\big(s,\frac{1}{2}\big) = (2^s-1)\zeta(s)$. By expanding both sides in a neighborhood of $s=1$ and by equating coefficients with same powers, we arrive at

$$\gamma_1\left(\frac{1}{2}\right) = -2\gamma\ln 2 - \ln^2 2 + \gamma_1 = -1.353459680\ldots \tag{60}$$

where as usually $\gamma_1 \equiv \gamma_1(1) = -0.07281584548\ldots$ Hence,

$$\begin{aligned}\gamma_1\left(\frac{1}{4}\right) &= 2\pi\ln\Gamma\left(\frac{1}{4}\right) - \frac{3\pi}{2}\ln\pi - \frac{7}{2}\ln^2 2 - (3\gamma+2\pi)\ln 2 - \frac{\gamma\pi}{2} + \gamma_1\\ &= -5.518076350\ldots.\end{aligned}$$

By the formula from exercise no. 63-b.2, we immediately get

$$\begin{aligned}\gamma_1\left(\frac{3}{4}\right) &= -2\pi\ln\Gamma\left(\frac{1}{4}\right) + \frac{3\pi}{2}\ln\pi - \frac{7}{2}\ln^2 2 - (3\gamma-2\pi)\ln 2 + \frac{\gamma\pi}{2} + \gamma_1\\ &= -0.3912989024\ldots.\end{aligned}$$

Our next step is the evaluation of $\gamma_1(1/3)$, $\gamma_1(2/3)$, $\gamma_1(1/6)$ and $\gamma_1(5/6)$. By using elementary transformations, one can show that $\zeta\big(s,\frac{1}{3}\big) + \zeta\big(s,\frac{2}{3}\big) = (3^s-1)\zeta(s)$, and hence

$$\gamma_1\left(\frac{1}{3}\right) + \gamma_1\left(\frac{2}{3}\right) = -3\gamma\ln 3 - \frac{3}{2}\ln^2 3 + 2\gamma_1. \tag{61}$$

By no. 63-b.2, it follows immediately that

$$\begin{aligned}\gamma_1\left(\frac{1}{3}\right) &= -\frac{3\gamma}{2}\ln 3 - \frac{3}{4}\ln^2 3 + \frac{\pi}{4\sqrt{3}}\left\{\ln 3 - 8\ln 2\pi - 2\gamma + 12\ln\Gamma\left(\frac{1}{3}\right)\right\} + \gamma_1\\ &= -3.259557515\ldots,\\ \gamma_1\left(\frac{2}{3}\right) &= -\frac{3\gamma}{2}\ln 3 - \frac{3}{4}\ln^2 3 - \frac{\pi}{4\sqrt{3}}\left\{\ln 3 - 8\ln 2\pi - 2\gamma + 12\ln\Gamma\left(\frac{1}{3}\right)\right\} + \gamma_1\\ &= -0.5989062842\ldots.\end{aligned}$$

By substituting $\gamma_1(1/3)$ into (59), we obtain

$$\begin{aligned}\gamma_1\left(\frac{1}{6}\right) &= -\frac{3\gamma}{2}\ln 3 - \frac{3}{4}\ln^2 3 - \ln^2 2 - (3\ln 3 + 2\gamma)\ln 2 + 3\pi\sqrt{3}\ln\Gamma\left(\frac{1}{3}\right)\\ &\quad + \frac{\pi}{2\sqrt{3}}\left\{\frac{3}{2}\ln 3 - 14\ln 2 - 12\ln\pi - 3\gamma\right\} + \gamma_1 = -10.74258252\ldots.\end{aligned}$$

In view of the fact that $\Gamma(1/6) = 3^{\frac{1}{2}}2^{-\frac{1}{3}}\pi^{-\frac{1}{2}}\Gamma^2(1/3)$, see e.g. [15, p. 31], the latter formula may be also written as

$$\gamma_1\left(\frac{1}{6}\right) = -\frac{3\gamma}{2}\ln 3 - \frac{3}{4}\ln^2 3 - \ln^2 2 - (3\ln 3 + 2\gamma)\ln 2 + \frac{3\pi\sqrt{3}}{2}\ln\Gamma\left(\frac{1}{6}\right) - \frac{\pi}{2\sqrt{3}}\left\{3\ln 3 + 11\ln 2 + \frac{15}{2}\ln\pi + 3\gamma\right\} + \gamma_1 = -10.74258252\ldots.$$

Hence, by no. 63-b.2, we also have

$$\gamma_1\left(\frac{5}{6}\right) = -\frac{3\gamma}{2}\ln 3 - \frac{3}{4}\ln^2 3 - \ln^2 2 - (3\ln 3 + 2\gamma)\ln 2 - \frac{3\pi\sqrt{3}}{2}\ln\Gamma\left(\frac{1}{6}\right) + \frac{\pi}{2\sqrt{3}}\left\{3\ln 3 + 11\ln 2 + \frac{15}{2}\ln\pi + 3\gamma\right\} + \gamma_1 = -0.2461690038\ldots.$$

However, further evaluation of $\gamma_1(k/n)$ faces much more difficulties. To illustrate this point, we first generalize equations (60) and (61). From elementary transformations, it follows that

$$\sum_{l=1}^{n-1}\zeta\left(s,\frac{l}{n}\right) = \left(n^s - 1\right)\zeta(s), \qquad n = 2, 3, 4, \ldots.$$

Writing the Laurent series about $s = 1$ for both sides and equating coefficients with same powers, one obtains[42]

$$\sum_{l=1}^{n-1}\gamma_1\left(\frac{l}{n}\right) = -n\gamma\ln n - \frac{n}{2}\ln^2 n + (n-1)\gamma_1, \qquad n = 2, 3, 4, \ldots. \tag{62}$$

Moreover, analogously it can be shown that

$$\sum_{l=0}^{n-1}\zeta\left(s, v + \frac{l}{n}\right) = n^s\zeta(s, nv), \qquad n = 2, 3, 4, \ldots$$

and hence

$$\sum_{l=0}^{n-1}\gamma_1\left(v + \frac{l}{n}\right) = n\ln n\cdot\Psi(nv) - \frac{n}{2}\ln^2 n + n\gamma_1(nv), \qquad n = 2, 3, 4, \ldots. \tag{63}$$

The latter formula represents a kind of the multiplication theorem for the first Stieltjes constants. It can be extended to the higher-order Stieltjes constants as follows:

$$\sum_{l=0}^{n-1}\gamma_p\left(v + \frac{l}{n}\right) = (-1)^p n\left[\frac{\ln n}{p+1} - \Psi(nv)\right]\ln^p n + n\sum_{r=0}^{p-1}(-1)^r C_p^r\gamma_{p-r}(nv)\cdot\ln^r n,$$

$n = 2, 3, 4$, where, as usually, C_p^r denotes the binomial coefficient $C_p^r = \frac{p!}{r!(p-r)!}$. A particular case of this formula for $v = 1/n$ was already found by Mark Coffey [18, p. 1830, Eq. (3.28)].

