Journal of Number Theory 148 (2015) 537–592

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

A theorem for the closed-form evaluation of the first generalized Stieltjes constant at rational arguments and some related summations

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ARTICLE INFO

Article history:
Received 14 January 2014
Received in revised form 31 July 2014
Accepted 19 August 2014
Available online 7 October 2014
Communicated by David Goss

Keywords:
Stieltjes constants
Generalized Euler's constants
Special constants
Number theory
Zeta function
Gamma function
Digamma function
Psi function
Malmsten
Rational arguments
Logarithmic integrals
Logarithmic series
Complex analysis
Orthogonal expansions

ABSTRACT

Recently, it was conjectured that the first generalized Stieltjes constant at rational argument may be always expressed by means of Euler's constant, the first Stieltjes constant, the Γ-function at rational argument(s) and some relatively simple, perhaps even elementary, function. This conjecture was based on the evaluation of $\gamma_1(1/2)$, $\gamma_1(1/3)$, $\gamma_1(2/3)$, $\gamma_1(1/4)$, $\gamma_1(3/4)$, $\gamma_1(1/6)$, $\gamma_1(5/6)$, which could be expressed in this way. This article completes this previous study and provides an elegant theorem which allows to evaluate the first generalized Stieltjes constant at any rational argument. Several related summation formulae involving the first generalized Stieltjes constant and the Digamma function are also presented. In passing, an interesting integral representation for the logarithm of the Γ-function at rational argument is also obtained. Finally, it is shown that similar theorems may be derived for higher Stieltjes constants as well; in particular, for the second Stieltjes constant the theorem is provided in an explicit form.

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http://dx.doi.org/10.1016/j.jnt.2014.08.009

1. Introduction and notations

1.1. Introduction

The ζ-functions are one of more important special functions in modern analysis and theory of functions. The most known and frequently encountered ζ-functions are Riemann and Hurwitz ζ-functions. They are classically introduced as the following series

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}, \qquad \zeta(s,v)=\sum_{n=0}^{\infty}\frac{1}{(n+v)^s}, \quad v\neq 0,-1,-2,\ldots$$

convergent for $\operatorname{Re} s>1$, and may be extended to other domains of s by the principle of analytic continuation. It is well known that $\zeta(s)$ and $\zeta(s,v)$ are meromorphic on the entire complex s-plane and that their only pole is a simple pole at $s=1$ with residue 1. They can be, therefore, expanded in the Laurent series in a neighborhood of $s=1$ in the following way

$$\zeta(s)=\frac{1}{s-1}+\sum_{n=0}^{\infty}\frac{(-1)^n(s-1)^n}{n!}\gamma_n, \quad s\neq 1, \tag{1}$$

and

$$\zeta(s,v)=\frac{1}{s-1}+\sum_{n=0}^{\infty}\frac{(-1)^n(s-1)^n}{n!}\gamma_n(v), \quad s\neq 1, \tag{2}$$

respectively. Coefficients γ_n appearing in the regular part of expansion (1) are called *Stieltjes constants* or *generalized Euler's constants*, while those appearing in the regular part of (2), $\gamma_n(v)$, are called *generalized Stieltjes constants*. It is obvious that $\gamma_n(1)=\gamma_n$ since $\zeta(s,1)=\zeta(s)$.

The study of these coefficients is an interesting subject and may be traced back to the works of Thomas Stieltjes and Charles Hermite [25, vol. I, letter 71 and following]. In 1885, first Stieltjes and then Hermite, proved that

$$\gamma_n=\lim_{m\to\infty}\left\{\sum_{k=1}^{m}\frac{\ln^n k}{k}-\frac{\ln^{n+1} m}{n+1}\right\}, \quad n=0,1,2,\ldots \tag{3}$$

Later, this formula was also obtained or simply stated in works of Johan Jensen [47, 50], Jørgen Gram [37], Godfrey Hardy [42], Srinivasa Ramanujan [19] and many others. From (3), it is visible that γ_0 is Euler's constant γ. However, the study of other Stieltjes constants revealed to be more difficult and, at the same time, interesting. In 1895 Franel [33], by using contour integration techniques, showed that[1]

[1] There was a priority dispute between Jensen, Kluyver and Franel related to this formula [33,50]. In fact, it can be straightforwardly deduced from the first integral formula for the ζ-function (88) which was first

$$\gamma_n = \frac{1}{2}\delta_{n,0} + \frac{1}{i}\int\limits_0^{\infty} \frac{dx}{e^{2\pi x}-1}\left\{\frac{\ln^n(1-ix)}{1-ix} - \frac{\ln^n(1+ix)}{1+ix}\right\}, \qquad n=0,1,2,\ldots \tag{4}$$

Ninety years later this integral formula was discovered independently by Ainsworth and Howell who also provided a very detailed proof of it [4]. Following Franel's line of reasoning, one can also obtain these formulae[2]

$$\gamma_n = -\frac{\pi}{2(n+1)}\int\limits_{-\infty}^{+\infty} \frac{\ln^{n+1}\left(\frac{1}{2}\pm ix\right)}{\operatorname{ch}^2\pi x}\,dx\,, \qquad n=0,1,2,\ldots \tag{5}$$

and

$$\begin{aligned}
\gamma_1 &= -\left[\gamma - \frac{\ln 2}{2}\right]\ln 2 + i\int\limits_0^{\infty}\frac{dx}{e^{\pi x}+1}\left\{\frac{\ln(1-ix)}{1-ix} - \frac{\ln(1+ix)}{1+ix}\right\}\\
\gamma_2 &= -\left[2\gamma_1 + \gamma\ln 2 - \frac{\ln^2 2}{3}\right]\ln 2 + i\int\limits_0^{\infty}\frac{dx}{e^{\pi x}+1}\left\{\frac{\ln^2(1-ix)}{1-ix} - \frac{\ln^2(1+ix)}{1+ix}\right\}\\
\gamma_3 &= -\left[3\gamma_2 + 3\gamma_1\ln 2 + \gamma\ln^2 2 - \frac{\ln^3 2}{4}\right]\ln 2 + i\int\limits_0^{\infty}\frac{dx}{e^{\pi x}+1}\left\{\frac{\ln^3(1-ix)}{1-ix} - \frac{\ln^3(1+ix)}{1+ix}\right\}\\
&\ldots
\end{aligned} \tag{6}$$

first of which is particularly simple.[3] Other important results concerning the Stieltjes constants lie in the field of rational expressions of natural numbers, as well as in the closely related field of integer parts of functions. In 1790 Lorenzo Mascheroni [69, p. 23], by using some previous findings of Gregorio Fontana, showed that[4]

$$\gamma = \sum_{k=1}^{\infty}\frac{|a_k|}{k}, \quad \text{where} \quad \frac{z}{\ln(1+z)} = 1 + \sum_{k=1}^{\infty} a_k z^k, \ |z|<1 \tag{7}$$

i.e. a_k are coefficients in the Maclaurin expansion of $z/\ln(1+z)$ and are usually referred to as *(reciprocal) logarithmic numbers* or *Gregory's coefficients* (in particular $a_1 = \frac{1}{2}$,

obtained by Jensen in 1893 [49]. By the way, we corrected the original Franel's formula which was not valid for $n=0$ [this correction comes from (13) and (14) here after].

[2] The proof is analogous to that given for the formula (13) here after, except that the Hermite representation should be replaced by the third and second Jensen's formulae for $\zeta(s)$ (88) respectively.

[3] Despite the surprising simplicity of these formulae, we have not found them in the literature devoted to Stieltjes constants. In contrast, formula (5) seems to be known; at least its variant for the generalized Stieltjes constant may be found in [17].

[4] The series itself was given by Fontana, who, however, failed to find a constant to which it converges (he only proved that it should be lesser than 1). Mascheroni identified this *Fontana's constant* and showed that it equals Euler's constant [69, pp. 21–23]. Taking into account that both Fontana and Mascheroni did practically the equal work, we propose to call (7) *Fontana–Mascheroni's series* for Euler's constant.

$a_2 = -\frac{1}{12}$, $a_3 = \frac{1}{24}$, $a_4 = -\frac{19}{720}$, $a_5 = \frac{3}{160}$, $a_6 = -\frac{863}{60\,480}$, ...).[5] Fontana–Mascheroni's series (7) seems to be the first known series representation for Euler's constant containing rational coefficients only and was subsequently rediscovered several times, in particular by Kluyver in 1924 [52], by Kenter in 1999 [51] and by Kowalenko in 2008 [55] (this list is far from exhaustive, see e.g. [56]). In 1897 Niels Nielsen [73, Eq. (6)] showed that

$$\gamma = 1 - \sum_{k=1}^{\infty} \sum_{l=2^{k-1}}^{2^k-1} \frac{k}{(2l+1)(2l+2)} \tag{8}$$

This formula was also the subject of several rediscoveries, e.g. by Addison in 1967 [3] and by Gerst in 1969 [35].[6] In 1906 Ernst Jacobsthal [48, Eq. (9)] and in 1910 Giovanni Vacca [84], apparently independently, derived a closely related series

$$\gamma = \sum_{k=2}^{\infty} \frac{(-1)^k}{k} \lfloor \log_2 k \rfloor \tag{9}$$

which was also rediscovered in numerous occasions, in particular by H.F. Sandham in 1949 [76], by D.F. Barrow, M.S. Klamkin and N. Miller in 1951 [6] or by Gerst in 1969 [35].[7] In 1910 James Glaisher [36] proposed yet another proof of the same result and derived a number of other series with rational terms for γ. In 1912 Hardy [42] extended (9) to the first Stieltjes constant

$$\gamma_1 = \frac{\ln 2}{2} \sum_{k=2}^{\infty} \frac{(-1)^k}{k} \lfloor \log_2 k \rfloor \cdot \bigl(2 \log_2 k - \lfloor \log_2 2k \rfloor\bigr) \tag{10}$$

However, this expression is not a full generalization of (9) since it also contains irrational coefficients. In 1924 Jan Kluyver [52] generalized Jacobsthal–Vacca's series (9) in the another direction and showed that

$$\gamma = \sum_{k=m}^{\infty} \frac{\beta_k}{k} \lfloor \log_m k \rfloor, \qquad \beta_k = \begin{cases} m-1, & k = \text{multiple of } m \\ -1, & k \neq \text{multiple of } m \end{cases} \tag{11}$$

[5] These coefficients have a venerable history and were named after James Gregory who gave first six of them in November 1670 in a letter to John Collins [83, vol. 1, p. 46] (although in the fifth coefficient there is an error or misprint: $\frac{3}{164}$ should be replaced by $\frac{3}{160}$). Coefficients a_k are also closely related to the *Cauchy numbers* of the first kind $C_{1,k}$, to the *generalized Bernoulli numbers*, to the *Stirling polynomials* and to the *signed Stirling numbers* of the first kind $S_1(k,l)$. In particular, $a_k = \frac{C_{1,k}}{k!} = \frac{1}{k!}\sum \frac{S_1(k,l)}{l+1}$, where the summation extends over $l = [1,k]$, see e.g. [26], [20, pp. 293–294], [7, vol. III, p. 257], [39, p. 45, n° 370], [56], [13].

[6] The actual Addison's formula [3] is slightly different, but it straightforwardly reduces to (8) by partial fraction decomposition. In [3], we also find a misprint: the upper bound in the second sum on p. 823 should be the same as in (8). As regards Gerst's formula [35], it is exactly the same as (8).

[7] Series (9), thanks to the error of Glaisher, Hardy and Kluyver, was long-time attributed to Giovanni Vacca and is widely known as *Vacca's series*, see e.g. [36,42,52]. It was only in 1993 that Stefan Krämer found that this series was first obtained by Jacobsthal in 1906. Besides, Krämer also showed that Nielsen's series (8) and Jacobsthal–Vacca's series (9) are closely related and can be derived one from another via a simple geometrical progression $\frac{1}{2} = \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$ [56].

where m is an arbitrary chosen positive integer.[8] In 1924–1927 Kluyver [53,54] also tried to obtain series with rational coefficients for higher Stieltjes constants, but these attempts were not successful. Currently, apart from γ_0, no closed-form expressions are known for γ_n. However, there are works devoted to their estimations and to the asymptotic series representations for them [8,46,58,60,81]. Besides, there are also works devoted to the behavior of their sign [11,71]. In particular, Briggs in 1955 [11] demonstrated that there are infinitely many changes of sign for them. Finally, aspects related to the computation of Stieltjes constants were considered in [4,37,57,64].

As regards generalized Stieltjes constants, they are much less studied than the usual Stieltjes constants. In 1972 Berndt, by employing the Euler–Maclaurin summation formula and by proceeding analogously to Lammel [58], showed that $\gamma_n(v)$ can be given by an asymptotic representation of the same kind as (3)

$$\gamma_n(v) = \lim_{m\to\infty}\left\{\sum_{k=0}^{m}\frac{\ln^n(k+v)}{k+v} - \frac{\ln^{n+1}(m+v)}{n+1}\right\}, \qquad \begin{array}{l} n=0,1,2,\ldots \\ v\neq 0,-1,-2,\ldots \end{array} \tag{12}$$

see [8].[9] Similarly to Franel's method of the derivation of (4), one may also derive the following integral representation for the nth generalized Stieltjes constant

$$\gamma_n(v) = \left[\frac{1}{2v} - \frac{\ln v}{n+1}\right]\ln^n v - i\int_0^\infty \frac{dx}{e^{2\pi x}-1}\left\{\frac{\ln^n(v-ix)}{v-ix} - \frac{\ln^n(v+ix)}{v+ix}\right\} \tag{13}$$

$n=0,1,2,\ldots$, $\operatorname{Re} v > 0$.[10] This formula was rediscovered several times, for example, by Mark Coffey in 2009 [17,24]. From both latter formulae, it follows that $\gamma_0(v) = -\Psi(v)$. Consider, for instance, (13) and put $n=0$. Then, the latter equation takes the form

$$\gamma_0(v) = \frac{1}{2v} - \ln v + 2\underbrace{\int_0^\infty \frac{x\,dx}{(e^{2\pi x}-1)(v^2+x^2)}}_{-\frac{1}{4v}+\frac{1}{2}\ln v - \frac{1}{2}\Psi(v)} = -\Psi(v) \tag{14}$$

where the last integral was first calculated by Legendre.[11] The demonstration of the same result from formula (12) may be found, for example, in [72]. For rational v, the 0th Stieltjes constant may be, therefore, expressed by means of Euler's constant γ and a finite

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[8] For example, if $m=2$ then $\beta_k = 1$.

[9] We, however, note that Wilton, by using Vallée–Poussin's expansion of the Hurwitz ζ-function, provided a similar formula already in 1927 [88].

[10] This formula follows straightforwardly from the well-known Hermite representation for $\zeta(s,v)$, see e.g. [44, p. 66], [65, p. 106], [7, vol. I, p. 26, Eq. 1.10(7)]. First, recall that $2(v^2+x^2)^{-s/2}\sin[s\operatorname{arctg}(x/v)] = -i[(v-ix)^{-s} - (v+ix)^{-s}]$, and then, expand $\frac{1}{2}v^{-s} + (s-1)^{-1}v^{1-s}$ into the Laurent series about $s=1$. Performing the term-by-term comparison of the derived expansion with the Laurent series (2) yields (13).

[11] And not by Binet as stated in [7, vol. I, p. 18, Eq. 1.7.2(27)], see [61, tome II, p. 190] and [10, p. 83, n° 40, Eq. (55)].

combination of elementary functions [thanks to the Gauss' Digamma theorem (B.4)(a, b)]. However, things are much more complicated for higher generalized Stieltjes constants; currently, no closed-form expressions are known for them and little is known as to their arithmetical properties. Basic properties, such as the multiplication theorem

$$\sum_{l=0}^{n-1}\gamma_p\left(v+\frac{l}{n}\right)=(-1)^p n\left[\frac{\ln n}{p+1}-\Psi(nv)\right]\ln^p n+n\sum_{r=0}^{p-1}(-1)^r C_p^r\gamma_{p-r}(nv)\cdot\ln^r n,$$

$n=2,3,4,\ldots$, where C_p^r denotes the binomial coefficient $C_p^r=\frac{p!}{r!(p-r)!}$, and the recurrent relationship

$$\gamma_p(v+1)=\gamma_p(v)-\frac{\ln^p v}{v},\qquad \begin{array}{l}p=1,2,3,\ldots\\ v\neq 0,-1,-2,\ldots\end{array}\tag{15}$$

may be both straightforwardly derived from those for the Hurwitz ζ-function, see e.g. [10, pp. 101–102].[12] In attempt to obtain other properties, several summation relations involving single and double infinite series were quite recently obtained in [15,16]. Also, many important aspects regarding the Stieltjes constants were considered by Donal Connon [21,23,24].