[42] This formula appears with an error in [18, p. 1836, Eq. (3.54)]: in the right part $\frac{1}{2}$ should be replaced by $\frac{1}{2}\ln q$.

Now, another useful property of the generalized Stieltjes constants may be derived from the recurrence formula for the Hurwitz ζ-function:

$$\zeta(s, v+1) = \zeta(s, v) - v^{-s}.$$

Expanding both sides in the Laurent series about $s = 1$, one can easily see that

$$\gamma_1(v+1) = \gamma_1(v) - \frac{\ln v}{v}, \tag{64}$$

and more generally

$$\gamma_p(v+1) = \gamma_p(v) - \frac{\ln^p v}{v}, \quad p = 1, 2, 3, \ldots. \tag{65}$$

This is the recurrence relationship for the pth generalized Stieltjes constant. Thus, equations (63), (64), and no. 63-b.2 constitute the multiplication theorem, the recurrence relationship and the reflection formula, respectively, for the first Stieltjes constant[43] and may be quite useful for the exact determination of some values of γ_1.

Now, we try to evaluate the set of $\gamma_1(k/8)$, $k = 1, 2, \ldots, 7$. Taking into account previously calculated values, there are 4 unknowns in this set. By making use of various formulas derived before, we may construct the following system of equations for these unknowns:

$$\begin{cases} \gamma_1(1/8) + \gamma_1(3/8) = \cdots \text{see Eq. (58) for } m = 1, n = 4, \\ \gamma_1(1/8) + \gamma_1(3/8) + \gamma_1(5/8) + \gamma_1(7/8) = \cdots \text{ see Eq. (62) for } n = 8, \\ \gamma_1(1/8) - \gamma_1(7/8) = \cdots \text{ see no. 63-b.2 for } m = 1, n = 8, \\ \gamma_1(3/8) - \gamma_1(5/8) = \cdots \text{ see no. 63-b.2 for } m = 3, n = 8, \end{cases}$$

where all quantities in the right part may be expressed in terms of γ, γ_1, the Γ-function and elementary functions. However, this system cannot be solved because the corresponding matrix is not of full rank:

$$\operatorname{rank} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{pmatrix} = 3.$$

The same problem of rank arises when trying to evaluate families $\gamma_1(k/5)$, $k = 1, 2, 3, 4$, and many others.[44] For instance, one may try to reuse the procedure of determination of $\Gamma(1/12)$ in terms of $\Gamma(1/3)$ and $\Gamma(1/4)$ described in [15, p. 31] for $\gamma_1(1/12)$. An observation of the system of equations II and III [15, pp. 30–31] shows that the rank of the corresponding matrix is 8, and thus, the system can be

[43] Note that these relationships are quite similar to those for the logarithm of the Γ-function.

[44] An attentive analysis shows that equations no. 63-b.2, (58) and (63) for $n = 2$ and $v = m/(2n)$ [variable n corresponds to (58)] are linearly dependent.

solved in terms of $\Gamma(1/3)$ and $\Gamma(1/4)$, the value $\Gamma(1/2)$ being known. An equivalent system of equations for the Stieltjes constants has the rank equal to 7 (due to the fact that the reflection formula for the first Stieltjes constant no. 63-b.2 slightly differs from that for the Γ-function), and hence, it cannot be solved. Thus, for the evaluation of other $\gamma_1(k/n)$, we need to get more independent equations than we currently possess. By the way, there is an interesting analogy between our results and those obtained by Miller and Adamchik [45]. Those authors proposed a method for the evaluation of $\zeta'(-2k+1, p)$, $k \in \mathbb{N}$, for some rational values of p in terms of diverse transcendental functions. In particular, they provided an explicit expression for the case $p = \frac{1}{3}$, and concluded that the same method can be equally applied to cases $p = \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{2}{3}, \frac{1}{6}$ and $\frac{5}{6}$. At the same time, they also reported that the evaluation of $\zeta'(-2k+1, p)$, $k \in \mathbb{N}$, for other rational values of p faces some major problems.

Notwithstanding, taking into account previously derived results, we may conjecture that *any generalized Stieltjes constant of the form $\gamma_1(k/n)$, where k and n are positive integers such that $k < n$, may be expressed by means of the Euler's constant γ, the first Stieltjes constant γ_1, the logarithm of the Γ–function at rational argument(s) and some relatively simple, perhaps elementary, function*. This statement may be written as follows:

$$\gamma_1\left(\frac{k}{n}\right) = f(k,n,\gamma) + \sum_{l=1}^{n-1} \alpha_l(k,n) \cdot \ln \Gamma\left(\frac{l}{n}\right) + \gamma_1, \quad k = 1, 2, \ldots, n-1, \tag{66}$$

Note that a similar relationship exists for the 0th generalized Stieltjes constant, which is simply $\gamma_0(k/n) = -\Psi(k/n)$, see e.g. [48, p. 153, Eq. (4.7)].

The results above suggest another interesting conjecture: *the sum of the first Stieltjes constant at a rational argument with its reflected version may be expressed in terms of some relatively simple function g (possibly elementary), the Euler's constant γ and the first Stieltjes constant γ_1*. In other words

$$\gamma_1\left(\frac{k}{n}\right) + \gamma_1\left(1 - \frac{k}{n}\right) = g + 2\gamma_1, \quad k = 1, 2, \ldots, n-1, \tag{67}$$

An interesting way to confirm or to refute this hypothesis could be to study the integral from exercise no. 66 for rational values of p.

Finally, we should say that above results seem to be novel though we do not know for sure. For example, Mark Coffey [18] already remarked that constants $\gamma_1(1/3)$ and $\gamma_1(2/3)$ may be separately written in terms of γ_1 [18, p. 1830, Collolary 1]; however, he did not provide explicit expressions for them.[45] In the same work, one may also find several formulas for the differences of Stieltjes constants expressed in terms of the second-order derivatives of the Hurwitz ζ-function, as well as several series and integral representations for them.

[45] Moreover, the exact determination of $\gamma_1(1/3)$ and $\gamma_1(2/3)$ is based on a particular case of (3.28) at $k = 1$; such a case is given by (3.54) [18]. The latter equation, as we already noticed in footnote 42, contains an error.