Let now focus our attention on the first generalized Stieltjes constant. The most strong and pertinent results in the field of its closed-form evaluation is the formula for the difference between the first generalized Stieltjes constant at rational argument and its reflected version

$$\gamma_1\left(\frac{m}{n}\right)-\gamma_1\left(1-\frac{m}{n}\right)=2\pi\sum_{l=1}^{n-1}\sin\frac{2\pi ml}{n}\cdot\ln\Gamma\left(\frac{l}{n}\right)-\pi(\gamma+\ln 2\pi n)\operatorname{ctg}\frac{m\pi}{n}\tag{16}$$

In the literature devoted to Stieltjes constants this result is usually attributed to Almkvist and Meurman who obtained it by deriving the functional equation for $\zeta(s,v)$, Eq. (33), with respect to s at rational v, see e.g. [2], [5, p. 261, §12.9], [70, Eq. (6)]. However, it was comparatively recently that we discovered that this formula, albeit in a slightly different form, was obtained by Carl Malmsten already in 1846. On pp. 20 and 38 [67], we, *inter alia*, find the following expression

$$\sum_{l=0}^{\infty}\left\{\frac{\ln[(2l+1)n-m]}{(2l+1)n-m}-\frac{\ln[(2l+1)n+m]}{(2l+1)n+m}\right\}=$$

[12] As regards the multiplication theorem, see e.g. [23, Eq. (6.6)] or [10, p. 101]. We can also find its particular case for $v=1/n$ in [18, p. 1830, Eq. (3.28)].

$$= \begin{cases} -\dfrac{\pi(\gamma+\ln 2\pi)}{2n}\operatorname{tg}\dfrac{\pi m}{2n} - \dfrac{\pi}{n}\cdot\displaystyle\sum_{l=1}^{n-1}(-1)^{l-1}\sin\dfrac{\pi m l}{n}\cdot\ln\left\{\dfrac{\Gamma\left(\frac{n+l}{2n}\right)}{\Gamma\left(\frac{l}{2n}\right)}\right\} \\ \hfill \text{if } m+n \text{ is odd,} \\ -\dfrac{\pi(\gamma+\ln \pi)}{2n}\operatorname{tg}\dfrac{\pi m}{2n} - \dfrac{\pi}{n}\cdot\displaystyle\sum_{l=1}^{\lfloor\frac{1}{2}(n-1)\rfloor}(-1)^{l-1}\sin\dfrac{\pi m l}{n}\cdot\ln\left\{\dfrac{\Gamma\left(\frac{n-l}{n}\right)}{\Gamma\left(\frac{l}{n}\right)}\right\} \\ \hfill \text{if } m+n \text{ is even,} \end{cases} \tag{17}$$

where m and n are integers such that $m < n$.[13] It is visible that the left part of this equality contains the difference of two first-order derivatives of $\zeta(s,v)$ at $s \to 1$ and $v = \frac{1}{2} \pm \frac{m}{2n}$. Putting $2m - n$ instead of m and using the Laurent series expansion (2) yields, after some algebra, formula (16). A somewhat different way to get (16) is to directly apply the Mittag–Leffler theorem to one of Malmsten's integrals at rational points; we developed such a method in our preceding study [10, pp. 97–98, n° 63 and pp. 106–107, n° 67].

Recently, Coffey [18] derived several formulae for the linear combination of the first generalized Stieltjes constants at some rational arguments. From these expressions, one may conjecture that in some cases (author gave only two examples of such cases [18, p. 1821, Eqs. (3.33)–(3.34)]), not only the Γ-function, but also the second-order derivative of the Hurwitz ζ-function could be related, in some way, to the first generalized Stieltjes constant. However, these preliminary findings do not permit to precisely identify their roles in the general problem of the closed-form evaluation of the first Stieltjes constant at any rational argument (the problem which we come to solve here).

Very recently, it has been conjectured in [10, p. 103] that similarly to the Digamma theorem for $\gamma_0(v)$, the first generalized Stieltjes constant $\gamma_1(v)$ at rational v may be expressed by means of the Euler's constant γ, the first Stieltjes constant γ_1, the Γ-function and some "relatively simple" function. For seven rational values of v in the range $(0,1)$, namely for $\frac{1}{6}$, $\frac{1}{4}$, $\frac{1}{3}$, $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$ and $\frac{5}{6}$, we showed in [10, pp. 98–101, n° 64] that this "relatively simple" function is elementary.[14] In this manuscript, we extend these preceding researches by providing a theorem which allows to evaluate the first generalized Stieltjes constant at any rational argument in a closed-form by precisely identifying this "relatively simple" function. The latter consists of elementary functions containing the Euler's constant γ and of the reflected sum of two second-order derivatives of the Hurwitz ζ-function at zero $\zeta''(0,p)+\zeta''(0,1-p)$, parameter p being rational in the range $(0,1)$. A close study of this reflected sum reveals that it has several important integral and series represen-

[13] Unfortunately, this Malmsten's work contains a huge quantity of misprints in formulae. We already corrected many of them in our previous work [10, Sections 2.1 & 2.3]. As regards the above-referenced Malmsten's original equation (55), case $m+n$ even, note that $\Gamma(\frac{n-i}{2n})$ should be replaced by $\Gamma(\frac{n-i}{n})$. Formula (56) also has an error: $\Gamma(\frac{n+i}{n})$ should be replaced by $\Gamma(\frac{n-i}{n})$.

[14] Further to remarks we received after the publication of [10], we note that similar closed-form expressions for $\gamma(1/4)$, $\gamma(3/4)$ and $\gamma(1/3)$ were also obtained in [21, pp. 17–18].

tations, one of which is quite similar to an integral representation for the logarithm of the Γ-function at rational argument (see Section 2.5 and Appendix C). Moreover, the derived theorem represents also the finite Fourier series for the first generalized Stieltjes constant, so that classic Fourier analysis tools may be used at their full strength. With the help of the latter, we derive several summation formulae including summation with trigonometric functions, summation with square, summation with the Digamma function and summation giving the first-order moment (see Section 2.4). Obviously, the same method can be applied to other discrete functions allowing similar representations. In particular, its application to a variant of Gauss' Digamma theorem yields several beautiful summation formulae for the Digamma function which are derived in Appendix B. We also derive, in passing [in Appendix C, Eq. (C.4)], an interesting integral representation for the logarithm of the Γ-function at rational argument. Finally, in Section 3, we discuss extensions of the derived theorem to the higher Stieltjes constants and provide closed-form expressions for the second generalized Stieltjes constant at rational arguments.

1.2. Notations

Throughout the manuscript, the following abbreviated notations are used: $\gamma = 0.5772156649\ldots$ for Euler's constant, γ_n for the nth Stieltjes constant, $\gamma_n(p)$ for the nth generalized Stieltjes constant at point p, $\lfloor x\rfloor$ for the integer part of x, $\operatorname{tg} z$ for the tangent of z, $\operatorname{ctg} z$ for the cotangent of z, $\operatorname{ch} z$ for the hyperbolic cosine of z, $\operatorname{sh} z$ for the hyperbolic sine of z, $\operatorname{th} z$ for the hyperbolic tangent of z.[15] In order to avoid any confusion between compositional inverse and multiplicative inverse, inverse trigonometric and hyperbolic functions are denoted as $\arccos, \arcsin, \operatorname{arctg}, \ldots$ and not as $\cos^{-1}, \sin^{-1}, \operatorname{tg}^{-1}, \ldots$. Writings $\Gamma(z)$, $\Psi(z)$, $\zeta(s)$ and $\zeta(s,v)$ denote respectively the Γ-function, the Ψ-function (or Digamma function), the Riemann ζ-function and the Hurwitz ζ-function. When referring to the derivatives of the Hurwitz ζ-function, we always refer to the derivative with respect to its first argument s (unless otherwise specified). $\operatorname{Re} z$ and $\operatorname{Im} z$ denote, respectively, real and imaginary parts of z. Natural numbers are defined in a traditional way as a set of positive integers, which is denoted by $\mathbb{N}$. Kronecker symbol of arguments l and k is denoted by $\delta_{l,k}$. Letter i is never used as index and is $\sqrt{-1}$. The writing $\operatorname{res}_{z=a} f(z)$ stands for the residue of the function $f(z)$ at the point $z = a$. By Malmsten's integral we mean any integral of the form

$$\int_0^\infty \frac{R(\operatorname{sh} px, \operatorname{ch} px)\cdot \ln x}{R(\operatorname{sh} x, \operatorname{ch} x)}\, dx$$

[15] Most of these notations come from Latin, e.g. "ch" stands for *cosinus hyperbolicus*, "sh" stands for *sinus hyperbolicus*, etc.

where R denotes a rational function and the parameter p is such that the convergence is guaranteed. Other notations are standard.

2. Evaluation of the first generalized Stieltjes constant at rational argument

2.1. Generalized Stieltjes constants and their relationships to Malmsten's integrals

Formula (16) provides a closed-form expression for the difference of two first Stieltjes constants at rational arguments. It should be therefore interesting to investigate if there could be some expressions containing other combinations of Stieltjes constants. In our previous work [10, pp. 97–107], we already demonstrated that some Malmsten's integrals are connected with the first generalized Stieltjes constants. This connection was quite fruitful and permitted not only to prove by another method formula (16), but also to evaluate the first generalized Stieltjes constant $\gamma_1(p)$ at $p = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6}, \frac{2}{3}, \frac{3}{4}, \frac{5}{6}$ by means of elementary functions, Euler's constant γ, the first Stieltjes constant γ_1 and the Γ-function, see for more details [10, pp. 98–101, nº 64]. Taking into account that aforementioned manuscript was quite long, many results and theorems were given as exercises with hints and without rigorous proofs. Below, we provide several useful proofs and unpublished results (given as lemmas and corollaries) showing that Malmsten's integrals of the first and second orders may be expressed by means of the first generalized Stieltjes constants. This connection between Malmsten's integrals and Stieltjes constants is crucial and plays the central role in the proof of the main theorem of this manuscript.

Lemma 1. *For any* $|\operatorname{Re} p| < 1$ *and* $\operatorname{Re} a > -1$,

$$\int_0^\infty \frac{x^{a-1}(\operatorname{ch} px - 1)}{\operatorname{sh} x}\,dx = \frac{\Gamma(a)}{2^a}\left\{\zeta\left(a, \frac{1}{2} - \frac{p}{2}\right) + \zeta\left(a, \frac{1}{2} + \frac{p}{2}\right) - 2(2^a - 1)\zeta(a)\right\} \tag{18}$$

Proof. From elementary analysis it is well-known that $\operatorname{sh}^{-1} x$, for $\operatorname{Re} x > 0$, may be represented by the following geometric series

$$\frac{1}{\operatorname{sh} x} = 2\sum_{n=0}^{\infty} e^{-(2n+1)x}, \qquad \operatorname{Re} x > 0.$$

This series, being uniformly convergent, can be integrated term-by-term. Hence

$$\begin{aligned}\int_0^\infty \frac{x^{a-1}(\operatorname{ch} px - 1)}{\operatorname{sh} x}\,dx &= \sum_{n=0}^{\infty}\int_0^\infty x^{a-1}\left\{e^{-(2n+1-p)x} + e^{-(2n+1+p)x} - 2e^{-(2n+1)x}\right\}dx \\ &= \Gamma(a)\sum_{n=0}^{\infty}\left\{\frac{1}{(2n+1-p)^a} + \frac{1}{(2n+1+p)^a} - \frac{2}{(2n+1)^a}\right\} =\end{aligned}$$

$$= \frac{\Gamma(a)}{2^a}\left\{\zeta\left(a, \frac{1}{2} - \frac{p}{2}\right) + \zeta\left(a, \frac{1}{2} + \frac{p}{2}\right) - 2\zeta\left(a, \frac{1}{2}\right)\right\},$$

where the integral on the left converges if $|\operatorname{Re} p| < 1$ and $\operatorname{Re} a > -1$. In order to obtain (18), it suffices to notice that $\zeta(a, \frac{1}{2}) = (2^a - 1)\zeta(a)$. □

Corollary 1. *For any p lying in the strip $|\operatorname{Re} p| < 1$, we always have*

$$\int_0^\infty \frac{(\operatorname{ch} px - 1)\ln x}{\operatorname{sh} x}\, dx = (\gamma + \ln 2)\cdot\left\{\Psi\left(\frac{1}{2} + \frac{p}{2}\right) + \ln 2 - \frac{\pi}{2}\operatorname{tg}\frac{\pi p}{2}\right\}$$
$$+ \gamma^2 + \gamma_1 - \frac{1}{2}\gamma_1\left(\frac{1}{2} + \frac{p}{2}\right) - \frac{1}{2}\gamma_1\left(\frac{1}{2} - \frac{p}{2}\right). \qquad (19)$$

This result is straightforwardly obtained from Lemma 1 by differentiating (18) with respect to a, and then by making $a \to 1$. In order to evaluate the limit in the right-hand side, we make use of Laurent series (1) and (2).

Another Malmsten's integral of the first order which also contains Stieltjes constants appear in the next lemma.

Lemma 2. *For any $|\operatorname{Re} p| < 1$ and $\operatorname{Re} a > -1$,*

$$\int_0^\infty \frac{x^{a-1}\operatorname{sh} px}{\operatorname{ch} x}\, dx = \frac{\Gamma(a)}{2^a}\left\{\zeta\left(a, \frac{1}{2} + \frac{p}{2}\right) - \zeta\left(a, \frac{1}{2} - \frac{p}{2}\right)\right.$$
$$\left. - 2^{1-a}\zeta\left(a, \frac{1}{4} + \frac{p}{4}\right) + 2^{1-a}\zeta\left(a, \frac{1}{4} - \frac{p}{4}\right)\right\}$$

Proof. Analogous to that of Lemma 1. □

Corollary 2. *For any $|\operatorname{Re} p| < 1$,*

$$\int_0^\infty \frac{\operatorname{sh} px \cdot \ln x}{\operatorname{ch} x}\, dx = \frac{1}{2}\left\{\pi(\gamma + \ln 2)\operatorname{tg}\frac{\pi p}{2} - (\gamma + 2\ln 2)\left[\Psi\left(\frac{1}{4} + \frac{p}{4}\right) - \Psi\left(\frac{1}{4} - \frac{p}{4}\right)\right]\right.$$
$$\left. + \gamma_1\left(\frac{1}{2} - \frac{p}{2}\right) - \gamma_1\left(\frac{1}{2} + \frac{p}{2}\right) - \gamma_1\left(\frac{1}{4} - \frac{p}{4}\right) + \gamma_1\left(\frac{1}{4} + \frac{p}{4}\right)\right\}.$$

This result can be shown in the same way as that in Corollary 1.

By the same line of argument, one may also prove that following logarithmic integrals may be expressed in terms of first generalized Stieltjes constants.

$$\int_0^\infty \frac{\operatorname{sh} px \cdot \ln x}{\operatorname{sh} x}\,dx = \frac{1}{2}\left\{\gamma_1\left(\frac{1}{2}+\frac{p}{2}\right) - \gamma_1\left(\frac{1}{2}-\frac{p}{2}\right) - \pi(\gamma+\ln 2)\operatorname{tg}\frac{\pi p}{2}\right\}$$

$$\int_0^\infty \frac{\operatorname{ch} px \cdot \ln x}{\operatorname{ch} x}\,dx = \frac{1}{2}\left\{\gamma_1\left(\frac{1}{2}+\frac{p}{2}\right) + \gamma_1\left(\frac{1}{2}-\frac{p}{2}\right) - \gamma_1\left(\frac{1}{4}+\frac{p}{4}\right) - \gamma_1\left(\frac{1}{4}-\frac{p}{4}\right)\right\}$$
$$- \frac{1}{2}\ln^2 2 + \ln 2 \cdot \Psi\left(\frac{1}{2}+\frac{p}{2}\right) + \frac{\pi}{2}(\gamma+\ln 2)\operatorname{tg}\frac{\pi p}{2}$$
$$- \frac{\pi}{2}(\gamma+2\ln 2)\operatorname{ctg}\left(\frac{\pi}{4}-\frac{\pi p}{4}\right)$$

$$\int_0^\infty \frac{\operatorname{sh}^2 px \cdot \ln x}{\operatorname{sh}^2 x}\,dx = \frac{1}{2}\big\{\ln\pi - \ln\sin\pi p + p\big[\gamma_1(p) - \gamma_1(1-p)\big]$$
$$- (\gamma+\ln 2)(1-\pi p\operatorname{ctg}\pi p)\big\}$$

$$\int_0^\infty \frac{\operatorname{ch} px \cdot \ln x}{\operatorname{ch}^2 x}\,dx = \frac{p}{2}\left\{\gamma_1\left(\frac{p}{2}\right) - \gamma_1\left(1-\frac{p}{2}\right) - \gamma_1\left(\frac{p}{4}\right) + \gamma_1\left(1-\frac{p}{4}\right)\right\}$$
$$+ \ln\operatorname{tg}\frac{\pi p}{4} - \frac{\pi p}{2}\left\{(\gamma+2\ln 2)\csc\frac{\pi p}{2} + \ln 2\cdot\operatorname{ctg}\frac{\pi p}{2}\right\} \tag{20}$$

where parameter p should be such that $|\operatorname{Re} p| < 1$ in the first three integrals and $|\operatorname{Re} p| < 2$ in the fourth one. Interestingly, higher Malmsten's integrals seem to not contain higher Stieltjes constants, but rather other ζ-function related constants. For instance, the evaluation of the third-order Malmsten's integral by the same method yields:

$$\int_0^\infty \frac{\operatorname{sh}^3 px \cdot \ln x}{\operatorname{sh}^3 x}\,dx = \frac{1}{4}\left\{3\zeta'\left(-1,\frac{1}{2}+\frac{p}{2}\right) - 3\zeta'\left(-1,\frac{1}{2}-\frac{p}{2}\right) - \zeta'\left(-1,\frac{1}{2}+\frac{3p}{2}\right)\right.$$
$$\left. + \zeta'\left(-1,\frac{1}{2}-\frac{3p}{2}\right)\right\} + \frac{3(1-p^2)}{16}\left\{\gamma_1\left(\frac{1}{2}+\frac{p}{2}\right) - \gamma_1\left(\frac{1}{2}-\frac{p}{2}\right)\right\}$$
$$- \frac{1-9p^2}{16}\left\{\gamma_1\left(\frac{1}{2}+\frac{3p}{2}\right) - \gamma_1\left(\frac{1}{2}-\frac{3p}{2}\right)\right\} - \frac{3p}{4}\ln(2\cos\pi p - 1)$$
$$+ \frac{\pi(\gamma+\ln 2)}{16}\left\{3\big(p^2-1\big)\operatorname{tg}\frac{\pi p}{2} - \big(9p^2-1\big)\operatorname{tg}\frac{3\pi p}{2}\right\} \tag{21}$$

in the strip $|\operatorname{Re} p| < 1$. In contrast, the evaluation of Malmsten's integrals containing higher powers of the logarithm in the numerator of the integrand[16] leads precisely to

[16] We propose to call such integrals *generalized Malmsten's integrals*.

higher Stieltjes constants. In fact, differentiating twice (18) with respect to a, and then making $a \to 1$, yields

$$\begin{aligned}\int\limits_0^\infty \frac{(\operatorname{ch} px - 1)\ln^2 x}{\operatorname{sh} x}\,dx &= -\gamma_2 + \frac{1}{2}\left\{\gamma_2\left(\frac{1}{2}+\frac{p}{2}\right) + \gamma_2\left(\frac{1}{2}-\frac{p}{2}\right)\right\} - 2\gamma_1(\gamma - \ln 2)\\ &\quad + (\gamma + \ln 2)\cdot\left\{\gamma_1\left(\frac{1}{2}+\frac{p}{2}\right) + \gamma_1\left(\frac{1}{2}-\frac{p}{2}\right)\right\} - \gamma^3 - \frac{\gamma}{6}(\pi^2 + 6\ln^2 2)\\ &\quad - \left[(\gamma+\ln 2)^2 + \frac{\pi^2}{6}\right]\cdot\left\{\Psi\left(\frac{1}{2}+\frac{p}{2}\right) - \frac{\pi}{2}\operatorname{tg}\frac{\pi p}{2}\right\} - \frac{\ln 2}{3}(\pi^2 + 2\ln^2 2)\end{aligned} \tag{22}$$

where $|\operatorname{Re} p| < 1$. More generally, the same integral containing $\ln^n x$ instead of $\ln^2 x$ will lead to the nth Stieltjes constants.

Nota Bene. As showed in [10, pp. 51–60, Sect. 4, n$^{\text{o}}$ 3, 6, 11, 13], integrals (20)–(21) for rational $p \in (0,1)$ may be reduced to the Γ-function and its logarithmic derivatives. Besides, integral (21), for any $|\operatorname{Re} p| < 1$, may be written in terms of antiderivatives of $\ln\Gamma(z)$ instead of $\zeta'(-1,z)$. We, however, noticed that currently there is no agree about the exact definition of $\Psi_{-2}(z) \equiv \int \ln\Gamma(z)\,dz$. From the well-known identity $\zeta'(0,z) = \ln\Gamma(z) - \frac{1}{2}\ln 2\pi$ and the fact that $\partial\zeta(s,z)/\partial z = -s\zeta(s+1,z)$, it clearly follows that

$$\Psi_{-2}(z) = \zeta'(-1,z) - \frac{z^2}{2} + \frac{z}{2}(1 + \ln 2\pi) + C \tag{23}$$

where C is the constant of integration.[17] Notwithstanding, we found that Maple 12 uses a different definition

$$\Psi_{-2}(z) = \zeta'(-1,z) - \frac{z^2}{2} + \frac{z}{2} - \frac{1}{12}$$

Yet, we remarked that Wolfram Alpha Pro employs another expression, which numerically corresponds to[18]

$$\Psi_{-2}(z) = z\ln\Gamma(z) - \frac{z^2}{2}\ln z + \frac{z^2}{4} + \frac{z}{2} + \frac{\ln 2\pi}{12} - \frac{1-\gamma}{12} - \frac{\zeta'(2)}{2\pi^2} + \int\limits_0^\infty \frac{x\ln(x^2+z^2)}{e^{2\pi x} - 1}\,dx$$

These three definitions are all different, but it may be easily seen that first definition (23) differs from the last one only by a constant of integration, while Maple's definition is really different.

[17] The Hurwitz ζ-function whose first argument is a negative integer may be trivially expressed in terms of Bernoulli polynomials. In particular $\zeta(-1,z) = -\frac{1}{2}z^2 + \frac{1}{2}z - \frac{1}{12}$.

[18] Wolfram Alpha Pro does not explain how $\Psi_{-2}(z)$ is evaluated. Expression given below is derived by the author by calculating the antiderivative of Binet's integral formula for the logarithm of the Γ-function subject to the initial condition $\Psi_{-2}(0) = 0$ (for Binet's formula for $\ln\Gamma(z)$, see e.g. [9, pp. 335–336], [86, pp. 250–251], [7, vol. I, p. 22, Eq. 1.9(9)], [10, p. 83, Eq. (54)]).

2.2. Malmsten's series and Hurwitz's reflection formula

We now show that the integral from Lemma 1 may be also evaluated via a trigonometric series.

Lemma 3. *In the vertical strip* $|\operatorname{Re} a| < 1$*, the following equality holds*

$$\int_0^\infty \frac{x^{a-1}(\operatorname{ch} px - 1)}{\operatorname{sh} x}\, dx = \pi^a \sec\frac{\pi a}{2} \sum_{n=1}^\infty (-1)^n \frac{\cos p\pi n - 1}{n^{1-a}} \tag{24}$$

for $-1 < p < +1$.

Proof. The Mittag–Leffler theorem is a fundamental theorem in the theory of functions of a complex variable and allows to expand meromorphic functions into a series accordingly to its poles.[19] Application of this theorem to the meromorphic function $(\operatorname{ch} pz - 1)/\operatorname{sh} z$, $p \in (-1, +1)$, having only first-order poles at $z = \pi n i$, $n \in \mathbb{Z}$, with residue $(-1)^n(\cos \pi pn - 1)$, leads to the following expansion

$$\frac{\operatorname{ch} pz - 1}{\operatorname{sh} z} = 2z \sum_{n=1}^\infty (-1)^n \frac{\cos p\pi n - 1}{z^2 + \pi^2 n^2}, \quad z \in \mathbb{C},\ z \neq \pi n i,\ n \in \mathbb{Z},$$

which is uniformly convergent on the entire complex z-plane except discs $|z - \pi i n| < \varepsilon$, $n \in \mathbb{Z}$, where the positive parameter ε can be made as small as we please. Therefore

$$\begin{aligned} \int_0^\infty \frac{x^{a-1}(\operatorname{ch} px - 1)}{\operatorname{sh} x}\, dx &= 2\sum_{n=1}^\infty (-1)^n (\cos p\pi n - 1) \underbrace{\int_0^\infty \frac{x^a}{x^2 + \pi^2 n^2}\, dx}_{\frac{1}{2}\pi^a n^{a-1} \sec \frac{1}{2}\pi a} \\ &= \pi^a \sec\frac{\pi a}{2} \sum_{n=1}^\infty (-1)^n \frac{\cos p\pi n - 1}{n^{1-a}} \end{aligned} \tag{25}$$

which holds only for $-1 < p < +1$ and $|\operatorname{Re} a| < 1$ (the elementary integral in the middle, whose evaluation is due to Euler, is convergent only in the strip $|\operatorname{Re} a| < 1$, see e.g. [85, p. 126, nº 880], [28, p. 197, nº 856.2], [1, p. 256, nº 6.1.17], [39, p. 67, nº 587], [65, p. 51]). However, the above equality can be analytically continued for other values of a: the integral is the analytic continuation of the sum for $\operatorname{Re} a \geqslant +1$, while the sum analytically continues the integral for $\operatorname{Re} a \leqslant -1$. We obviously have to expect trouble with the right-hand part at $a = \pm1, \pm3, \pm5, \ldots$ because of the secant. Since when $a = -1, -3, -5, \ldots$ the sum in the right-hand side converges, these points are poles of the

[19] For more details, see [68], [85, pp. 147–148, nº 994–1002], [32, Chap. V, §27, nº 27.10-2], [79, Chap. VII, p. 175], [65].

first order for the analytic continuation of integral (25). In contrast, for $a = 1, 3, 5, \ldots$, the integral on the left remains bounded, and thus, these points are removable singularities for the right-hand side of (25). In other words, formally $\sum(-1)^n(\cos p\pi n - 1)n^{a-1}$, $n \geqslant 1$, must vanish identically for any odd positive a (exactly as $\eta(1-a)$, the result which has been derived by Euler, see e.g. [30, p. 85]). These matters are treated in detail in the next corollary. □

Corollary 3. *For* $0 < p < 1$

$$\begin{cases} \displaystyle\sum_{n=1}^{\infty} \frac{\cos 2\pi pn}{n^{1-a}} = \Gamma(a)(2\pi)^{-a} \cos\frac{\pi a}{2}\big\{\zeta(a,p) + \zeta(a,1-p)\big\} & \text{(a)} \\ \displaystyle\sum_{n=1}^{\infty} \frac{\sin 2\pi pn}{n^{1-a}} = \Gamma(a)(2\pi)^{-a} \sin\frac{\pi a}{2}\big\{\zeta(a,p) - \zeta(a,1-p)\big\} & \text{(b)} \end{cases} \tag{26}$$

where both series on the left-hand side are uniformly convergent in $\operatorname{Re} a < 1$ *and are absolutely convergent in the half-plane* $\operatorname{Re} a < 0$*. These important formulae seem to be obtained for the first time by Malmsten in 1846.*

Proof. In view of the fact the alternating ζ-function $\eta(s)$ may be reduced to the ordinary ζ-function and by making use of Euler–Riemann's reflection formula for the ζ-function $\zeta(1-s) = 2\zeta(s)\Gamma(s)(2\pi)^{-s}\cos\frac{1}{2}\pi s$, we may continue (25) as follows

$$\begin{aligned} \pi^a \sec\frac{\pi a}{2}\sum_{n=1}^{\infty}(-1)^n\frac{\cos p\pi n - 1}{n^{1-a}} &= \pi^a \sec\frac{\pi a}{2}\left\{\sum_{n=1}^{\infty}(-1)^n\frac{\cos p\pi n}{n^{1-a}} - \big(2^a - 1\big)\zeta(1-a)\right\} \\ &= \pi^a \sec\frac{\pi a}{2}\sum_{n=1}^{\infty}(-1)^n\frac{\cos p\pi n}{n^{1-a}} - 2\big(1 - 2^{-a}\big)\Gamma(a)\zeta(a) \end{aligned}$$

Comparing the latter expression to the result of Lemma 1 gives

$$\sum_{n=1}^{\infty}(-1)^n\frac{\cos p\pi n}{n^{1-a}} = \Gamma(a)(2\pi)^{-a}\cos\frac{\pi a}{2}\left\{\zeta\left(a, \frac{1}{2} + \frac{p}{2}\right) + \zeta\left(a, \frac{1}{2} - \frac{p}{2}\right)\right\}$$

Writing in this expression $2p-1$ instead of p yields immediately (26)(a). Now, by partially differentiating (26)(a) with respect to p and by remarking that $a\Gamma(a) = \Gamma(a+1)$, and then, by writing a instead of $a+1$, we arrive at (26)(b). Note also that both sums (26)(a, b) may be analytically continued to other domains of a by means of expressions in corresponding right parts. □

Interestingly, nowadays, formulae (26)(a, b) seem to be not particularly well-known (for instance, advanced calculators such as Wolfram Alpha Pro expresses both series in terms of polylogarithms). Notwithstanding, Eq. (26)(b) can be found in an old

Malmsten's work published as early as 1849 [67, p. 17, Eq. (48)], and (26)(a) is a straightforward consequence of (26)(b).

Corollary 4. *If we notice that*

$$\Gamma(a) = \frac{\pi}{\sin \pi a \cdot \Gamma(1-a)} = \frac{\pi}{2\sin\frac{1}{2}\pi a \cdot \cos\frac{1}{2}\pi a \cdot \Gamma(1-a)}$$

then, the sum of (26)(a) with (26)(b) leads to an important formula

$$\zeta(a,p) = \frac{2\Gamma(1-a)}{(2\pi)^{1-a}}\left[\sin\frac{\pi a}{2}\sum_{n=1}^{\infty}\frac{\cos 2\pi pn}{n^{1-a}} + \cos\frac{\pi a}{2}\sum_{n=1}^{\infty}\frac{\sin 2\pi pn}{n^{1-a}}\right], \tag{27}$$

with $0 < p \leqslant 1$ *and* $\operatorname{Re} a < 1$*, which is usually attributed to Adolf Hurwitz who derived it in 1881, see [45, p. 93],*[20] *[86, p. 269], [65, p. 107], [82, p. 37], [8, p. 156], [7, vol. I, p. 26, Eq. 1.10(6)].*[21] *Sometimes, it is written in a complex form*

$$\zeta(a,p) = \frac{i\Gamma(1-a)}{(2\pi)^{1-a}}\left[e^{-\frac{1}{2}\pi i a}\sum_{n=1}^{\infty}\frac{e^{-2\pi i p n}}{n^{1-a}} - e^{+\frac{1}{2}\pi i a}\sum_{n=1}^{\infty}\frac{e^{+2\pi i p n}}{n^{1-a}}\right],$$

$0 < p \leqslant 1$, $\operatorname{Re} a < 1$, *see e.g. [14, p. 87], which is completely equivalent to (27).*

Nota Bene. It is quite rarely emphasized that latter representations coincide with the trigonometric Fourier series for $\zeta(a,p)$. Remarking this permits to immediately derive several integral formulae, whose demonstration by other means is more difficult

[20] Hurwitz derived all his results for the function $f(s,a)$ which is related to the modern Hurwitz ζ-function as $f(s,a) \equiv f_m(s,a) = m^{-s}\zeta(s,a/m)$, see [45, p. 89]. By the way, this famous Hurwitz's paper begins with several factual errors. The reflection formula for the L-function, which he attributed to Oscar Schlömilch [45, p. 86, first two formulae for $f(s)$], was first deduced by mathematical induction by Leonhard Euler in 1749 [30, p. 105]. Then, it was rigorously proved by two different methods by Malmsten in 1842 [66] and in 1846 [67]. As regards Schlömilch's contribution, he gave the same formula only in 1849 [77], and this, without the proof (the proof [78] was published 9 years later). Similarly, Hurwitz erroneously attributed the reflection formula for the ζ-function to Bernhard Riemann, although it was first given also by Euler [30, p. 94], albeit in a slightly form, and Riemann's contribution consists mainly in the more rigorous proof of it [75]. Further information about the history of these two important formulae may be found in [87], [43, p. 23], [27, p. 861], [10, pp. 35–37].