Hint: The technique of proof is similar to that from the preceding exercise. First, by using geometric series expansions, one can show that the following integral may be expressed by means of alternating Hurwitz ζ-functions

$$\int_0^\infty \frac{x^a \operatorname{sh} bx}{\operatorname{ch} x}\, dx = \frac{\Gamma(a+1)}{2^{a+1}}\left\{\eta\left(a+1, \frac{1}{2}-\frac{b}{2}\right) - \eta\left(a+1, \frac{1}{2}+\frac{b}{2}\right)\right\},$$

$a > -2$, $|\operatorname{Re} b| < 1$, and hence, in virtue of (4), by means of ordinary Hurwitz ζ-functions. Differentiating the above integral with respect to a, and then, letting $a \to -1$, yields

$$\begin{aligned}
&\int_0^\infty \frac{\operatorname{sh} bx}{x \operatorname{ch} x} \ln x\, dx \\
&\quad = \frac{1}{2}\left\{\zeta''\left(0, \frac{1}{2}+\frac{b}{2}\right) - \zeta''\left(0, \frac{1}{2}-\frac{b}{2}\right)\right\} + \zeta''\left(0, \frac{1}{4}-\frac{b}{4}\right) - \zeta''\left(0, \frac{1}{4}+\frac{b}{4}\right) \\
&\qquad + \frac{3b}{2}\ln^2 2 + \gamma b \ln 2 + 2\ln 2 \cdot \ln \Gamma\left(\frac{1}{2}+\frac{b}{2}\right) - (\gamma + \ln 2)\ln\cos\frac{\pi b}{2} \\
&\qquad - \ln 2 \cdot \ln \pi + (\gamma + 2\ln 2)\left\{(1-b)\ln 2 + 2\ln\sin\left(\frac{\pi}{4} - \frac{\pi b}{4}\right)\right\},
\end{aligned}$$

where derivatives of the Hurwitz ζ-function are taken with respect to its first argument and where $|\operatorname{Re} b| < 1$. Now, the derivative of the latter integral with respect to b coincides, at $b = m/n$, with Malmsten's integral from no. 3-b. Writing $2m - n$ instead of m, and then simplifying the formula from no. 3-b for $a = 0$, $b = 1$ and $p = (2m-n)/n$, produces the final result.

65* Prove that the following integral may be evaluated by means of generalized Stieltjes constants

$$\begin{aligned}
\int_0^\infty \frac{\operatorname{sh} px \cdot \ln x}{\operatorname{ch} x}\, dx = \frac{1}{2}\Bigg\{&\pi(\gamma + \ln 2)\operatorname{tg}\frac{\pi p}{2} \\
&- (\gamma + 2\ln 2)\left[\Psi\left(\frac{1}{4}+\frac{p}{4}\right) - \Psi\left(\frac{1}{4}-\frac{p}{4}\right)\right] \\
&+ \gamma_1\left(\frac{1}{2}-\frac{p}{2}\right) - \gamma_1\left(\frac{1}{2}+\frac{p}{2}\right) - \gamma_1\left(\frac{1}{4}-\frac{p}{4}\right) + \gamma_1\left(\frac{1}{4}+\frac{p}{4}\right)\Bigg\},
\end{aligned}$$

where $|\operatorname{Re} p| < 1$. Note that if p is rational, then $\gamma_1(1/2 - p/2) - \gamma_1(1/2 + p/2)$ may be expressed in terms of the Γ-function (see no. 63-b.1). This integral may be also important for the closed-form determination of the first generalized Stieltjes constant (see the *Nota bene* of the next exercise).

Hint: See previous exercises. At the final stage, use the reflection formula for the Ψ-function.

66* Analogously to the previous exercise, prove that for $|\operatorname{Re} p| < 1$

$$\int_0^\infty \frac{(\operatorname{ch} px - 1)\,\ln x}{\operatorname{sh} x}\,dx = (\gamma + \ln 2)\cdot\left\{\Psi\left(\frac{1}{2}+\frac{p}{2}\right) + \ln 2 - \frac{\pi}{2}\operatorname{tg}\frac{\pi p}{2}\right\}$$
$$+\gamma^2 + \gamma_1 - \frac{1}{2}\gamma_1\left(\frac{1}{2}+\frac{p}{2}\right) - \frac{1}{2}\gamma_1\left(\frac{1}{2}-\frac{p}{2}\right).$$

Show then that this integral for $p = \frac{1}{2}, \frac{1}{3}, \frac{2}{3}$ may be expressed in terms of elementary functions and the Euler's constant γ.

Nota bene: This integral plays an important role for the second conjecture (67) concerning the first generalized Stieltjes constant (see exercise no. 64). The formula given above is derived by making use of geometric series, which lead to the Hurwitz ζ-function (see the hint below). It is, therefore, highly desirable to find another method for the evaluation of this integral. One of the possibilities could be the application of the Mittag–Leffler theorem to the integrand. Accordingly, we may expand for any $-1 < p < 1$

$$\frac{\operatorname{ch} pz - 1}{\operatorname{sh} z} = 2z\sum_{n=1}^\infty (-1)^n \frac{\cos p\pi n - 1}{z^2 + \pi^2 n^2}, \qquad z \in \mathbb{C},\ z \neq \pi n i,\ n \in \mathbb{Z}$$

see e.g. [23, no. 27.10.2]. But the performance of term-by-term integration results in a divergent series

$$\int_0^\infty \frac{(\operatorname{ch} px - 1)\ln x}{\operatorname{sh} x}\,dx = 2\sum_{n=1}^\infty (-1)^n(\cos p\pi n - 1)\underbrace{\int_0^\infty \frac{x\ln x}{x^2+\pi^2 n^2}\,dx}_{\infty} = \cdots.$$

One can also try to evaluate a similar integral

$$\int_0^\infty \frac{(\operatorname{ch} px - 1)x^{a-1}}{\operatorname{sh} x}\,dx = 2\sum_{n=1}^\infty (-1)^n(\cos p\pi n - 1)\underbrace{\int_0^\infty \frac{x^a}{x^2+\pi^2 n^2}\,dx}_{\frac{1}{2}\pi^a n^{a-1}\sec\frac{1}{2}\pi a}$$
$$= \pi^a \sec\frac{\pi a}{2}\sum_{n=1}^\infty (-1)^n \frac{\cos p\pi n - 1}{n^{1-a}}. \tag{68}$$