[21] There is a slight error in this formula in the latter reference: it remains valid not only for $\operatorname{Re} a < 0$, but also for $\operatorname{Re} a < 1$.

$$\begin{cases} \displaystyle\int_0^1 \zeta(a,p)\,dp = 0 \\ \displaystyle\int_0^1 \zeta(a,p)\cos 2\pi pn\,dp = \Gamma(1-a)(2\pi n)^{a-1}\sin\frac{\pi a}{2} \qquad \operatorname{Re} a < 1 \\ \displaystyle\int_0^1 \zeta(a,p)\sin 2\pi pn\,dp = \Gamma(1-a)(2\pi n)^{a-1}\cos\frac{\pi a}{2} \end{cases} \tag{28}$$

for $n = 1, 2, 3, \ldots$. Furthermore, in virtue of Parseval's theorem, we also have

$$\int_0^1 \zeta^2(a,p)dp = 2\Gamma^2(1-a)(2\pi)^{2a-2}\zeta(2-2a), \quad \operatorname{Re} a < 1,\ a \neq \frac{1}{2} \tag{29}$$

Differentiating this formula with respect to a and then setting $a = 0$, yields:

$$2\int_0^1 \underbrace{\left(\frac{1}{2} - p\right)}_{\zeta(0,p)} \cdot \underbrace{\left(\ln\Gamma(p) - \frac{1}{2}\ln 2\pi\right)}_{\zeta'(0,p)} dp = \frac{\gamma + \ln 2\pi}{6} - \frac{\zeta'(2)}{\pi^2}$$

Whence, accounting for the well-known result[22]

$$\int_0^1 \ln\Gamma(p)\,dp = \frac{1}{2}\ln 2\pi \tag{30}$$

we obtain

$$\int_0^1 p\ln\Gamma(p)\,dp = \frac{\zeta'(2)}{2\pi^2} - \frac{\gamma - 2\ln 2\pi}{12}$$

Integration by parts of the latter expression leads to the antiderivatives of $\ln\Gamma(x)$ which are currently not well-studied yet (see the *Nota Bene* on p. 548). Similarly, differentiating twice (29) with respect to a at $a = 0$, and accounting for[21]

[22] The value of integral (30), as well as that of (31), may be both straightforwardly deduced from a similar Fourier series expansion for the logarithm of the Γ-function, see e.g. [7, vol. I, pp. 23–24, §1.9.1] or [80, p. 17, Eq. (36)]. This expansion, attributed erroneously to Ernst Kummer, was first derived by Malmsten and colleagues from the Uppsala University in 1842. This interesting historical question is discussed in details in [10, Sect. 2.2, Fig. 2 and exercise n° 20 on pp. 66–68]. By the way, the evaluation of integral (31) may be also found in several modern works, see e.g. [29, p. 177, Eq. (7.3)], [22, p. 14, Eq. (3.19)].

$$\begin{aligned}\int\limits_0^1 \ln^2\Gamma(p)\,dp &= \frac{\gamma^2}{12}+\frac{\pi^2}{48}+\frac{\gamma\ln 2\pi}{6}+\frac{\ln^2 2\pi}{3}-\frac{(\gamma+\ln 2\pi)\zeta'(2)}{\pi^2}+\frac{\zeta''(2)}{2\pi^2}\\ &= \frac{1}{6}+\frac{\pi^2}{36}+\frac{\ln^2 2\pi}{4}-2\zeta'(-1)-\zeta''(-1)\end{aligned} \tag{31}$$

yields another integral

$$\begin{aligned}\int\limits_0^1 p\zeta''(0,p)\,dp &= \frac{\pi^2}{144}-\frac{\gamma^2}{12}-\frac{\gamma\ln 2\pi}{6}-\frac{\ln^2 2\pi}{12}+\frac{(\gamma+\ln 2\pi)\zeta'(2)}{\pi^2}-\frac{\zeta''(2)}{2\pi^2}\\ &= -\frac{1}{6}+2\zeta'(-1)+\zeta''(-1)\end{aligned}$$

Some further results related to the Fourier series expansion of the Hurwitz ζ-function are provided in [29].[22]

Corollary 5. *In (27), the index n may be represented as $n = mk + l$, where for each $k = 0, 1, 2, \ldots, \infty$, the index l runs over $[1, 2, \ldots, m]$ and where m is some positive integer. Then, (27) may be written in the form:*

$$\zeta(a,p) = \frac{2\Gamma(1-a)}{(2\pi)^{1-a}}\left[\sin\frac{\pi a}{2}\sum_{l=1}^{m}\sum_{k=0}^{\infty}\frac{\cos 2\pi p(mk+l)}{(mk+l)^{1-a}}+\cos\frac{\pi a}{2}\sum_{l=1}^{m}\sum_{k=0}^{\infty}\frac{\sin 2\pi p(mk+l)}{(mk+l)^{1-a}}\right]$$

Now, let p be a rational part of m, i.e. $p = r/m$, where r and m are positive integers such that $r \leqslant m$. Then $\cos[2\pi p(mk + l)] = \cos(2\pi rl/m)$, and similarly for the sine. Hence, for positive rational p not greater than 1, the previous formula takes the form

$$\begin{aligned}\zeta\left(a,\frac{r}{m}\right) &= \frac{2\Gamma(1-a)}{(2\pi)^{1-a}}\left[\sin\frac{\pi a}{2}\sum_{l=1}^{m}\cos\frac{2\pi rl}{m}\underbrace{\sum_{k=0}^{\infty}\frac{1}{(mk+l)^{1-a}}}_{m^{a-1}\zeta(1-a,l/m)}\right.\\ &\qquad\left.+\cos\frac{\pi a}{2}\sum_{l=1}^{m}\sin\frac{2\pi rl}{m}\sum_{k=0}^{\infty}\frac{1}{(mk+l)^{1-a}}\right]\\ &= \frac{2\Gamma(1-a)}{(2\pi m)^{1-a}}\sum_{l=1}^{m}\sin\left(\frac{2\pi rl}{m}+\frac{\pi a}{2}\right)\cdot\zeta\left(1-a,\frac{l}{m}\right), \qquad r=1,2,\ldots,m.\end{aligned} \tag{32}$$

This equality holds in the entire complex a-plane for any positive integer $m \geqslant 2$. Furthermore, by putting in the latter formula $1 - a$ instead of a, it may be rewritten as

[22] However, in many formulae domains of validity remain unspecified, and sometimes, are incorrect (e.g. compare [29, Eqs. (1.26) and (3.5)] with (28) and (29) respectively).

$$\zeta\left(1-a,\frac{r}{m}\right)=\frac{2\Gamma(a)}{(2\pi m)^a}\sum_{l=1}^{m}\cos\left(\frac{2\pi rl}{m}-\frac{\pi a}{2}\right)\cdot\zeta\left(a,\frac{l}{m}\right),\qquad r=1,2,\ldots,m. \tag{33}$$

In the case $r=m$, the above formulae reduce to Euler–Riemann's reflection formulae for the ζ-function (simply use the multiplication theorem for the Hurwitz ζ-function, see e.g. [10, p. 101]). Formulae (32) and (33) are known as functional equations for the Hurwitz ζ-function and were both obtained by Hurwitz in the same article [45, p. 93] in 1881. By the way, the above demonstration also shows that they can be elementary derived from Malmsten's results (26)(a, b) obtained as early as 1840s.

Nota Bene. Malmsten's series (26)(a, b) are actually particular cases of a more general series

$$f(s)\equiv\sum_{n=1}^{\infty}\frac{a_n}{n^s},\qquad a_n\in\mathbb{C},\ |a_n|\leqslant 1, \tag{34}$$

which is uniformly and absolutely convergent in the region $\operatorname{Re}s>1$ (it may also converge, albeit non-absolutely, in the half-plane $\operatorname{Re}s>0$).[23] Such a series is known as the *Dirichlet series*. Let now focus our attention on a particular case of this series in which coefficients a_n are m-periodic, i.e. $a_n=a_{n+m}=a_{n+2m}=\ldots$ (period m being natural).[24] The first important consequence of such a particular case is that $f(s)$ may be reduced to a linear combination of Hurwitz ζ-functions at rational argument. Representing again the summation's index $n=mk+l$ yields

$$f(s)=\sum_{l=1}^{m}\sum_{k=0}^{\infty}\frac{a_{mk+l}}{(mk+l)^s}=\sum_{l=1}^{m}a_l\sum_{k=0}^{\infty}\frac{1}{(mk+l)^s}=\frac{1}{m^s}\sum_{l=1}^{m}a_l\zeta\left(s,\frac{l}{m}\right) \tag{35}$$

The right-hand side continues $f(s)$ to the entire complex s-plane, except possibly the point $s=1$.[25] In order to identify the character of the point $s=1$, we evaluate the corresponding residue

$$\operatorname*{res}_{s=1}f(s)=\lim_{s\to1}\big[(s-1)f(s)\big]=\frac{1}{m}\sum_{l=1}^{m}a_l\lim_{s\to1}\left[(s-1)\zeta\left(s,\frac{l}{m}\right)\right]=\frac{1}{m}\sum_{l=1}^{m}a_l$$

Therefore, if the coefficients $a_1,a_2,\ldots,a_m$ are chosen so that the latter sum vanishes, then $f(s)$ is holomorphic; otherwise $f(s)$ is a meromorphic function with a unique pole at $s=1$. Typical examples of cases when $f(s)$ is regular everywhere are Malmsten's series (26)(a, b) at rational p because

[23] Indeed $\sum|a_n n^{-s}|\leqslant\sum|n^{-s}|=\zeta(\operatorname{Re}s)$, the latter being uniformly and absolutely convergent in $\operatorname{Re}s>1$.
[24] If a_n is a character, the above series may be, in turn, an example of the Dirichlet L-function.
[25] This is the unique point where the Hurwitz ζ-function is not regular.

$$\sum_{l=1}^{m} a_l = \sum_{l=1}^{m} \sin\frac{2\pi rl}{m} = 0 \quad \text{and} \quad \sum_{l=1}^{m} \cos\frac{2\pi rl}{m} = 0\,, \qquad r = 1, 2, \ldots, m-1.$$

As to the reflection formula for the Dirichlet series $f(s)$, it may be easily deduced with the help of (33). Writing in (35) $1-s$ for s, and then, using (33), yields

$$\begin{aligned} f(1-s) &= \frac{1}{m^{1-s}} \sum_{l=1}^{m} a_l \zeta\left(1-s, \frac{l}{m}\right) = \frac{2\Gamma(s)}{m(2\pi)^s} \sum_{l=1}^{m} a_l \sum_{k=1}^{m} \cos\left(\frac{2\pi lk}{m} - \frac{\pi s}{2}\right) \cdot \zeta\left(s, \frac{k}{m}\right) \\ &= \frac{2\Gamma(s)}{m(2\pi)^s} \left[\sin\frac{\pi s}{2} \sum_{k=1}^{m} \alpha_k \zeta\left(s, \frac{k}{m}\right) + \cos\frac{\pi s}{2} \sum_{k=1}^{m} \beta_k \zeta\left(s, \frac{k}{m}\right)\right] \end{aligned} \tag{36}$$

where

$$\alpha_k = \sum_{l=1}^{m} a_l \sin\frac{2\pi lk}{m} \quad \text{and} \quad \beta_k = \sum_{l=1}^{m} a_l \cos\frac{2\pi lk}{m}$$

holding in the entire complex s-plane except at points $s = 1, 0, -1, -2, \ldots$. This formula is also very useful in that the expression on the right represents the analytic continuation for $f(1-s)$ to the domains where the series (34) does not converge. Finally, remark that the latter formula may be also written in a complex form

$$f(1-s) = \frac{\Gamma(s)}{m(2\pi)^s} \left[e^{+\frac{1}{2}\pi i s} \sum_{k=1}^{m} \widetilde{\alpha}_k \zeta\left(s, \frac{k}{m}\right) + e^{-\frac{1}{2}\pi i s} \sum_{k=1}^{m} \widetilde{\beta}_k \zeta\left(s, \frac{k}{m}\right)\right]$$

$s \neq 1, 0, -1, -2, \ldots$, where

$$\widetilde{\alpha}_k = \sum_{l=1}^{m} a_l e^{-2\pi i lk/m} \quad \text{and} \quad \widetilde{\beta}_k = \sum_{l=1}^{m} a_l e^{+2\pi i lk/m}$$

and some authors precisely prefer this form, see e.g. [14, pp. 88–91]. This form is more appropriated if one wishes to emphasize the Fourier series aspect (coefficients $\widetilde{\alpha}_k$ and $\widetilde{\beta}_k$ may be regarded as m-points Fourier transforms of coefficients a_l).

2.3. Closed-form evaluation of the first generalized Stieltjes constant at rational argument

We now state the main result of this manuscript allowing to evaluate in a closed-form the first generalized Stieltjes constant at any rational argument.

Theorem 1. *The first generalized Stieltjes constant of any rational argument in the range* $(0, 1)$ *may be expressed in a closed form via a finite combination of logarithms of the*

Γ-function, of second-order derivatives of the Hurwitz ζ-function at zero, of Euler's constant γ, of the first Stieltjes constant γ_1 and of elementary functions:

$$\begin{aligned}\gamma_1\left(\frac{r}{m}\right) = {} & \gamma_1 - \gamma\ln 2m - \frac{\pi}{2}(\gamma+\ln 2\pi m)\operatorname{ctg}\frac{\pi r}{m} - \ln^2 2 - \ln 2\cdot\ln\pi m \\ & - \frac{1}{2}\ln^2 m - \frac{(-1)^r}{4}\big[1-(-1)^{m+1}\big]\cdot(3\ln 2 + 2\ln\pi)\ln 2 \\ & - \pi\ln\pi\cdot\csc\frac{\pi r}{m}\cdot\sin\left(\frac{\pi r}{m}\left\lfloor\frac{m+1}{2}\right\rfloor\right)\cdot\sin\left(\frac{\pi r}{m}\left\lfloor\frac{m-1}{2}\right\rfloor\right) \\ & + 2(\gamma+\ln 2\pi m)\cdot\sum_{l=1}^{\lfloor\frac{1}{2}(m-1)\rfloor}\cos\frac{2\pi r l}{m}\cdot\ln\sin\frac{\pi l}{m} \\ & + \pi\sum_{l=1}^{\lfloor\frac{1}{2}(m-1)\rfloor}\sin\frac{2\pi r l}{m}\cdot\ln\sin\frac{\pi l}{m} + 2\pi\sum_{l=1}^{\lfloor\frac{1}{2}(m-1)\rfloor}\sin\frac{2\pi r l}{m}\cdot\ln\Gamma\left(\frac{l}{m}\right) \\ & + \sum_{l=1}^{\lfloor\frac{1}{2}(m-1)\rfloor}\cos\frac{2\pi r l}{m}\cdot\left\{\zeta''\left(0,\frac{l}{m}\right)+\zeta''\left(0,1-\frac{l}{m}\right)\right\}\end{aligned} \tag{37}$$

This elegant formula holds for any $r = 1, 2, 3, \ldots, m-1$, where m is positive integer greater than 1. The Stieltjes constants for other "periods" may be obtained from the recurrent relationship:

$$\gamma_1(v+1) = \gamma_1(v) - \frac{\ln v}{v}, \qquad v \neq 0, \tag{38}$$

see, e.g. [10, p. 102, Eq. (64)]. The above theorem is an equivalent of Gauss' Digamma theorem for the 0*th Stieltjes constant $\gamma_0(r/m) = -\Psi(r/m)$. Three alternative forms of the same theorem are given in Eqs. (50), (53) and (55) respectively.*

Proof. Consider the integral (18). Put $2p-1$ instead of p and denote the resulting integral via $J_a(p)$:

$$\begin{aligned}J_a(p) &\equiv \int_0^\infty \frac{x^{a-1}(\operatorname{ch}[(2p-1)x]-1)}{\operatorname{sh}x}\,dx \\ &= \frac{\Gamma(a)}{2^a}\big\{\zeta(a,p)+\zeta(a,1-p)-2\big(2^a-1\big)\zeta(a)\big\}\end{aligned} \tag{39}$$

converging in the strip $0 < \operatorname{Re} p < 1$. Let now p be rational $p = r/m$, where r and m are positive integers such that $r < m$. Then, the preceding equation becomes

$$J_a\left(\frac{r}{m}\right) = \frac{\Gamma(a)}{2^a}\left\{\zeta\left(a,\frac{r}{m}\right)+\zeta\left(a,1-\frac{r}{m}\right)-2\big(2^a-1\big)\zeta(a)\right\} \tag{40}$$

The sum of first two terms in curly brackets may be evaluated via Hurwitz's reflection formula (32):

$$\zeta\left(a,\frac{r}{m}\right)+\zeta\left(a,1-\frac{r}{m}\right)=$$
$$=\frac{2\Gamma(1-a)}{(2\pi m)^{1-a}}\sum_{l=1}^{m}\left[\sin\left(\frac{2\pi rl}{m}+\frac{\pi a}{2}\right)+\sin\left(\frac{2\pi(m-r)l}{m}+\frac{\pi a}{2}\right)\right]\zeta\left(1-a,\frac{l}{m}\right)$$
$$=\frac{4\Gamma(1-a)}{(2\pi m)^{1-a}}\sin\frac{\pi a}{2}\cdot\sum_{l=1}^{m}\cos\frac{2\pi rl}{m}\cdot\zeta\left(1-a,\frac{l}{m}\right)$$