The equality holds for $-1 < p < 1$ and $a \in (-1, +1)$, but it can be analytically continued for $a \notin (-1, +1)$. The integral is the analytic continuation of the sum for $a \geqslant 1$, while the sum analytically continues the integral for $a \leqslant -1$. We obviously have to expect trouble with the right-hand part at $a = \pm 1, \pm 3, \pm 5, \ldots$ because of the secant. Since when $a = -1, -3, -5, \ldots$ the sum in the right-hand side converges, these points are poles of the first order for the analytic continuation of integral (68). In contrast, for $a = 1, 3, 5, \ldots$, the integral on the left remains bounded, and thus, these points are removable singularities for the right-hand side of (68). In other words, formally $\sum (-1)^n (\cos p\pi n - 1) n^{a-1}$, $n \geqslant 1$, must vanish identically for any odd positive a (exactly as $\eta(1-a)$, the result which has been derived by Euler, see e.g. [20, p. 85]). Thus, the partial differentiation of the right-hand side of (68) with respect to a at $a \to 1$ leads to an A-summable divergent series (see [31, p. 7]). When using divergent series methods, the major difficulty consists in the evaluation of $\sum (-1)^n (\cos p\pi n - 1) \ln^2 n$, $n \geqslant 1$, the series $\sum (-1)^n (\cos p\pi n - 1) \ln n$, $n \geqslant 1$, being easily reducible to $\frac{1}{2}\Psi(\frac{1}{2} + \frac{1}{2}p) - \frac{1}{4}\pi \operatorname{tg} \frac{1}{2}\pi p + \frac{1}{2}\gamma + \ln 2$ (see e.g. [40, p. 58, Eq. (72)] or differentiate no. 20-b with respect to φ).

Another approach consists in the use either of polylogarithms, or of the Lerch transcendent or of the hypergeometric function, but the price will be a high transcendence. The search for more suitable analytic continuations of (68) remains, therefore, relevant.

Hint: Differentiate formula (52) with respect to a, and then, let $a \to 0$. In order to perform the latter limiting procedure, expand ζ-functions in the Laurent series (as we did in no. 63). The final result is obtained by using the reflection formula for the Ψ-function. As regards the values of the integral at $p = \frac{1}{2}, \frac{1}{3}, \frac{2}{3}$, use results obtained in exercise no. 64.

67* In exercises no. 63–66, we saw that there is a connection between Malmsten's integrals of the first order and the first Stieltjes constants. Prove now that

$$\int_0^\infty \frac{\operatorname{sh}^2 px \cdot \ln x}{\operatorname{sh}^2 x}\, dx$$

$$= \frac{1}{2}\Big\{\ln \pi - \ln \sin \pi p + p\big[\gamma_1(p) - \gamma_1(1-p)\big] - (\gamma + \ln 2)(1 - \pi p \operatorname{ctg} \pi p)\Big\}$$

for $|\operatorname{Re} p| < 1$, and therefore, such a connection exists also between Malmsten's integrals of the second order and the first Stieltjes constants.

Hint: From elementary analysis, it is well known that

$$\frac{1}{y^2 - 2y + 1} = \sum_{n=1}^\infty n y^{n-1}, \qquad |y| < 1.$$

By putting in the latter expansion $y = e^{-2x}$, one can easily show that

$$\int_0^\infty \frac{x^a \cdot e^{-px}}{\operatorname{sh}^2 x}\, dx = 4\Gamma(a+1) \sum_{n=1}^\infty \frac{n}{(2n+p)^{a+1}}$$

$$= \frac{\Gamma(a+1)}{2^{a-1}}\left\{\zeta\left(a,\frac{p}{2}\right)-\frac{p}{2}\zeta\left(a+1,\frac{p}{2}\right)\right\}, \qquad a>1,\ \operatorname{Re} p>-2$$

and hence,

$$\int_0^\infty \frac{x^a \operatorname{ch} px}{\operatorname{sh}^2 x}\,dx = \frac{\Gamma(a+1)}{2^a}\left\{\zeta\left(a,\frac{p}{2}\right)-\frac{p}{2}\zeta\left(a+1,\frac{p}{2}\right)+\zeta\left(a,-\frac{p}{2}\right)\right.$$
$$\left.+\frac{p}{2}\zeta\left(a+1,-\frac{p}{2}\right)\right\}, \qquad a>1,\ |\operatorname{Re} p|<2.$$

Furthermore, it can be analogously shown that

$$\int_0^\infty \frac{x^a \operatorname{sh}^2 px}{\operatorname{sh}^2 x}\,dx = \frac{\Gamma(a+1)}{2^{a+1}}\big\{\zeta(a,p)-p\zeta(a+1,p)+\zeta(a,-p)$$
$$+p\zeta(a+1,-p)-2\zeta(a)\big\}, \qquad a>-1,\ |\operatorname{Re} p|<1.$$

Differentiating this integral with respect to a, and then letting $a\to 0$, as well as using (65), produces the wanted result. By the way, by putting $p=m/n$, and by using no. 63-b.2, we arrive at the result obtained in exercise no. 11-b.

References

1. Abramowitz, M., Stegun, I.A.: Handbook of Mathematical Functions with Formula, Graphs and Mathematical Tables. Applied Mathematics Series, vol. 55. National Bureau of Standards, Washington (1961)
2. Adamchik, V.: A class of logarithmic integrals. In: Proceedings of the 1997 International Symposium on Symbolic and Algebraic Computation, pp. 1–8 (1997)
3. Ahlfors, L.: Complex Analysis, 3rd edn. McGraw-Hill Science, New York (1979)
4. Amdeberhan, T., Espinosa, O., Moll, V.H.: The Laplace transform of the digamma function: an integral due to Glasser, Manna and Oloa. Proc. Am. Math. Soc. **136**(9), 3211–3221 (2008)
5. Apostol, T.M.: Introduction to Analytic Number Theory. Springer, New York (1976)
6. Artin, E.: Einführung in die Theorie der Gammafunktion. Teubner, Leipzig (1931)
7. Bailey, D.H., Borwein, J.M., Calkin, N.J., Girgensohn, R., Luke, R.D., Moll, V.H.: Experimental Mathematics in Action. AK Peters, Wellesley (2007)
8. Bassett, M.E.: Integrals, L functions, prime numbers: a short tour of analytic number theory. Report on a talk given at UCL (University College London) Undergrad Maths Colloquium on Analytic Number Theory, London's Global University (2010)
9. Bateman, H., Erdélyi, A.: Higher Transcendental Functions. McGraw-Hill Book, New York (1955) [in 3 volumes]
10. Bermant, A.F.: In: Course of Mathematical Analysis. Part II, 5th edn., Gosudarstvennoe izdatel'stvo tehniko–teoreticheskoj literatury, Moscow (1954) [in Russian]
11. Berndt, B.C.: The Gamma function and the Hurwitz Zeta-function. Am. Math. Mon. **92**(2), 126–130 (1985)
12. Binet, M.J.: Mémoire sur les intégrales définies eulériennes et sur leur application à la théorie des suites, ainsi qu'à l'évaluation des fonctions des grands nombres. J. Éc. Polytech. **XVI**(27), 123–343 (1839)
13. Boros, G., Moll, V.: Irresistible Integrals: Symbolics, Analysis and Experiments in the Evaluation of Integrals. Cambridge University Press, Cambridge (2004)