Thus, by noticing that $\Gamma(a)\Gamma(1-a)=\frac{1}{2}\pi\csc\frac{1}{2}\pi a\cdot\sec\frac{1}{2}\pi a$, the integral $J_a(r/m)$ takes the form:

$$J_a\left(\frac{r}{m}\right)=\underbrace{\frac{\pi}{(\pi m)^{1-a}}\sec\frac{\pi a}{2}}_{f_1}\cdot\underbrace{\sum_{l=1}^{m}\cos\frac{2\pi rl}{m}\cdot\zeta\left(1-a,\frac{l}{m}\right)}_{f_2}-\underbrace{\frac{\Gamma(a)(2^a-1)\zeta(a)}{2^{a-1}}}_{f_3} \tag{41}$$

which is third expression for the integral J_a, other two expressions being given by (18) and (24). Let now study each term of the right part, denoted for brevity f_1, f_2 and f_3 respectively, in a neighborhood of $a=1$. The first and the third terms have poles of the first order at this point, while the second term f_2 is analytic at $a=1$. Thus, in a neighborhood of $a=1$, terms f_1 and f_3 may be expanded in the Laurent series as follows

$$f_1=-\frac{2}{a-1}-2\ln\pi m-\left(\frac{\pi^2}{12}+\ln^2\pi m\right)\cdot(a-1)+O(a-1)^2 \tag{42}$$

and

$$f_3=\frac{1}{a-1}+\ln 2+\left(\frac{\pi^2}{12}-\frac{\ln^2 2}{2}-\frac{\gamma^2}{2}-\gamma_1\right)\cdot(a-1)+O(a-1)^2 \tag{43}$$

while f_2 may be represented by the following Taylor series

$$f_2=\sum_{l=1}^{m}\cos\frac{2\pi rl}{m}\cdot\underbrace{\zeta\left(0,\frac{l}{m}\right)}_{\frac{1}{2}-l/m}-(a-1)\sum_{l=1}^{m}\cos\frac{2\pi rl}{m}\cdot\underbrace{\zeta'\left(0,\frac{l}{m}\right)}_{\ln\Gamma(l/m)-\frac{1}{2}\ln 2\pi}+$$
$$+\frac{(a-1)^2}{2}\sum_{l=1}^{m}\cos\frac{2\pi rl}{m}\cdot\zeta''\left(0,\frac{l}{m}\right)+O(a-1)^3 \tag{44}$$
$$=-\frac{1}{2}-(a-1)\sum_{l=1}^{m}\cos\frac{2\pi rl}{m}\cdot\ln\Gamma\left(\frac{l}{m}\right)+\frac{(a-1)^2}{2}\sum_{l=1}^{m}\cos\frac{2\pi rl}{m}\cdot\zeta''\left(0,\frac{l}{m}\right)+O(a-1)^3$$

because

$$\begin{cases} \displaystyle\sum_{l=1}^{m} \cos\frac{2\pi rl}{m} = 0 & r = 1, 2, 3, \ldots, m-1 \\ \displaystyle\sum_{l=1}^{m} l\cdot\cos\frac{2\pi rl}{m} = \frac{m}{2}, & r = 1, 2, 3, \ldots, m-1 \end{cases} \tag{45}$$

In the final analysis, the substitution of (42), (43) and (44) into (41), yields the following representation for the integral $J_a(r/m)$ in a neighborhood of $a = 1$:

$$\begin{aligned} J_a\left(\frac{r}{m}\right) = \ln\frac{\pi m}{2} + 2A_m(r) + (a-1)\cdot\biggl[-B_m(r) + 2A_m(r)\ln\pi m - \frac{\pi^2}{24} + \frac{\ln^2\pi m}{2} \\ + \frac{\gamma^2}{2} + \frac{\ln^2 2}{2} + \gamma_1\biggr] + O(a-1)^2 \end{aligned} \tag{46}$$

where

$$\begin{cases} A_m(r) \equiv \displaystyle\sum_{l=1}^{m} \cos\frac{2\pi rl}{m}\cdot\ln\Gamma\left(\frac{l}{m}\right) \\ B_m(r) \equiv \displaystyle\sum_{l=1}^{m} \cos\frac{2\pi rl}{m}\cdot\zeta''\left(0, \frac{l}{m}\right) \end{cases}$$

Now, if we look at the integral $J_a(r/m)$ defined in (39), we see that it is uniformly convergent and regular near $a = 1$, and hence, may be expanded in the following Taylor series

$$J_a(r/m) = J_1(r/m) + (a-1)\left.\frac{\partial J_a(r/m)}{\partial a}\right|_{a=1} + O(a-1)^2 \tag{47}$$

Equating right-hand sides of (46) and (47), and then, searching for terms with same powers of $(a-1)$, gives

$$\begin{cases} \displaystyle\int_0^\infty \frac{\operatorname{ch}[(2p-1)x]-1}{\operatorname{sh}x}\,dx = \ln\frac{\pi m}{2} + 2A_m(r) \\ \displaystyle\int_0^\infty \frac{(\operatorname{ch}[(2p-1)x]-1)\ln x}{\operatorname{sh}x}\,dx = \gamma_1 - B_m(r) + 2A_m(r)\ln\pi m - \frac{\pi^2}{24} + \frac{\ln^2\pi m}{2} \\ \qquad\qquad + \dfrac{\ln^2 2}{2} + \dfrac{\gamma^2}{2} \end{cases}$$

where $p \equiv r/m$. Remarking that the reflection formula for the logarithm of the Γ-function reduces the sum $A_m(r)$ to elementary functions[26]

$$A_m(r) \equiv \sum_{l=1}^{m} \cos\frac{2\pi rl}{m} \cdot \ln\Gamma\left(\frac{l}{m}\right) = -\frac{1}{2}\left\{\ln\pi + \sum_{l=1}^{m-1}\cos\frac{2\pi rl}{m} \cdot \ln\sin\frac{\pi l}{m}\right\}$$

yields for the first integral

$$\int_0^\infty \frac{\operatorname{ch}[(2p-1)x] - 1}{\operatorname{sh} x}\, dx = \ln\frac{m}{2} - \sum_{l=1}^{m-1}\cos\frac{2\pi rl}{m} \cdot \ln\sin\frac{\pi l}{m}, \qquad p \equiv \frac{r}{m}$$

while the second one reads

$$\int_0^\infty \frac{(\operatorname{ch}[(2p-1)x] - 1)\ln x}{\operatorname{sh} x}\, dx = \ln^2 2 + \ln 2 \cdot \ln\pi + \frac{1}{2}\ln^2 m - \sum_{l=1}^{m-1}\cos\frac{2\pi rl}{m} \cdot \zeta''\left(0, \frac{l}{m}\right)$$
$$- \ln\pi m \cdot \sum_{l=1}^{m-1}\cos\frac{2\pi rl}{m} \cdot \ln\sin\frac{\pi l}{m}, \qquad p \equiv \frac{r}{m} \tag{48}$$

where, at the final stage, we separate the last term in the sum $B_m(r)$ whose value is known $\zeta''(0,1) = \zeta''(0) = \gamma_1 + \frac{1}{2}\gamma^2 - \frac{1}{24}\pi^2 - \frac{1}{2}\ln^2 2\pi$. But the integral (48) was also evaluated in (19) by means of first generalized Stieltjes constants. Hence, the comparison of (19) to (48) yields

$$\gamma_1\left(\frac{r}{m}\right) + \gamma_1\left(1 - \frac{r}{m}\right) = 2\gamma_1 - 2\gamma\ln 2m + 2\sum_{l=1}^{m-1}\cos\frac{2\pi rl}{m} \cdot \zeta''\left(0, \frac{l}{m}\right) - 2\ln 2 \cdot \ln\pi m$$
$$+ 2(\gamma + \ln 2\pi m)\sum_{l=1}^{m-1}\cos\frac{2\pi rl}{m} \cdot \ln\sin\frac{\pi l}{m} - 2\ln^2 2 - \ln^2 m \tag{49}$$

for each $r = 1, 2, \ldots, m-1$. Adding this to Malmsten's reflection formula for the first generalized Stieltjes constant (16) finally gives

$$\gamma_1\left(\frac{r}{m}\right) = \gamma_1 - \gamma\ln 2m - \frac{\pi}{2}(\gamma + \ln 2\pi m)\operatorname{ctg}\frac{\pi r}{m} + \sum_{l=1}^{m-1}\cos\frac{2\pi rl}{m} \cdot \zeta''\left(0, \frac{l}{m}\right)$$
$$+ \pi\sum_{l=1}^{m-1}\sin\frac{2\pi rl}{m} \cdot \ln\Gamma\left(\frac{l}{m}\right) + (\gamma + \ln 2\pi m)\sum_{l=1}^{m-1}\cos\frac{2\pi rl}{m} \cdot \ln\sin\frac{\pi l}{m}$$
$$- \ln^2 2 - \ln 2 \cdot \ln\pi m - \frac{1}{2}\ln^2 m \tag{50}$$

[26] By using Malmsten's representation for the Digamma function, see (B.4)(c), the sum $A_m(r)$ may be also written in terms of the Ψ-function and Euler's constant γ.

This is the most simple form of the theorem which we are stating here and can be used as is. It can be also written in several other forms. For instance, one may notice that sums over $l \in [1, m-1]$ may be further simplified. Since each pair of terms which occupy symmetrical positions relatively to the center (except for $l = m/2$ when m is even) may be grouped together, the first sum may be reduced to

$$\sum_{l=1}^{m-1} \cos\frac{2\pi rl}{m}\cdot\zeta''\left(0,\frac{l}{m}\right) = \frac{1}{2}\sum_{l=1}^{m-1} \cos\frac{2\pi rl}{m}\cdot\left\{\zeta''\left(0,\frac{l}{m}\right)+\zeta''\left(0,1-\frac{l}{m}\right)\right\} =$$

$$= \begin{cases} \displaystyle\sum_{l=1}^{\frac{1}{2}(m-1)} \cos\frac{2\pi rl}{m}\cdot\left\{\zeta''\left(0,\frac{l}{m}\right)+\zeta''\left(0,1-\frac{l}{m}\right)\right\}, & \text{if } m \text{ is odd} \\ \displaystyle\sum_{l=1}^{\frac{1}{2}m-1} \cos\frac{2\pi rl}{m}\cdot\left\{\zeta''\left(0,\frac{l}{m}\right)+\zeta''\left(0,1-\frac{l}{m}\right)\right\} + (-1)^r\zeta''\left(0,\frac{1}{2}\right), & \text{if } m \text{ is even} \end{cases}$$

$$= \sum_{l=1}^{\lfloor\frac{1}{2}(m-1)\rfloor} \cos\frac{2\pi rl}{m}\cdot\left\{\zeta''\left(0,\frac{l}{m}\right)+\zeta''\left(0,1-\frac{l}{m}\right)\right\} - \\ - \frac{(-1)^r}{4}\left[1-(-1)^{m+1}\right]\cdot(3\ln 2+2\ln\pi)\ln 2 \qquad (51)$$

because $\zeta''\left(0,\frac{1}{2}\right) = -\frac{3}{2}\ln^2 2 - \ln\pi\ln 2$, see e.g. [10, p. 72, n° 24]. Analogously, the second sum may be written as

$$\sum_{l=1}^{m-1} \sin\frac{2\pi rl}{m}\cdot\ln\Gamma\left(\frac{l}{m}\right) = \sum_{l=1}^{\lfloor\frac{1}{2}(m-1)\rfloor} \sin\frac{2\pi rl}{m}\cdot\underbrace{\left\{\ln\Gamma\left(\frac{l}{m}\right)-\ln\Gamma\left(1-\frac{l}{m}\right)\right\}}_{2\ln\Gamma(l/m)+\ln\sin(\pi l/m)-\ln\pi}$$

$$= 2\sum_{l=1}^{\lfloor\frac{1}{2}(m-1)\rfloor} \sin\frac{2\pi rl}{m}\cdot\ln\Gamma\left(\frac{l}{m}\right) + \sum_{l=1}^{\lfloor\frac{1}{2}(m-1)\rfloor} \sin\frac{2\pi rl}{m}\cdot\ln\sin\frac{\pi l}{m} \qquad (52)$$

$$- \ln\pi\cdot\csc\frac{\pi r}{m}\cdot\sin\left(\frac{\pi r}{m}\left\lfloor\frac{m+1}{2}\right\rfloor\right)\cdot\sin\left(\frac{\pi r}{m}\left\lfloor\frac{m-1}{2}\right\rfloor\right)$$

because for natural n

$$\sum_{l=1}^{n} \sin(lx) = \csc\frac{x}{2}\cdot\sin\frac{nx}{2}\cdot\sin\left[\frac{x}{2}(n+1)\right]$$

see e.g. [39, n° 58, p. 12]. In like manner

$$\sum_{l=1}^{m-1} \cos\frac{2\pi rl}{m}\cdot\ln\sin\frac{\pi l}{m} = 2\sum_{l=1}^{\lfloor\frac{1}{2}(m-1)\rfloor} \cos\frac{2\pi rl}{m}\cdot\ln\sin\frac{\pi l}{m}$$

By using the last three identities, Eq. (50) reduces to (37).

The theorem may be also written by means of the Digamma function. In fact, by recalling that Gauss' Digamma theorem (B.4)(b) provides a connection between the last sum in (50) and the Ψ-function, formula (50) may be also written in the following form:

$$\gamma_1\left(\frac{r}{m}\right) = \gamma_1 + \gamma^2 + \gamma\ln 2\pi m + \ln 2\pi\cdot\ln m + \frac{1}{2}\ln^2 m + (\gamma+\ln 2\pi m)\cdot\Psi\left(\frac{r}{m}\right)$$
$$+\,\pi\sum_{l=1}^{m-1}\sin\frac{2\pi rl}{m}\cdot\ln\Gamma\left(\frac{l}{m}\right) + \sum_{l=1}^{m-1}\cos\frac{2\pi rl}{m}\cdot\zeta''\left(0,\frac{l}{m}\right) \tag{53}$$

In some cases, it may be more advantageous to have the complete finite Fourier series form. For this aim, it suffices to take again (50) and to recall that

$$\sum_{l=1}^{m-1} l\cdot\sin\frac{2\pi rl}{m} = -\frac{m}{2}\operatorname{ctg}\frac{\pi r}{m},\qquad r=1,2,\ldots,m-1. \tag{54}$$

This yields the following expression

$$\gamma_1\left(\frac{r}{m}\right) = \gamma_1 - \gamma\ln 2m - \ln^2 2 - \ln 2\cdot\ln\pi m - \frac{1}{2}\ln^2 m$$
$$+\,\pi\sum_{l=1}^{m-1}\sin\frac{2\pi rl}{m}\cdot\left\{\ln\Gamma\left(\frac{l}{m}\right) + \frac{l(\gamma+\ln 2\pi m)}{m}\right\}$$
$$+\sum_{l=1}^{m-1}\cos\frac{2\pi rl}{m}\cdot\left\{\zeta''\left(0,\frac{l}{m}\right) + (\gamma+\ln 2\pi m)\ln\sin\frac{\pi l}{m}\right\} \tag{55}$$

where $r = 1,2,3,\ldots,m-1$, and m is positive integer greater than 1. □

Formulae (37) (50), (53), (55) and (38) permit to readily obtain closed-form expressions for $\gamma_1(v)$ at any rational v. We, however, remark in passing that in some cases, these expressions may be further simplified so that the resulting formulae may not contain at all $\zeta''(0,l/m)+\zeta''(0,1-l/m)$, or contain only one combination (or transcendent) of them. More detailed information related to these two special cases are provided in Appendix A.

2.4. Summation formulae with the first generalized Stieltjes constant at rational argument

The derived theorem is very useful for many purposes, and in particular, for the derivation of summation formulae involving the first generalized Stieltjes constant at rational argument.