14. Briggs, W.E.: Some constants associated with the Riemann Zeta-function. Mich. Math. J. **3**(2), 117–121 (1955)
15. Campbell, R.: Les intégrales eulériennes et leurs applications. Dunod, Paris (1966)
16. Carathéodory, C.: Theory of Functions of a Complex Variable. Chelsea, New York (1954) [in 2 volumes]
17. Coffey, M.W.: Series representations for the Stieltjes constants. arXiv:0905.1111v2 (2009)
18. Coffey, M.W.: On representations and differences of Stieltjes coefficients, and other relations. Rocky Mt. J. Math. **41**(6), 1815–1846 (2011)
19. Davis, P.J.: Leonhard Euler's integral: a historical profile of the Gamma function. Am. Math. Mon. **66**, 849–869 (1959)
20. Euler, L.: Remarques sur un beau rapport entre les séries des puissances tant directes que réciproques. Histoire de l'Académie Royale des Sciences et Belles-Lettres, année MDCCLXI, Tome 17, pp. 83–106, A Berlin, chez Haude et Spener, Libraires de la Cour et de l'Académie Royale (1768 [read in 1749])
21. Eulero, L.: Institutiones calculi differentialis cum eius usu in analysi finitorum ac doctrina serierum. Academiæ Imperialis Scientiarum Petropolitanæ, Saint-Petersburg (1755)
22. Evgrafov, M.A.: Analytic Functions. Saunders, Philadelphia (1966)
23. Evgrafov, M.A., Sidorov, Y.V., Fedoriuk, M.V., Shabunin, M.I., Bezhanov, K.A.: A Collection of Problems in the Theory of Analytic Functions. Nauka, Moscow (1969) [in Russian]
24. Frullani, G.: Sopra gli integrali definiti. Mem. Soc. Ital. Sci. **XX**(II), 448–467 (1829)
25. Fuchs, B.A., Shabat, B.V.: Functions of a Complex Variable and Some of Their Applications. International Series of Monographs in Pure and Applied Mathematics. Pergamon, Oxford (1961/1964) [in 2 volumes]
26. Gauss, C.F.: Disquisitiones generales circa seriem infinitam $1 + \frac{\alpha\beta}{1\cdot\gamma}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1\cdot 2\cdot\gamma(\gamma+1)}xx + \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1\cdot 2\cdot 3\cdot\gamma(\gamma+1)(\gamma+2)}x^3 + \text{etc.}$ Commentationes Societatis Regiae Scientiarum Gottingensis recentiores, Classis Mathematicæ, vol. II, pp. 3–46 [republished later in "Carl Friedrich Gauss Werke", vol. 3, pp. 265–327, Königliche Gesellschaft der Wissenschaften, Göttingen, 1866] (1813)
27. Godefroy, M.: La fonction Gamma; Théorie, Histoire, Bibliographie. Gauthier-Villars, Imprimeur Libraire du Bureau des Longitudes, de l'École Polytechnique, Quai des Grands-Augustins, 55, Paris (1901)
28. Gradshteyn, I.S., Ryzhik, I.M.: Tables of Integrals, Series and Products, 4th edn. Academic Press, New York (1980)
29. Gunther, N.M., Kuzmin, R.O.: A Collection of Problems on Higher Mathematics, vol. 3, 4th edn. Gosudarstvennoe izdatel'stvo tehniko–teoreticheskoj literatury, Leningrad (1951) [in Russian]
30. Hamming, R.W.: Numerical Methods for Scientists and Engineers. McGraw-Hill, New York (1962)
31. Hardy, G.H.: Divergent Series. Clarendan, Oxford (1949)
32. Henrici, P.: Applied and Computational Complex Analysis. Wiley, New York (1974/1977/1986) [in 3 volumes]
33. Hurwitz, A., Courant, R.: Vorlesungen über allgemeine Funktionentheorie und elliptische Funktionen. Herausgegeben und ergänzt durch einen Abschnitt über geometrische Funktionentheorie, 4th edn. Springer, Berlin (1964)
34. Jeffreys, H., Jeffreys, B.S.: Methods of Mathematical Physics, 2nd edn. Cambridge University Press, Cambridge (1950)
35. Kummer, E.E.: Beitrag zur theorie der function $\Gamma(x)$. J. Reine Angew. Math. **35**, 1–4 (1847)
36. Leche, V., Meijer, B., Nyström, J.F., Warburg, K., Westrin, T., Meijer, B.: Nordisk familjebok, 2nd edn., Nordisk familjeboks förlags aktiebolag, Stockholm (1904–1926). http://runeberg.org/nf (in 38 volumes)
37. Lerch, M.: Další studie v oboru Malmsténovských řad. Rozpravy České akademie císare Františka Josefa pro vedy, slovesnost a umení. Trída II, Mathematicko-prírodnická, ročník 3, číslo 28, pp. 1–63 (1894)
38. Liang, J.J.Y., Todd, J.: The Stieltjes constants. J. Res. Natl. Bur. Stand. B, Math. Sci. **76B**(3–4), 161–178 (1972)
39. Lindelöf, E.: Le calcul des résidus et ses applications à la théorie des fonctions. Gauthier-Villars, Imprimeur Libraire du Bureau des Longitudes, de l'École Polytechnique, Quai des Grands-Augustins, 55, Paris (1905)
40. Malmsten, C.J., Almgren, T.A., Camitz, G., Danelius, D., Moder, D.H., Selander, E., Grenander, J.M.A., Themptander, S., Trozelli, L.M., Föräldrar, Ä., Ossbahr, G.E., Föräldrar, D.H., Ossbahr, C.O.,

Lindhagen, C.A., Moder, D.H., Syskon, Ä., Lemke, O.V., Fries, C., Laurenius, L., Leijer, E., Gyllenberg, G., Morfader, M.V., Linderoth, A.: Specimen analyticum, theoremata quædam nova de integralibus definitis, summatione serierum earumque in alias series transformatione exhibens (Eng. trans.: "Some new theorems about the definite integral, summation of the series and their transformation into other series") [Dissertation, in 8 parts]. Upsaliæ, excudebant Regiæ academiæ typographi. Uppsala, Sweden (April–June 1842)
41. Malmstén, C.J.: De integralibus quibusdam definitis seriebusque infinitis (Eng. trans.: "On some definite integrals and series"). J. Reine Angew. Math., **38**, 1–39 (1849) [work dated May 1, 1846]
42. Markushevich, A.I.: Theory of Functions of a Complex Variable, 2nd edn. AMS Chelsea/American Mathematical Society, New York/Providence (2005) [in 3 volumes]
43. McLachlan, N.W.: Complex Variable and Operational Calculus with Technical Applications. Cambridge University Press, London (1942)
44. Medina, L.A., Moll, V.H.: A class of logarithmic integrals. Ramanujan J. **20**(1), 91–126 (2009)
45. Miller, J., Adamchik, V.S.: Derivatives of the Hurwitz Zeta function for rational arguments. J. Comput. Appl. Math. **100**, 201–206 (1998)
46. Moll, V.H.: The integrals in Gradshteyn and Ryzhik. Part 1: A family of logarithmic integrals. Scentia, Ser. A, Math. Sci. **14**, 1–6 (2007)
47. Moll, V.H., Amdeberhan, T.: Contemporary Mathematics: Tapas in Experimental Mathematics. American Mathematical Society, Providence (2007)
48. Nan-You, Z., Williams, K.S.: Some results on the generalized Stieltjes constant. Analysis **14**, 147–162 (1994)
49. Nielson, N.: Handbuch der Theorie der Gammafunktion. Teubner, Leipzig (1906)
50. Ostrowski, A.M.: On some generalisations of the Cauchy–Frullani integral. Proc. Natl. Acad. Sci. USA **35**, 612–616 (1949)
51. Poisson, S.D.: Second mémoire sur la distribution de l'électricité à la surface des corps conducteurs (lu le 6 septembre 1813). Mémoires de la classe des sciences mathématiques et physiques de l'Institut Impérial de France, partie II Chez Fermin Didot, Imprimeur de l'Institut Impérial de France et Libraire pour les mathématiques, pp. 163–274, rue Jacob, no. 24, Paris, France (1814)
52. Poisson, S.D.: Suite du mémoire sur les intégrales définies, inséré dans les deux précédents volumes de ce journal. J. Éc. Polytech. **XI**(18), 295–341 (1820)
53. Prudnikov, A.P., Brychkov, Y.A., Marichev, O.I.: Integrals and Series, vols. I–IV. Gordon and Breach, New York (1992)
54. Riemann, B.: Ueber die Anzahl der Primzahlen unter einer gegebenen Grosse. Monatsberichte der Königlich Preußischen Akademie der Wissenschaften zu Berlin, pp. 136–144 (1859)
55. Schlömilch, O.: Uebungsaufgaben für Schüler. Lehrsatz. Grunert Arch. Math. Phys. **XII**(IV), 415 (1849), part XXXV
56. Schlömilch, O.: Ueber eine Eigenschaft gewisser Reihen. Z. Math. Phys. **III**, 130–132 (1858)
57. Sloane, N.J.A.: Sequence A115252. The On-Line Encyclopedia of Integer Sequences (2006). http://oeis.org/A115252
58. Smirnov, V.I.: A Course of Higher Mathematics. vol. I–V. Pergamon, London (1964)
59. Spiegel, M.R.: Theory and Problems of Complex Variables with an Introduction to Conformal Mapping and Its Application. McGraw-Hill, New York (1968)
60. Sveshnikov, A.G., Tikhonov, A.N.: Theory of Functions of a Complex Variable. Nauka, Moscow (1967). [in Russian]
61. Bierens de Haan, D.: Tables d'intégrales définies. Verhandelingen der Koninklijke Nederlandse Akademie van Wetenschappen, Deel IV, Amsterdam (1858)
62. Bierens de Haan, D.: Nouvelles tables d'intégrales définies. P. Engels, Libraire Éditeur, Leide (1867)
63. Gårding, L.: Mathematics and Mathematicians: Mathematics in Sweden Before 1950. American Mathematical Society/London Mathematical Society, Providence (1994)
64. Le Gendre, A.M.: Exercices de calcul intégral sur divers ordres de transcendantes et sur les quadratures Tomes I–III. M$^{\text{me}}$ V$^{\text{e}}$ Courcier, Imprimeur–Libraire pour les Mathématiques, rue du Jardinet, no. 12, quartier Saint-André-des-Arc, Paris (1811–1817)
65. Titchmarsh, E.C.: The Theory of Functions, 2nd edn. Oxford University Press, Oxford (1939)
66. Titchmarsh, E.C.: The Theory of the Riemann Zeta-Function, 2nd edn. Clarendon, Oxford (1986)
67. Vardi, I.: Integrals, an introduction to analytic number theory. Am. Math. Mon. **95**, 308–315 (1988)
68. Vilceanu, R.O.: An application of Dirichlet L-series to the computation of certain integrals. Bull. Math. Soc. Sci. Math. Roum. **51(99)**(2), 159–173 (2008)

69. Volkovyskii, L.I., Lunts, G.L., Aramanovich, I.G.: A Collection of Problems on Complex Analysis. Pergamon, London (1965)
70. Weisstein, E.W.: Vardi's integral. From MathWorld—a Wolfram web resource (2006). http://mathworld.wolfram.com/VardisIntegral.html
71. Whittaker, E., Watson, G.N.: A Course of Modern Analysis. an Introduction to the General Theory of Infinite Processes and of Analytic Functions, with an Account of the Principal Transcendental Functions, 3rd edn. Cambridge University Press, Cambridge (1920)

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ERRATUM

Erratum and Addendum to: Rediscovery of Malmsten's integrals, their evaluation by contour integration methods and some related results [Ramanujan J. (2014), 35:21–110]

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Addendum to Section 2.2

The historical analysis of functional Eqs. (20)–(22) on pp. 35–37 is far from exhaustive. In order to give a larger vision of this subject, several complimentary remarks may be needed.