Theorem 2. *For the first generalized Stieltjes constant at rational argument take place following summation formulae*

$$\begin{cases} \displaystyle\sum_{r=1}^{m-1} \gamma_1\left(\frac{r}{m}\right) \cdot \cos\frac{2\pi rk}{m} = -\gamma_1 + m(\gamma + \ln 2\pi m)\ln\left(2\sin\frac{k\pi}{m}\right) \\ \displaystyle\qquad\qquad + \frac{m}{2}\left\{\zeta''\left(0,\frac{k}{m}\right) + \zeta''\left(0, 1-\frac{k}{m}\right)\right\} & \text{(a)} \\ \displaystyle\sum_{r=1}^{m-1} \gamma_1\left(\frac{r}{m}\right) \cdot \sin\frac{2\pi rk}{m} = \frac{\pi}{2}(\gamma + \ln 2\pi m)(2k-m) - \frac{\pi m}{2}\left\{\ln\pi - \ln\sin\frac{k\pi}{m}\right\} \\ \displaystyle\qquad\qquad + m\pi\ln\Gamma\left(\frac{k}{m}\right) & \text{(b)} \end{cases} \tag{56}$$

for $k = 1, 2, 3, \ldots, m-1$, where m is natural greater than 1.[27]

Proof. Formula (55) represents the finite Fourier series of the type (B.1). Comparing (55) to (B.1), we immediately identify

$$\begin{cases} a_m(0) = \gamma_1 - \gamma\ln 2m - \ln^2 2 - \ln 2\cdot\ln\pi m - \dfrac{1}{2}\ln^2 m, \\ a_m(l) = \zeta''\left(0, \dfrac{l}{m}\right) + (\gamma + \ln 2\pi m)\ln\sin\dfrac{\pi l}{m}, \quad l = 1, 2, 3, \ldots, m-1 \\ b_m(l) = \pi\left\{\ln\Gamma\left(\dfrac{l}{m}\right) + \dfrac{l(\gamma + \ln 2\pi m)}{m}\right\}, \quad l = 1, 2, 3, \ldots, m-1 \end{cases} \tag{57}$$

Thus, in virtue of (B.2), for any $k = 1, 2, 3, \ldots, m-1$,

$$\begin{aligned} \sum_{r=1}^{m-1} \gamma_1\left(\frac{r}{m}\right) \cdot \cos\frac{2\pi rk}{m} &= -\gamma_1 + \gamma\ln 2m + \ln^2 2 + \ln 2\cdot\ln\pi m + \frac{1}{2}\ln^2 m \\ &\quad -\underbrace{\sum_{l=1}^{m-1}\zeta''\left(0,\frac{l}{m}\right)}_{-\frac{1}{2}\ln^2 m - \ln m\cdot\ln 2\pi} + \frac{m(\gamma+\ln 2\pi m)}{2}\left[\ln\sin\frac{\pi k}{m} + \ln\sin\frac{\pi(m-k)}{m}\right] \\ &\quad -(\gamma + \ln 2\pi m)\underbrace{\sum_{l=1}^{m-1}\ln\sin\frac{\pi l}{m}}_{(1-m)\ln 2 + \ln m} + \frac{m}{2}\left\{\zeta''\left(0,\frac{k}{m}\right) + \zeta''\left(0, 1-\frac{k}{m}\right)\right\} \\ &= -\gamma_1 + m(\gamma + \ln 2\pi m)\cdot\ln\left(2\sin\frac{k\pi}{m}\right) + \frac{m}{2}\left\{\zeta''\left(0,\frac{k}{m}\right) + \zeta''\left(0, 1-\frac{k}{m}\right)\right\} \end{aligned}$$

[27] One of these formulae also appears in an unpublished work sent to the author by Donal Conon.

where we respectively used the multiplication theorem for the Hurwitz ζ-function

$$\sum_{l=1}^{m-1} \zeta''\left(0, \frac{l}{m}\right) = \frac{d^2}{ds^2}\left[(n^s - 1)\zeta(s)\right]\bigg|_{s=0} = -\frac{1}{2}\ln^2 m - \ln m \cdot \ln 2\pi \tag{58}$$

see e.g. [10, p. 101], and the well-known formula from elementary mathematical analysis

$$\prod_{l=1}^{m-1} \sin\frac{\pi l}{m} = \frac{m}{2^{m-1}} \tag{59}$$

which is, by the way, due to Euler [31, tomus I, art. 240, p. 204], [62, tome II, art. 99, p. 445] or [74, vol. I, p. 752, n° 6.1.2-2]. Analogously, by (B.3), we deduce

$$\begin{aligned}\sum_{r=1}^{m-1} \gamma_1\left(\frac{r}{m}\right) \cdot \sin\frac{2\pi rk}{m} &= \frac{\pi m}{2}\left\{\ln\Gamma\left(\frac{k}{m}\right) - \ln\Gamma\left(1 - \frac{k}{m}\right) + \frac{\gamma + \ln 2\pi m}{m}\big[k - (m-k)\big]\right\} \\ &= \frac{\pi}{2}(\gamma + \ln 2\pi m)(2k - m) - \frac{\pi m}{2}\left\{\ln\pi - \ln\sin\frac{\pi k}{m}\right\} + m\pi\ln\Gamma\left(\frac{k}{m}\right)\end{aligned}$$

which holds for $k = 1, 2, 3, \ldots, m-1$. $\square$

Theorem 3. *Parseval's theorem for the first generalized Stieltjes constant at rational argument has the following form*

$$\begin{aligned}\sum_{r=1}^{m-1} \gamma_1^2\left(\frac{r}{m}\right) &= (m-1)\gamma_1^2 - m\gamma_1(2\gamma + \ln m)\ln m + \frac{m}{4}\sum_{l=1}^{m-1}\left\{\zeta''\left(0, \frac{l}{m}\right) + \zeta''\left(0, 1 - \frac{l}{m}\right)\right\}^2 \\ &+ m(\gamma + \ln 2\pi m)\sum_{l=1}^{m-1}\left\{\zeta''\left(0, \frac{l}{m}\right) + \zeta''\left(0, 1 - \frac{l}{m}\right)\right\} \cdot \ln\sin\frac{\pi l}{m} + m\pi^2\sum_{l=1}^{m-1}\ln^2\Gamma\left(\frac{l}{m}\right) \\ &+ 2\pi^2(\gamma + \ln 2\pi m)\sum_{l=1}^{m-1} l \cdot \ln\Gamma\left(\frac{l}{m}\right) + \frac{m}{4}\big[4(\gamma + \ln 2\pi m)^2 - \pi^2\big]\sum_{l=1}^{m-1}\ln^2\sin\frac{\pi l}{m} + C_m\end{aligned} \tag{60}$$

where, for convenience in writing, by C_m we designated an elementary function depending on m and containing Euler's constant γ

$$\begin{aligned}C_m \equiv\; &-m(m-1)\ln^4 2 - m(m-1)(2\ln m + 2\gamma + 3\ln\pi)\ln^3 2 - m(m-2)\ln^2 m \cdot \ln^2 2 \\ &+ \frac{1}{4}m\ln^4 m + m\ln^3 m \cdot \ln 2 - 2m\big[2(m-1)\ln\pi + \gamma(m-2)\big]\ln m \cdot \ln^2 2 \\ &- m(m-1)\left[3\ln^2\pi + 4\gamma\ln\pi + \gamma^2 + \frac{5}{12}\pi^2 + \frac{1}{6m}\pi^2\right]\ln^2 2 -\end{aligned}$$

$$
\begin{aligned}
&- m\left[\left(m-\frac{5}{2}\right)\ln\pi - 3\gamma\right]\ln m^2\cdot\ln 2\\
&+ 2m\left[(1-m)\ln^2\pi - \left(m-\frac{5}{2}\right)\gamma\ln\pi\right]\ln m\cdot\ln 2\\
&+ \frac{1}{12}\bigg[\big((6\pi^2+24\gamma^2)m+4\pi^2(1-m^2)\big)\ln m\\
&- 4(m-1)\Big\{3m\ln^3\pi + 6m\gamma\ln^2\pi + \gamma\pi^2(m+1)\\
&+ \left(\left(\frac{13}{4}\pi^2+3\gamma^2\right)m+\pi^2\right)\ln\pi\Big\}\bigg]\ln 2 + m\left(\gamma+\frac{1}{2}\ln\pi\right)\ln^3 m\\
&+ \frac{1}{12}\big\{6m\ln^2\pi + 18\gamma m\ln\pi + \pi^2 m^2 + \big(12\gamma^2+3\pi^2\big)m + 2\pi^2\big\}\ln^2 m\\
&+ \frac{1}{12}\big[12m\gamma\ln^2\pi + \big((12\gamma^2+9\pi^2)m+4\pi^2(1-m^2)\big)\ln\pi + 2\pi^2\big(2+m^2\big)\gamma\big]\ln m\\
&- \frac{1}{12}(m-1)\big[2\pi^2(4m+1)\ln^2\pi + 4\gamma\pi^2(m+1)\ln\pi - \pi^2\gamma^2(m-2)\big]\\
&- \frac{1}{4}m\big[4(\gamma+\ln 2\pi m)^2-\pi^2\big]\cdot\big[(1-m)\ln 2+\ln m\big]\ln\pi\\
&+ m(\gamma+\ln 2\pi m)\left(\frac{1}{2}\ln m+\ln 2\pi\right)\ln\pi\cdot\ln m
\end{aligned}
$$

and where m is natural greater than 1.

Proof. Inserting Fourier series coefficients (57) into (B.4) and proceeding analogously to (B.8)–(B.9), yields, after several pages of careful calculations and simplifications, the above result. The unique formula that should be used in addition to those employed in derivations (B.8)–(B.9) is

$$
\sum_{l=1}^{m-1} l\cdot\ln\sin\frac{\pi l}{m} = \frac{m}{2}\sum_{l=1}^{m-1}\ln\sin\frac{\pi l}{m} = \frac{m[(1-m)\ln 2+\ln m]}{2} \tag{61}
$$

Also, the fact that the reflected sum $\zeta''(0,l/m)+\zeta''(0,1-l/m)$, as well as the function $\ln\sin(\pi l/m)$, are both invariant with respect to a change of summation's index $l\to m-l$ greatly helps when simplifying formula (60). □

Analogously, (55) allows us to obtain a number of other interesting summation formulae for the first generalized Stieltjes constant at rational argument. For instance, with the help of (B.10), we easily deduce these results

$$\left\{\begin{aligned}
&\sum_{r=0}^{m-1}\cos\frac{(2r+1)\pi k}{m}\cdot\gamma_1\left(\frac{2r+1}{2m}\right)=m(\gamma+\ln 4\pi m)\ln\operatorname{tg}\frac{\pi k}{2m}+ &&\text{(a)}\\
&\quad+\frac{m}{2}\left\{\zeta''\left(0,\frac{k}{2m}\right)+\zeta''\left(0,1-\frac{k}{2m}\right)\right\}-\frac{m}{2}\left\{\zeta''\left(0,\frac{1}{2}+\frac{k}{2m}\right)+\zeta''\left(0,\frac{1}{2}-\frac{k}{2m}\right)\right\}\\
&\sum_{r=0}^{m-1}\sin\frac{(2r+1)\pi k}{m}\cdot\gamma_1\left(\frac{2r+1}{2m}\right)= &&\text{(b)}\\
&\quad=m\pi\left\{\ln\Gamma\left(\frac{k}{2m}\right)+\ln\Gamma\left(\frac{1}{2}-\frac{k}{2m}\right)+\frac{1}{2}\ln\sin\frac{\pi k}{m}\right\}-\frac{\pi m}{2}(3\ln 2\pi+\ln m+\gamma)
\end{aligned}\right. \tag{62}$$

for $k=1,2,3,\ldots,m-1$, where m is natural greater than 1. By a similar line of argument, we also deduce

$$\left\{\begin{aligned}
&\sum_{r=1}^{m-1}\cos\frac{(2k+1)\pi r}{m}\cdot\gamma_1\left(\frac{r}{m}\right)=\\
&\qquad=-\pi\sum_{r=1}^{m-1}\frac{\sin\frac{2\pi r}{m}}{\cos\frac{2\pi r}{m}-\cos\frac{(2k+1)\pi}{m}}\left\{\ln\Gamma\left(\frac{r}{m}\right)+\frac{r(\gamma+\ln 2\pi m)}{m}\right\}\\
&\sum_{r=1}^{m-1}\sin\frac{(2k+1)\pi r}{m}\cdot\gamma_1\left(\frac{r}{m}\right)=\left[\gamma_1-\gamma\ln 2m-\ln^2 2-\ln 2\cdot\ln\pi m-\frac{1}{2}\ln^2 m\right]\times\\
&\quad\times\operatorname{ctg}\frac{(2k+1)\pi}{2m}+(\gamma+\ln 2\pi m)\sin\frac{(2k+1)\pi}{m}\cdot\sum_{r=1}^{m-1}\frac{1}{\cos\frac{2\pi r}{m}-\cos\frac{(2k+1)\pi}{m}}\cdot\ln\sin\frac{\pi r}{m}\\
&\quad+\frac{1}{2}\sin\frac{(2k+1)\pi}{m}\cdot\sum_{r=1}^{m-1}\frac{1}{\cos\frac{2\pi r}{m}-\cos\frac{(2k+1)\pi}{m}}\cdot\left\{\zeta''\left(0,\frac{r}{m}\right)+\zeta''\left(0,1-\frac{r}{m}\right)\right\}
\end{aligned}\right.$$

which are valid for any $k\in\mathbb{Z}$. By the way, two particular cases of (56)(a) and (62)(b) may represent some special interest. Thus putting in the former $k=m/2$ when m is even yields

$$\sum_{r=1}^{2m-1}(-1)^r\cdot\gamma_1\left(\frac{r}{2m}\right)=-\gamma_1+m(2\gamma+\ln 2+2\ln m)\ln 2 \tag{63}$$

However, the same relationship may be also derived from the multiplication theorem for the first Stieltjes constant[28]

[28] This is a particular case of the multiplication theorem for the first generalized Stieltjes constant. More general case of this theorem and equivalent theorems for higher Stieltjes constants were derived in exercise n° 64 [10, p. 101, Eqs. (62)–(63)]. Some particular cases of these theorems appear also in [18, Eqs. (3.28), (3.54)]; Eq. (3.54) contains, unfortunately, an error (see footnote 42 [10, p. 101]).

$$\sum_{r=1}^{m-1} \gamma_1\left(\frac{r}{m}\right) = (m-1)\gamma_1 - m\gamma\ln m - \frac{m}{2}\ln^2 m \tag{64}$$

Putting $2m$ instead of m, and then, treating separately odd and even terms, we have

$$\sum_{r=0}^{m-1} \gamma_1\left(\frac{2r+1}{2m}\right) = m\left\{\gamma_1 - \gamma\ln 4m - \frac{1}{2}\ln^2 m - \ln^2 2 - 2\ln 2\cdot\ln m\right\} \tag{65}$$

Subtracting from the above sum even terms $\gamma_1(2r/2m)$ for $r = 1, 2, \ldots, m-1$, immediately yields (63). In other words, (63) may be also regarded as a direct consequence of the multiplication theorem for the first Stieltjes constant. In contrast, the particular case of Eq. (62)(b) corresponding to $k = m/2$ when m is even

$$\sum_{r=0}^{2m-1} (-1)^r\cdot\gamma_1\left(\frac{2r+1}{4m}\right) = m\left\{4\pi\ln\Gamma\left(\frac{1}{4}\right) - \pi(4\ln 2 + 3\ln\pi + \ln m + \gamma)\right\} \tag{66}$$

appears to be more interesting and cannot be derived solely from (64). Moreover, we can also combine (66) with (65) by putting in the later $2m$ instead of m. Adding and subtracting them respectively yields:

$$\sum_{r=0}^{m-1} \gamma_1\left(\frac{4r+1}{4m}\right) = \frac{m}{2}\left\{2\gamma_1 - \gamma(\pi + 6\ln 2 + 2\ln m) + 4\pi\ln\Gamma\left(\frac{1}{4}\right) - 4\pi\ln 2\right.$$
$$\left. - 3\pi\ln\pi - \pi\ln m - 7\ln^2 2 - 6\ln 2\cdot\ln m - \ln^2 m\right\} \tag{67a}$$

$$\sum_{r=0}^{m-1} \gamma_1\left(\frac{4r+3}{4m}\right) = \frac{m}{2}\left\{2\gamma_1 + \gamma(\pi - 6\ln 2 - 2\ln m) - 4\pi\ln\Gamma\left(\frac{1}{4}\right) + 4\pi\ln 2\right.$$
$$\left. + 3\pi\ln\pi + \pi\ln m - 7\ln^2 2 - 6\ln 2\cdot\ln m - \ln^2 m\right\} \tag{67b}$$

From these equations it follows, *inter alia*, that sums $\gamma_1(1/8) + \gamma_1(5/8)$ and $\gamma_1(1/12) + \gamma_1(5/12)$ may be expressed in terms of $\Gamma(1/4)$, γ_1, γ and elementary functions.[29] Besides, the role of $\ln\Gamma(1/4)$ in three latter identities seems quite intriguing because the logarithm of the Γ-function possesses very similar properties

$$\sum_{r=0}^{2m-1} (-1)^r\cdot\ln\Gamma\left(\frac{2r+1}{4m}\right) = 2\ln\Gamma\left(\frac{1}{4}\right) - \frac{1}{2}(\ln 2 + 2\ln\pi - \ln m)$$

$$\sum_{r=0}^{m-1} \ln\Gamma\left(\frac{4r+1}{4m}\right) = \ln\Gamma\left(\frac{1}{4}\right) + \frac{1}{2}(m-1)\ln 2\pi + \frac{1}{4}\ln m$$

[29] For the value of $\gamma_1(3/4)$, see [10, p. 100].

$$\sum_{r=0}^{m-1} \ln\Gamma\left(\frac{4r+3}{4m}\right) = -\ln\Gamma\left(\frac{1}{4}\right) + \frac{m}{2}\ln 2\pi + \frac{1}{4}\ln\frac{\pi^2}{m}$$

Particular cases of (56)(b) corresponding to $k = m/3$ and $k = m/6$ are also interesting. Put in (56)(b) $3m$ instead of m, and then, set $k = m$. This yields:

$$\gamma_1\left(\frac{1}{3m}\right) - \gamma_1\left(\frac{2}{3m}\right) + \gamma_1\left(\frac{4}{3m}\right) - \gamma_1\left(\frac{5}{3m}\right) + \ldots + \gamma_1\left(\frac{3m-2}{3m}\right) - \gamma_1\left(\frac{3m-1}{3m}\right)$$
$$= \frac{\pi m}{\sqrt{3}}\left\{6\ln\Gamma\left(\frac{1}{3}\right) - \gamma - 4\ln 2\pi + \frac{1}{2}\ln 3 - \ln m\right\} \tag{68}$$

But the multiplication theorem (64) rewritten for $3m$ in place of m gives

$$\gamma_1\left(\frac{1}{3m}\right) + \gamma_1\left(\frac{2}{3m}\right) + \gamma_1\left(\frac{4}{3m}\right) + \gamma_1\left(\frac{5}{3m}\right) + \ldots + \gamma_1\left(\frac{3m-2}{3m}\right) + \gamma_1\left(\frac{3m-1}{3m}\right)$$
$$= 2m\gamma_1 - m\gamma(2\ln m + 3\ln 3) - \frac{m}{2}\left(3\ln^2 3 + 6\ln 3\cdot\ln m + 2\ln^2 m\right) \tag{69}$$

and hence

$$\sum_{r=0}^{m-1} \gamma_1\left(\frac{3r+1}{3m}\right) = m\left\{\gamma_1 - \gamma\left(\frac{\pi}{2\sqrt{3}} + \ln m + \frac{3}{2}\ln 3\right) + \pi\sqrt{3}\ln\Gamma\left(\frac{1}{3}\right)\right.$$
$$\left. - \frac{\pi}{2\sqrt{3}}\left(4\ln 2\pi - \frac{1}{2}\ln 3 + \ln m\right) - \frac{1}{4}\left(3\ln^2 3 + 6\ln 3\cdot\ln m + 2\ln^2 m\right)\right\}$$

Consider now the particular case of (56)(a) corresponding to $k = m/6$. Recalling that $\ln\Gamma\left(\frac{1}{6}\right) = \frac{1}{2}\ln 3 - \frac{1}{3}\ln 2 - \frac{1}{2}\ln\pi + 2\ln\Gamma\left(\frac{1}{3}\right)$, we have

$$\gamma_1\left(\frac{1}{6m}\right) + \gamma_1\left(\frac{2}{6m}\right) - \gamma_1\left(\frac{4}{6m}\right) - \gamma_1\left(\frac{5}{6m}\right) + \ldots - \gamma_1\left(\frac{3m-2}{6m}\right) - \gamma_1\left(\frac{3m-1}{6m}\right)$$
$$= \frac{2\pi m}{\sqrt{3}}\left\{12\ln\Gamma\left(\frac{1}{3}\right) - 2\gamma - 9\ln 2 + \ln 3 - 8\ln\pi - 2\ln m\right\} \tag{70}$$

By adding this to (69) rewritten for $2m$ instead of m, and then, by subtracting (70) results in another summation relation

$$\sum_{r=0}^{m-1} \gamma_1\left(\frac{6r+1}{6m}\right) = m\left\{\gamma_1 - \gamma\left(\frac{\sqrt{3}\pi}{2} + 2\ln 2 + \frac{3}{2}\ln 3 + \ln m\right) + 3\pi\sqrt{3}\ln\Gamma\left(\frac{1}{3}\right)\right.$$
$$- \frac{\pi}{2\sqrt{3}}\left(14\ln 2 - \frac{3}{2}\ln 3 + 12\ln\pi + 3\ln m\right) - \ln^2 2 - \frac{3}{4}\ln^2 3$$
$$\left. - 3\ln 2\cdot\ln 3 - 2\ln 2\cdot\ln m - \frac{3}{2}\ln 3\cdot\ln m - \frac{1}{2}\ln^2 m\right\} \tag{71}$$

Previous relationships permit to derive several summation formulae for $\gamma_1(\ldots/12m)$. Put in (67a) $3m$ instead of m and then represent the summation index r as $3l+k$, where the new summation index l runs through 0 to $m-1$ for each $k=0,1,2$. Then (67a) may be written as a sum of three terms last of which equals (67b). Hence

$$\begin{aligned}\sum_{l=0}^{m-1}\gamma_1\left(\frac{12l+1}{12m}\right)+\sum_{l=0}^{m-1}\gamma_1\left(\frac{12l+5}{12m}\right)=\frac{m}{2}\bigg\{&4\gamma_1-\gamma(4\pi+12\ln 2+6\ln 3+4\ln m)\\&+16\pi\ln\Gamma\left(\frac{1}{4}\right)-\pi(16\ln 2+12\ln\pi+3\ln 3+4\ln m)-14\ln^2 2\\&-3\ln^2 3-18\ln 2\cdot\ln 3-12\ln 2\cdot\ln m-6\ln 3\cdot\ln m-2\ln^2 m\bigg\}\end{aligned}\tag{72}$$

Similarly, by separately treating odd and even terms in (71) written for $2m$ instead of m, we have

$$\begin{aligned}\sum_{l=0}^{m-1}\gamma_1\left(\frac{12l+1}{12m}\right)+\sum_{l=0}^{m-1}\gamma_1\left(\frac{12l+7}{12m}\right)=2m\bigg\{&\gamma_1-\gamma\left(\frac{\sqrt{3}\pi}{2}+3\ln 2+\frac{3}{2}\ln 3+\ln m\right)\\&+3\pi\sqrt{3}\ln\Gamma\left(\frac{1}{3}\right)-\frac{\pi}{2\sqrt{3}}\left(17\ln 2-\frac{3}{2}\ln 3+12\ln\pi+3\ln m\right)\\&-\frac{7}{2}\ln^2 2-\frac{3}{4}\ln^2 3-\frac{9}{2}\ln 2\cdot\ln 3-3\ln 2\cdot\ln m-\frac{3}{2}\ln 3\cdot\ln m-\frac{1}{2}\ln^2 m\bigg\}\end{aligned}$$

From these relationships, it appears that the sum $\gamma_1(1/12)+\gamma_1(5/12)$ may be expressed in terms of $\Gamma(1/4)$, γ_1, γ and elementary functions, while $\gamma_1(1/12)+\gamma_1(7/12)$ contains $\Gamma(1/3)$ instead of $\Gamma(1/4)$.[30] This is certainly correlated with the fact that $\Gamma(1/12)$ may be written in terms of product $\Gamma(1/3)\cdot\Gamma(1/4)$, see e.g. [12, p. 31]. Many particular cases of equations from pp. 562–565 will also imply $\zeta''(0,p)+\zeta''(0,1-p)$ at different rational p. For instance, setting in (56)(a) $k=m/5$ and recalling that $\cos\frac{2}{5}\pi=\frac{1}{4}(\sqrt{5}-1)$, $\cos\frac{4}{5}\pi=-\frac{1}{4}(\sqrt{5}+1)$ and $\sin\frac{1}{5}\pi=\frac{1}{4}\sqrt{10-2\sqrt{5}}$, as well as using several times the multiplication theorem (64), yields

$$\begin{aligned}\sum_{l=0}^{m-1}\gamma_1\left(\frac{5l+1}{5m}\right)+\sum_{l=0}^{m-1}\gamma_1\left(\frac{5l+4}{5m}\right)=\frac{m}{2\sqrt{5}}\bigg\{&4\gamma_1\sqrt{5}+10\left[\zeta''\left(0,\frac{1}{5}\right)+\zeta''\left(0,\frac{4}{5}\right)\right]\\&-\gamma\big(4\sqrt{5}\ln m+10\ln(1+\sqrt{5})-10\ln 2+5\sqrt{5}\ln 5\big)\\&-10(\ln 2+\ln 5+\ln\pi+\ln m)\cdot\ln(1+\sqrt{5})+10\ln^2 2-\frac{10}{1+\sqrt{5}}\ln^2 5-2\sqrt{5}\ln^2 m+\end{aligned}$$

[30] At the same time, the difference $\gamma_1(1/12)-\gamma_1(7/12)$ may be written as function of $\Gamma(1/4)$ and $\zeta''(0,1/12)+\zeta''(0,11/12)$. This follows from the argument developed here later.

$$+ 15\ln 2\cdot\ln 5 + 10\ln 2\cdot\ln\pi + 5\ln 5\cdot\ln\pi + 10\ln 2\cdot\ln m - 5\sqrt{5}\ln 5\cdot\ln m\Big\}$$

Interestingly, the golden ratio ϕ seems to play a certain role in the above formula.

Let now consider the case $k = m/8$, where k should be positive integer. Eq. (56)(b), employed together with both Eqs. (67a) and (67b), provides

$$\gamma_1\left(\frac{1}{8m}\right) + \gamma_1\left(\frac{3}{8m}\right) - \gamma_1\left(\frac{5}{8m}\right) - \gamma_1\left(\frac{7}{8m}\right) + \ldots + \gamma_1\left(\frac{8m-7}{8m}\right) + \gamma_1\left(\frac{8m-5}{8m}\right)$$
$$- \gamma_1\left(\frac{8m-3}{8m}\right) - \gamma_1\left(\frac{8m-1}{8m}\right) = \pi m\sqrt{2}\Big\{8\ln\Gamma\left(\frac{1}{8}\right) - 4\ln\Gamma\left(\frac{1}{4}\right) - 2\gamma - 11\ln 2$$
$$- 4\ln\pi - 2\ln m - 2\ln(1+\sqrt{2})\Big\}$$

At the same time, Eq. (56)(a) for $k = m/8$, used together with (63), leads to

$$\gamma_1\left(\frac{1}{8m}\right) - \gamma_1\left(\frac{3}{8m}\right) - \gamma_1\left(\frac{5}{8m}\right) + \gamma_1\left(\frac{7}{8m}\right) + \ldots + \gamma_1\left(\frac{8m-7}{8m}\right) - \gamma_1\left(\frac{8m-5}{8m}\right)$$
$$- \gamma_1\left(\frac{8m-3}{8m}\right) + \gamma_1\left(\frac{8m-1}{8m}\right) = m\sqrt{2}\Big\{4\Big[\zeta''\left(0,\frac{1}{8}\right) + \zeta''\left(0,\frac{7}{8}\right)\Big]$$
$$- 4(\gamma + 4\ln 2 + \ln\pi + \ln m)\cdot\ln(1+\sqrt{2}) + 7\ln^2 2 + 2\ln 2\cdot\ln\pi\Big\}$$

Adding both equations together results in another summation relation

$$\sum_{r=0}^{m-1}\gamma_1\left(\frac{8r+1}{8m}\right) - \sum_{r=0}^{m-1}\gamma_1\left(\frac{8r+5}{8m}\right) = \frac{m}{\sqrt{2}}\Big\{4\Big[\zeta''\left(0,\frac{1}{8}\right) + \zeta''\left(0,\frac{7}{8}\right)\Big] + 8\pi\ln\Gamma\left(\frac{1}{8}\right)$$
$$- 4\pi\ln\Gamma\left(\frac{1}{4}\right) - 2\gamma\big[\pi + 2\ln(1+\sqrt{2})\big] - 2(\pi + 8\ln 2 + 2\ln\pi + 2\ln m)\cdot\ln(1+\sqrt{2})$$
$$+ 7\ln^2 2 + 2\ln 2\cdot\ln\pi - \pi(11\ln 2 + 4\ln\pi + 2\ln m)\Big\}$$

Analogous relation with "+" instead of "−" in the left part has much more simple form and follows directly from (67a) rewritten for $2m$ in place of m

$$\sum_{r=0}^{m-1}\gamma_1\left(\frac{8r+1}{8m}\right) + \sum_{r=0}^{m-1}\gamma_1\left(\frac{8r+5}{8m}\right) = m\Big\{2\gamma_1 - \gamma(\pi + 8\ln 2 + 2\ln m) + 4\pi\ln\Gamma\left(\frac{1}{4}\right)$$
$$-5\pi\ln 2 - 3\pi\ln\pi - \pi\ln m - 14\ln^2 2 - 8\ln 2\cdot\ln m - \ln^2 m\Big\}$$

Similarly, one can obtain equations for $\sum[\gamma_1(\frac{8r+3}{8m}) \pm \gamma_1(\frac{8r+7}{8m})]$.

The above summation formulae are not only interesting in themselves, but also may be useful for the closed-form determination of certain first Stieltjes constants (expressions in Appendix A are obtained precisely by means of such formulae). Besides, summation formulae akin to (65), (67a), (70), (71) may be often more easily obtained by the direct summation of (50). For the derivation of such a formula, we, first, write in (50) mn for m and $rn+k$ for r, where $n \in \mathbb{N}$ and $k = 1, 2, \ldots, n-1$. Then, we remark that for $l = 1, 2, 3, \ldots, mn-1$, we have

$$\begin{aligned}
\sum_{r=0}^{m-1} \cos\frac{2\pi l(nr+k)}{nm} &= m\cos\frac{2\pi lk}{nm} \cdot \{\delta_{l,m} + \delta_{l,2m} + \delta_{l,3m} + \ldots + \delta_{l,(n-1)m}\} \\
\sum_{r=0}^{m-1} \sin\frac{2\pi l(nr+k)}{nm} &= m\sin\frac{2\pi lk}{nm} \cdot \{\delta_{l,m} + \delta_{l,2m} + \delta_{l,3m} + \ldots + \delta_{l,(n-1)m}\} \\
\sum_{r=0}^{m-1} \operatorname{ctg}\frac{\pi(nr+k)}{nm} &= m\operatorname{ctg}\frac{\pi k}{n}
\end{aligned}$$

see e.g. [38, p. 8, n° 33], whence

$$\begin{aligned}
\sum_{r=0}^{m-1} \gamma_1\left(\frac{nr+k}{nm}\right) &= m\left(\gamma_1 - \gamma\ln 2mn - \ln^2 2 - \ln 2\cdot\ln\pi mn - \frac{1}{2}\ln^2 mn\right) \\
&\quad + m\sum_{\lambda=1}^{n-1}\cos\frac{2\pi\lambda k}{n}\cdot\zeta''\left(0,\frac{\lambda}{n}\right) + m\pi\sum_{\lambda=1}^{n-1}\sin\frac{2\pi\lambda k}{n}\cdot\ln\Gamma\left(\frac{\lambda}{n}\right) \\
&\quad - \frac{m\pi}{2}(\gamma+\ln 2\pi mn)\operatorname{ctg}\frac{\pi k}{n} + m(\gamma+\ln 2\pi mn)\sum_{\lambda=1}^{n-1}\cos\frac{2\pi\lambda k}{n}\cdot\ln\sin\frac{\pi\lambda}{n}
\end{aligned}$$

Comparing the right-hand side of this equation with the parent equation (50) finally yields

$$\begin{aligned}
&\frac{1}{m}\sum_{r=0}^{m-1}\gamma_1\left(\frac{nr+k}{nm}\right) = \qquad (73) \\
&= \gamma_1\left(\frac{k}{n}\right) - \left\{\gamma + \ln 2n + \frac{1}{2}\ln m + \frac{\pi}{2}\operatorname{ctg}\frac{\pi k}{n} - \sum_{\lambda=1}^{n-1}\cos\frac{2\pi\lambda k}{n}\cdot\ln\sin\frac{\pi\lambda}{n}\right\}\ln m
\end{aligned}$$

This relationship represents a special variant of the generalized multiplication theorem for the first generalized Stieltjes constant.[31]

Another summation formula with the first generalized Stieltjes constants may be obtained by using respectively (54), (56)(b), (B.9) and (61)

[31] This variant may be also obtained from [23, Eq. (6.6)] or [10, p. 101, Eq. (63)] by making use of Gauss' Digamma theorem (B.4).

$$\sum_{r=1}^{m-1} \operatorname{ctg}\frac{\pi r}{m}\cdot\gamma_1\left(\frac{r}{m}\right) = \frac{\pi}{3}\left\{\frac{(1-m)(m-2)\gamma}{2} + \left(m^2-1\right)\ln 2\pi - \frac{\left(m^2+2\right)\ln m}{2}\right\} \\ - 2\pi\sum_{l=1}^{m-1} l\cdot\ln\Gamma\left(\frac{l}{m}\right) \tag{74}$$

The normalized first-order moment of the first generalized Stieltjes constant may be derived from (55) by making use of (B.9), (54), (45), (59), as well as (58). This yields

$$\sum_{r=1}^{m-1} \frac{r}{m}\cdot\gamma_1\left(\frac{r}{m}\right) = \frac{1}{2}\left\{(m-1)\gamma_1 - m\gamma\ln m - \frac{m}{2}\ln^2 m\right\} \tag{75}$$

$$- \frac{\pi}{2m}(\gamma+\ln 2\pi m)\sum_{l=1}^{m-1} l\cdot\operatorname{ctg}\frac{\pi l}{m} - \frac{\pi}{2}\sum_{l=1}^{m-1}\operatorname{ctg}\frac{\pi l}{m}\cdot\ln\Gamma\left(\frac{l}{m}\right)$$

More complicated summation relations may be obtained if considering other functions. For example, the summation formula with the Digamma function reads

$$\sum_{r=1}^{m-1} \Psi\left(\frac{r}{m}\right)\cdot\gamma_1\left(\frac{r}{m}\right) = \big[\gamma(1-m) - m\ln m\big]\gamma_1 + m\gamma^2\ln m + \left\{\frac{(m-1)(m-2)\pi^2}{12}\right. \\ \left. -m(m-1)\ln^2 2 + 2m\ln 2\cdot\ln m + \frac{3m}{2}\ln^2 m\right\}\gamma - m(m-1)\ln^3 2 + \frac{m}{2}\ln^3 m \\ -\big[m(m-2)\ln m + m(m-1)\ln\pi\big]\ln^2 2 + \frac{3m}{2}\ln 2\cdot\ln^2 m + m\ln 2\cdot\ln\pi\cdot\ln m \\ -\frac{(m^2-1)\pi^2}{6}\ln 2\pi + \frac{(m^2+2)\pi^2}{12}\ln m + m(\gamma+\ln 2\pi m)\sum_{l=1}^{m-1}\ln^2\sin\frac{\pi l}{m} \\ +\frac{m}{2}\sum_{l=1}^{m-1}\left\{\zeta''\left(0,\frac{l}{m}\right) + \zeta''\left(0,1-\frac{l}{m}\right)\right\}\cdot\ln\sin\frac{\pi l}{m} + \pi^2\sum_{l=1}^{m-1} l\cdot\ln\Gamma\left(\frac{l}{m}\right) \tag{76}$$

In order to obtain this expression we start from (50) and we successively employ (B.6), (B.11), (58), (59) as well as multiplication theorems for the logarithm of the Γ-function and for the Ψ-function

$$\sum_{r=1}^{m-1}\ln\Gamma\left(\frac{r}{m}\right) = \frac{1}{2}(m-1)\ln 2\pi - \frac{1}{2}\ln m, \qquad \sum_{r=1}^{m-1}\Psi\left(\frac{r}{m}\right) = \gamma(1-m) - m\ln m \tag{77}$$

Note that, generally, when summing the first generalized Stieltjes constants with an odd function, one arrives at the logarithm of the Γ-function, while summing with an even function leads to a reflected sum of two second-order derivatives of the Hurwitz ζ-function. The latter sum is the subject of a more detailed study presented in the next section.