First, on p. 37, lines 1–5, the text "By the way, the above reflection formula (21) for $L(s)$ was also obtained by Oscar Schlömilch; in 1849 he presented it as an exercise for students [55], and then, in 1858, he published the proof [56]. Yet, it should be recalled that an analog of formula (20) for the alternating…" may be replaced by the following one: "By the way, between 1849 and 1858, the above reflection formula for $L(s)$ was also obtained by several other mathematicians, including Oscar Schlömilch [55, 56], Gotthold Eisenstein [73, 84], and Thomas Clausen [75].[1] Yet, it should be noted that formula (20) itself was rigorously proved by Kinkelin a year before Riemann [79, p. 100], [78], and its analog for the alternating…"

[1] In 1849, Schlömilch presented the theorem as an exercise for students [55]. In 1858, Clausen [75] published the proof to this exercise. The same year, Schlömilch published his own proof [56]. Eisenstein did not publish the proof, but left some drafts dating back to 1849, see e.g. [73, 84].

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Second, on p. 37, in formula (22), the confusing part "$n = 1, 2, 3, \ldots$" should be replaced by "$n \in \mathbb{R}$". In fact, by comparing values of $\eta(1-n)$ to $\eta(n)$ at positive integers and by noticing that both of them contain the same Bernoulli numbers, Euler deduced Eq. (22). After that, he carried out a number of complimentary verifications, which suggested that Eq. (22) should hold not only for integer values of n, but also for fractional and continuous values of n. Whence, he conjectured that (22) should be true for any value of argument, including continuous values of n. In particular, on p. 94 of [20], Euler wrote: *"Par cette raison j'hazarderai la conjecture suivante, que quelque soit l'exposant n, cette equation a toujours lieu :*

$$\frac{1 - 2^{n-1} + 3^{n-1} - 4^{n-1} + 5^{n-1} - 6^{n-1} + \ldots}{1 - 2^{-n} + 3^{-n} - 4^{-n} + 5^{-n} - 6^{-n} + \ldots}
= \frac{-1 \cdot 2 \cdot 3 \cdots (n-1)\,(2^n - 1)}{(2^{n-1} - 1)\,\pi} \cos\frac{\pi n}{2} \text{."}$$

The latter is our Eq. (22) and is also equivalent to (20). By the way, Hardy's exposition of Euler's achievements, which we cited in footnote 15, Ref. [31, pp. 23–26], is also far from exhaustive. For instance, Hardy did not mention the fact that Euler have conjectured that formula (22) remains true for any value of n. Moreover, Hardy says that it was comparatively recently that it was observed, first by Cahen and then by Landau, that the reflection formulas for $L(s)$ and $\eta(s)$ both stand in Euler's paper written in 1749. This is, however, not true. Thus, Malmsten in 1846 remarked [40, p. 18] that (21) were obtained by Euler by induction, and Hardy cited this work of Malmsten. Unfortunately, Hardy did not notice that Malmsten also quoted Euler.

Third, on p. 37, after the last sentence in the first paragraph ending by "…requires the notion of analytic continuation.", the following footnote may be added

> "An alternative historical analysis of functional Eqs. (20)–(21) in the context of contributions of various authors may be found in [85], [31, p. 23], [84], [82, p. 4], [78, p. 193], [74, pp. 326–328], [83, p. 298], [73]. Note, however, that Butzer et al.'s statement [74, p. 328] "Malmstén included the functional equation without proof" is rather incorrect. Thus, André Weil [84, p. 8] points out that "Malmstén included the proof in a long paper written in May 1846". Moreover, our investigations show that this proof was not only included in his paper [41] written in 1846, but also was present in an earlier work [40] published in 1842. By the way, Malmsten remarked that reflection formulas of such kind were first announced by Euler in 1749, the fact which was not mentioned by Schlömilch [55, 56], nor by Clausen [75], nor by Kinkelin [79], nor by Riemann [54]."

Addendum to Section 4.1.2, Exercise no. 18

Results of this exercise also permit to evaluate some very curious integrals containing $\cos\ln\ln x$ and $\sin\ln\ln x$ in the numerator. Putting in the last unnumbered equation in

Exercise no. 18 on p. 66 $a = i\alpha$, $\alpha \in \mathbb{R}$, and $b = 1$, we have

$$\begin{aligned}
\int_0^\infty \frac{\cos(\alpha \ln x)}{\operatorname{ch} x}\, dx &= 2\int_1^\infty \frac{\cos(\alpha \ln\ln x)}{1+x^2}\, dx \\
&= -\alpha \operatorname{Im}\left[\frac{\Gamma(i\alpha)}{2^{2i\alpha}}\left\{\zeta\left(1+i\alpha, 1/4\right) - 2^{i\alpha}\left(2^{1+i\alpha}-1\right)\zeta(1+i\alpha)\right\}\right], \\
\int_0^\infty \frac{\sin(\alpha \ln x)}{\operatorname{ch} x}\, dx &= 2\int_1^\infty \frac{\sin(\alpha \ln\ln x)}{1+x^2}\, dx \\
&= \alpha \operatorname{Re}\left[\frac{\Gamma(i\alpha)}{2^{2i\alpha}}\left\{\zeta\left(1+i\alpha, 1/4\right) - 2^{i\alpha}\left(2^{1+i\alpha}-1\right)\zeta(1+i\alpha)\right\}\right],
\end{aligned}$$

These integrals are, in some sense, complimentary to basic Malmsten's integrals (1)–(2), which were evaluated in Sect. 3.4 and 4.1.2, no. 18-g, and readily permit to evaluate integrals

$$\int_0^\infty \frac{\ln^n x}{\operatorname{ch} x}\, dx = 2\int_1^\infty \frac{\ln^n \ln x}{1+x^2}\, dx = 2\int_0^1 \frac{\ln^n \ln \frac{1}{x}}{1+x^2}\, dx\,, \qquad n = 1, 2, 3, \ldots$$

in terms of Stieltjes constants (first two such expressions were given in Exercises no. 18-g and 18-h). It is also interesting that right parts of both expressions contain $\zeta(1+i\alpha)$, which was found to be connected with the nontrivial zeros of the ζ-function.[2]

Addendum to Section 4.2, Exercise no. 29

In right parts of formulas (d)–(g), it may be more preferable to have $\ln(1+\sqrt{2})$ rather than $\ln(2 \pm \sqrt{2})$

$$\begin{aligned}
\text{(d)}\quad \int_0^1 \frac{\ln\ln \frac{1}{x}}{1+\sqrt{2}\,x+x^2}\, dx &= \int_1^\infty \frac{\ln\ln x}{1+\sqrt{2}\,x+x^2}\, dx \\
&= \frac{\pi}{4\sqrt{2}}\left\{5\ln\pi + 4\ln 2 - 2\ln(1+\sqrt{2}) - 8\ln\Gamma\left(\frac{3}{8}\right)\right\}, \\
\text{(e)}\quad \int_0^1 \frac{\ln\ln \frac{1}{x}}{1-\sqrt{2}\,x+x^2}\, dx &= \int_1^\infty \frac{\ln\ln x}{1-\sqrt{2}\,x+x^2}\, dx \\
&= \frac{\pi}{4\sqrt{2}}\left\{7\ln\pi + 6\ln 2 + 2\ln(1+\sqrt{2}) - 8\ln\Gamma\left(\frac{1}{8}\right)\right\},
\end{aligned}$$

[2] Estimation of $\left|\zeta(1+i\alpha)\right|$ was found to be connected with $\operatorname{Re}\rho$, where ρ are the zeros of $\zeta(s)$ in the critical strip $0 \leqslant \operatorname{Re} s \leqslant 1$, see e.g. [80, p. 128].

$$\text{(f)}\quad \int_0^1 \frac{x\ln\ln\frac{1}{x}}{1+\sqrt{2}\,x^2+x^4}\,\mathrm{d}x = \int_1^\infty \frac{x\ln\ln x}{1+\sqrt{2}\,x^2+x^4}\,\mathrm{d}x$$
$$= \frac{\pi}{8\sqrt{2}}\left\{5\ln\pi + 3\ln 2 - 2\ln(1+\sqrt{2}) - 8\ln\Gamma\left(\frac{3}{8}\right)\right\},$$

$$\text{(g)}\quad \int_0^1 \frac{x\ln\ln\frac{1}{x}}{1-\sqrt{2}\,x^2+x^4}\,\mathrm{d}x = \int_1^\infty \frac{x\ln\ln x}{1-\sqrt{2}\,x^2+x^4}\,\mathrm{d}x$$
$$= \frac{\pi}{8\sqrt{2}}\left\{7\ln\pi + 3\ln 2 + 2\ln(1+\sqrt{2}) - 8\ln\Gamma\left(\frac{1}{8}\right)\right\}.$$

Addendum to Section 4.5, Exercise no. 62-b

On p. 96, in Exercise no. 62-b, in the unnumbered formula after Eq. (56), in the first line the last term

$$-\frac{1}{2}\sum_{l=1}^{n-1}\alpha_{l,n}\varsigma_{l,n}$$

may be removed. Strictly speaking, the actual expression for $\sum \Upsilon_{k,n}\ln\Gamma\left(\frac{k}{n}\right)$ is correct. However, because of the symmetry, the function $\varsigma_{l,n}$ identically vanishes for any integer $l = 1, 2, \ldots, n-1$, and hence, so does the last term in the first line of this formula.

Some minor corrections and additions

- p. 42, line 20: "has no branch points." should read "has no branch points except at poles of $\Gamma(z)$."
- p. 42, line 27: "points at all, which allows" should read "points at all in the right half–plane, which allows".
- p. 66, in *Nota Bene* of exercise no. 19: "derived by Malmsten in [41, unnumbered" should read "derived by Malmsten in [40, p. 24, Eq. (37)], [41, unnumbered".
- p. 68, first line: "no. 21-e" should read "no. 21-d".
- p. 73, last line, "$|\mathrm{Re} r| < 2\pi$" should read "$|\mathrm{Im} r| < 2\pi$".
- p. 82, line 7, "no. 39-c is given" should read "no. 39-e is given".
- p. 83, exercise no. 40: formula given in exercise no. 40-b, as well as formula (55), were also obtained by Nørlund in [81, p. 107].
- p. 97, footnote 40 "formula (c) was" should read "formula (b.2) was".[3]

[3] It may also noted that in a later work [72, pp. 542–543], we showed that (b.1), which is a shifted version of (b.2), was already known to Malmsten in 1846.

- p. 100, exercise no. 64: closed-form expressions equivalent to those we gave for $\gamma_1(1/2)$, $\gamma_1(1/4)$, $\gamma_1(3/4)$ and $\gamma_1(1/3)$ were also obtained by Connon in [76, pp. 1, 50, 53, 54–55], [77, pp. 17–18].

References

72. Blagouchine, Ia.V.: A theorem for the closed–form evaluation of the first generalized Stieltjes constant at rational arguments and some related summations, J. Number Theory **148**, 537–592 and **151**, 276–277, arXiv:1401.3724 (2015)
73. Bombieri, E.: The classical theory of zeta and L-functions. Milan J. Math. **78**(1), 11–59 (2010)
74. Butzer, P.M., Jansche, S.: A direct approach to the Mellin transform. J. Fourier Anal. Appl. **3**(4), 325–376 (1997)
75. Clausen, T.: Beweis des von Schlömilch Archiv Bd. XII. No. XXXV. aufgestellten Lehssatzes; — über die Ableitung des Differentials von log Γx; und — über eine allgemeine Aufgabe über die Funktionen von Abel, Grunert Archiv der Mathematik und Physik, vol. XXX, pp. 166–170 (1858)
76. Connon, D.F.: Some series and integrals involving the Riemann zeta function, binomial coefficients and the harmonic numbers. vol. II(b), arXiv:0710.4024 (2007)
77. Connon, D.F.: The difference between two Stieltjes constants, arXiv:0906.0277 (2009)
78. Dutka, J.: On the summation of some divergent series of Euler and the zeta functions. Arch. Hist. Exact Sci. **50**(2), 187–200 (1996)
79. Kinkelin, H.: Ueber einige unendliche Reihen. Mitteilungen der Naturforschenden Gesellschaft in Bern, no. **419–420**, 89–104 (1858)
80. Linnik, Y.V.: On Hilbert’s eighth problem. In: Alexandrov, P.S. (ed.) Hilbert’s Problems (in Russian), pp. 128–130. Nauka, Moscow (1969)
81. Nörlund, N.E.: Vorlesungen über Differenzenrechnung. Springer, Berlin (1924)
82. Patterson, S.J.: An Introduction to the Theory of the Riemann Zeta-function. Cambridge Studies in Advanced Mathematics, vol. 14. Cambridge University Press, Cambridge (1988)
83. Srinivasan, G.K.: The gamma function: an eclectic tour. Am. Math. Mon. **114**(4), 297–315 (2007)
84. Weil, A.: Prehistory of the zeta-function. In: Aubert, K., Bombieri, E., Goldfeld, D. (eds.) Number Theory, Trace Formulas and Discrete Groups, pp. 1–9. Academic Press, London (1989)
85. Wieleitner, H.: Geschichte der Mathematik (in 2 vols.), Berlin (1922–1923)