2.5. Several remarks related to the sum $\zeta''(0,p)+\zeta''(0,1-p)$

From the above formulae it appears that the sum of $\zeta''(0,p)$ with its reflected version $\zeta''(0,1-p)$, at positive rational p less than 1, plays the fundamental role for the first generalized Stieltjes constant at rational argument. We do not know which is the transcendence of such a sum, but it is not unreasonable to expect that it is lower than that of solely $\zeta''(0,p)$. Furthermore, in our previous work [10, pp. 66–71], we demonstrated that this sum has several comparatively simple integral and series representations; below, we briefly present some of them. In exercises n° 20–21, we dealt with integral $\Phi(\varphi)$, which we, unfortunately, could not reduce to elementary functions (despite of its simple and naive appearance). Written in terms of this integral, the above sum reads[32]

$$\begin{aligned}\zeta''(0,p)+\zeta''(0,1-p) &= \pi\operatorname{ctg}2\pi p\cdot\bigl\{2\ln\Gamma(p)+\ln\sin\pi p+(2p-1)\ln 2\pi-\ln\pi\bigr\}\\ &\quad -2\ln 2\pi\cdot\ln(2\sin\pi p)+\int_0^\infty\frac{e^{-x}\ln x}{\operatorname{ch}x-\cos 2\pi p}\,dx\end{aligned}\tag{78}$$

where parameter p should lie within the strip $0<\operatorname{Re}p<1$. By a simple change of variable, the last integral may be rewritten in a variety of other forms. For instance,

$$\begin{aligned}\int_0^\infty\frac{e^{-x}\ln x}{\operatorname{ch}x-\cos 2\pi p}\,dx &= 2\int_0^1\frac{x\ln\ln\frac{1}{x}}{x^2-2x\cos 2\pi p+1}\,dx = 2\int_1^\infty\frac{\ln\ln x}{x(x^2-2x\cos 2\pi p+1)}\,dx\\ &= \frac{\pm 2}{\sin 2\pi p}\operatorname{Im}\int_0^\infty\frac{\ln x}{e^x-e^{\pm 2\pi i p}}\,dx = \frac{\pm 2}{\sin 2\pi p}\operatorname{Im}\int_0^1\frac{x\ln\ln\frac{1}{x}}{x-e^{\pm 2\pi i p}}\,dx\\ &= \frac{\pm 2}{\sin 2\pi p}\operatorname{Im}\int_1^\infty\frac{\ln\ln x}{x(x-e^{\pm 2\pi i p})}\,dx\end{aligned}\tag{79}$$

The latter forms are particularly simple and display the close connection to the polylogarithms. Let now focus our attention on the last integral from the first line. By partial fraction decomposition it may be written in terms of three other integrals

$$\int_1^\infty\frac{\ln\ln x}{x^n(x^2-2x\cos 2\pi p+1)}\,dx,$$

$$\int_1^\infty\frac{\ln\ln x}{x^k}\,dx\quad\text{and}\quad\int_1^\infty\frac{\ln\ln x}{x^2-2x\cos 2\pi p+1}\,dx\tag{80}$$

[32] Put in [10, p. 69, Eq. 49] $\varphi=\pi(2p-1)$.

where n and k are positive integers greater than 1. The values of the last two integrals, thanks to Euler, Legendre and Malmsten, are known,[33] so that the problem of the evaluation of (78) may be reduced to the first integral. We, however, note that the success of this technique depends on the appropriate choice of p and n. Indeed, by expanding the integrand of the first integral in (80) into partial fractions, we have

$$\frac{1}{x^n(x^2-2x\cos 2\pi p+1)}=\frac{a_0}{x(x^2-2x\cos 2\pi p+1)}+\frac{a_1}{x^2-2x\cos 2\pi p+1}+\sum_{l=2}^{n}\frac{a_l}{x^l} \tag{81}$$

with coefficients a_l given by

$$a_0=\frac{\sin 2\pi pn}{\sin 2\pi p},\qquad a_1=-\frac{\sin 2\pi p(n-1)}{\sin 2\pi p},\qquad a_2=+\frac{\sin 2\pi p(n-1)}{\sin 2\pi p},\qquad \ldots,$$

$$a_l=\frac{\sin 2\pi p(n-l+1)}{\sin 2\pi p},\qquad \ldots,\qquad a_{n-1}=2\cos 2\pi p,\qquad a_n=1.$$

But if parameter p is such that $a_0=0$, the wanted integral cannot be collared. The most unpleasant is that this situation occurs precisely when $p=k/n$, where k is positive integer or demi-integer not greater than n. We, in turn, are able to evaluate

$$\int\limits_1^\infty \frac{\ln\ln x}{x^n(x^2-2x\cos 2\pi p+1)}\,dx \tag{82}$$

only for those p which may be written as k/n, in which case it can be expressed in terms of $\ln\Gamma(k/n)$ [see Appendix C]. Thus, the evaluation of the integral

$$\int\limits_0^\infty \frac{e^{-nx}\cdot\ln x}{\operatorname{ch} x-\cos\frac{2\pi k}{m}}\,dx=2\int\limits_0^1 \frac{x^n\ln\ln\frac{1}{x}}{x^2-2x\cos\frac{2\pi k}{m}+1}\,dx=2\int\limits_1^\infty \frac{\ln\ln x}{x^n\big(x^2-2x\cos\frac{2\pi k}{m}+1\big)}\,dx \tag{83}$$

with $n=2,3,4,\ldots$, number m being positive integer such that $m\neq 2kn/l$ for $l=\pm1,\pm2,\pm3,\ldots$, could bring the solution to our problem, but currently we do not know if this integral can be evaluated in terms of lower transcendental functions. However, it should be noted that integrals closely related to (83) and (C.3) were a subject of several investigations appeared already in the XIXth century. The most significant contribution seems to belong to Malmsten who showed in 1842 that

$$\frac{\sin a}{\Gamma(s)}\int\limits_0^1 \frac{x^y\cdot\ln^{s-1}\frac{1}{x}}{x^2+2x\cos a+1}\,dx=\int\limits_0^\infty \frac{\operatorname{sh} ax}{\operatorname{sh}\pi x}\cdot\frac{\cos\big(s\operatorname{arctg}\frac{x}{y}\big)}{(x^2+y^2)^{s/2}}\,dx=\sum_{l=1}^\infty(-1)^{l-1}\frac{\sin al}{(y+l)^s} \tag{84}$$

[33] See [67, p. 24], [10, Sect. 4, n° 2, 29-h, 30]) or (C.5) in Appendix C.

$y, s \in \mathbb{C}$, $-\pi < a < +\pi$, see [66, pp. 20–25] and [67, p. 12]. He studied these integrals for different values of parameters y, s and a, and evaluated some of them in a closed form. The above equality permitted to Malmsten to derive numerous fascinating results, such as, for example, formulae (17) and (26)(b). Furthermore, his investigations devoted to the cases $y = 0$, $a = \pi/2$ and $y = 0$, $a = \pi/3$ resulted in two important reflection formulae for the L- and M-functions respectively [66, p. 23, Eq. (36)], [67, pp. 17–18, Eqs. (51)–(52)], [10, pp. 35–36, Eq. (21), Fig. 3] (these formulae are similar to Euler–Riemann's reflection formula for the ζ-function, see also footnote 20). Notwithstanding, Malmsten failed to show that more generally, when a is a rational multiple of π, one has

$$\sum_{l=1}^{\infty}(-1)^{l-1}\frac{\sin al}{(y+l)^s} = \frac{1}{(2n)^s}\sum_{l=1}^{2n-1}(-1)^{l-1}\sin\frac{\pi ml}{n}\cdot\zeta\left(s,\frac{y+l}{2n}\right),\qquad a\equiv\frac{m\pi}{n} \tag{85}$$

$m = 1, 2, 3, \ldots, n-1$, which may be obtained by applying Hurwitz's method used in (27),(28),(30)–(35) to series (84).[34,35] Now, Malmsten's integrals from (84) are related to ours from (79) as follows

$$\int_0^1 \frac{x\ln\ln\frac{1}{x}}{x^2-2x\cos 2\pi p+1}\,dx = \lim_{s\to 1}\left\{\frac{\partial}{\partial s}\int_0^1\frac{x\cdot\ln^{s-1}\frac{1}{x}}{x^2-2x\cos 2\pi p+1}\,dx\right\} \tag{86}$$

Therefore, by (84) we have

$$\begin{aligned}\int_0^1 \frac{x\cdot\ln^{s-1}\frac{1}{x}}{x^2-2x\cos 2\pi p+1}\,dx &= -\frac{\Gamma(s)}{\sin 2\pi p}\int_0^{\infty}\frac{\operatorname{sh}[\pi(2p-1)x]}{\operatorname{sh}\pi x}\cdot\frac{\cos(s\operatorname{arctg}x)}{(x^2+1)^{s/2}}\,dx\\ &= -\frac{\Gamma(s)}{2\sin 2\pi p}\int_{-\infty}^{+\infty}\frac{\operatorname{sh}[\pi(2p-1)x]}{\operatorname{sh}\pi x}\cdot\frac{dx}{(1\pm ix)^s},\qquad \operatorname{Re}s>0.\end{aligned} \tag{87}$$

Integrals appearing on the right-hand side are also quite similar to Jensen's formulae for $\zeta(s)$ derived between 1893 and 1895 by contour integration methods, see [49] and [50]. Taking into account that these references are hard to find and that the same formulae were later reprinted with misprints,[36] we find it useful to reproduce them here as well

[34] Actually, Malmsten also studied the case $a = m\pi/n$, but quite superficially and mainly for $y = 0$.
[35] Note that for $s = 1, 2, 3, \ldots$ the right part of (85) reduces to polygamma functions, see e.g. [10, pp. 71–72, n° 23].
[36] In the well-known monograph [7, vol. I], in formula (13) on p. 33, "$(e^{2\pi t}+1)^{-t}$" should be replaced by "$(e^{\pi t}+1)^{-t}$".

$$\zeta(s) = \frac{1}{s-1} + \frac{1}{2} + 2\int\limits_0^{\pi/2} \frac{(\cos\theta)^{s-2}\sin s\theta}{e^{2\pi\operatorname{tg}\theta}-1}\,d\theta = \frac{1}{s-1} + \frac{1}{2} + 2\int\limits_0^{\infty} \frac{\sin(s\operatorname{arctg} x)\,dx}{(e^{2\pi x}-1)(x^2+1)^{s/2}}$$

$$\zeta(s) = \frac{2^{s-1}}{s-1} + i2^{s-1}\int\limits_0^{\infty} \frac{(1+ix)^s-(1-ix)^s}{(e^{\pi x}+1)(x^2+1)^s}\,dx = \frac{2^{s-1}}{s-1} - 2^s\int\limits_0^{\infty} \frac{\sin(s\operatorname{arctg} x)\,dx}{(e^{\pi x}+1)(x^2+1)^{s/2}}$$

$$\zeta(s) = \frac{\pi}{2(s-1)}\int\limits_{-\infty}^{+\infty} \frac{1}{\operatorname{ch}^2\pi x}\cdot\frac{dx}{(\frac{1}{2}+ix)^{s-1}} = \frac{\pi 2^{s-2}}{s-1}\int\limits_0^{\infty} \frac{\cos[(s-1)\operatorname{arctg} x]}{(x^2+1)^{(s-1)/2}\operatorname{ch}^2\frac{1}{2}\pi x}\,dx \tag{88}$$

$s\in\mathbb{C}$, $s\neq 1$, where final simplifications were done later by Lindelöf [65, p. 103] who also gave details of their derivation.[37] Application of contour integration methods to integrals (87) seems quite attractive as well (especially if p is rational), but the branch point at $\pm i$ is really annoying.

Other representations for $\zeta''(0,p)+\zeta''(0,1-p)$ may also involve integrals

$$\int\limits_0^{\infty} \frac{\ln(x^2+p^2)\cdot\operatorname{arctg}(x/p)}{e^{2\pi x}-1}\,dx \quad\text{or}\quad \int\limits_0^{\infty} \frac{\ln^2(ip+x)-\ln^2(ip-x)}{e^{2\pi x}-1}\,dx$$

which directly follow from the well-known Hermite representation for the Hurwitz ζ-function [44, p. 66], [65, p. 106], [7, vol. I, p. 26, Eq. 1.10(7)].

The sum $\zeta''(0,p)+\zeta''(0,1-p)$ may be also reduced to an important logarithmic–trigonometric series

$$\zeta''(0,p)+\zeta''(0,1-p) = -2(\gamma+\ln 2\pi)\ln(2\sin\pi p) + 2\sum_{n=1}^{\infty}\frac{\cos 2\pi pn\cdot\ln n}{n}$$

see [10, p. 69, n° 22]. This series, unlike the similar sine-series, is not known to be reducible to any elementary or classical function of analysis; however, it was remarked by Landau [59, pp. 180–182] that it has some common properties with the logarithm of the Γ-function. Besides, it also appeared in works of Lerch [63] and Gut [40].

Another way to treat the problem could be to use the antiderivatives of the first generalized Stieltjes constant $\Gamma_1(p)$. In [10, p. 69, n° 22], we showed that the sum $\zeta''(0,p)+\zeta''(0,1-p)$ may be also written in terms of such functions

$$\zeta''(0,p)+\zeta''(0,1-p) = -(3\ln 2+2\ln\pi)\ln 2 - 4\Gamma_1(1/2) + 2\Gamma_1(p) + 2\Gamma_1(1-p)$$

[37] Jensen did not provide proofs for these formulae; he only stated that he had found them in his notes,[38] and added that they can be easily derived by Cauchy's residue theorem. By the way, the first of these three formulae was also obtained by Franel [33,49,50].

[38] *Je trouve encore, dans mes notes, entre autres, les formules...* [50].

The latter formula, inserted into (50), gives an equation which is in some way analogous to Malmsten's representation for the Digamma function (B.4)(c) [in the sense that for rational arguments it provides a connection between the function and its derivative].

Finally, note that almost all above expressions remain valid everywhere in the strip $0 < \operatorname{Re} p < 1$, so it is not impossible that for rational p they could be further simplified or reduced to less transcendental forms. Thus, the question of the possibility to express any first generalized Stieltjes constant of a rational argument not only via the Γ-function, γ_1, γ and some "relatively simple" function, but solely via the Γ-function, γ_1, γ and elementary functions remains open and is directly connected to the transcendence of the reflected sum $\zeta''(0,p) + \zeta''(0,1-p)$ at rational p, which is currently not sufficiently well studied.

3. Extensions of the theorem to the second and higher Stieltjes constants

It can be reasonably expected that similar theorems could be derived for the higher Stieltjes constants. Such a demonstration could be carried out again with the help of $J_a(p)$ and integral (19) where $\ln x$ is replaced with $\ln^n x$ [see below how integral (22) is used for the determination of the second Stieltjes constant]. As regards the equation for the difference between generalized Stieltjes constants, which is also necessary, it is simply sufficient to note that from (2) and (32) it follows that

$$\gamma_n\left(\frac{r}{m}\right) - \gamma_n\left(1-\frac{r}{m}\right) = (-1)^n \lim_{a\to 1}\left\{\zeta^{(n)}\left(a,\frac{r}{m}\right) - \zeta^{(n)}\left(a,1-\frac{r}{m}\right)\right\} =$$
$$= 4(-1)^n \lim_{a\to 1}\frac{\partial^n}{\partial a^n}\left\{\frac{\Gamma(1-a)}{(2\pi m)^{1-a}}\cos\frac{\pi a}{2}\cdot\sum_{l=1}^{m-1}\sin\frac{2\pi r l}{m}\cdot\zeta\left(1-a,\frac{l}{m}\right)\right\}$$

$n = 1,2,3,\ldots$ and where r and m are positive integers such that $r < m$. In particular, for the second generalized Stieltjes constant, the latter formula takes the form[39]

$$\gamma_2\left(\frac{r}{m}\right) - \gamma_2\left(1-\frac{r}{m}\right) = 2\pi\sum_{l=1}^{m-1}\sin\frac{2\pi r l}{m}\cdot\zeta''\left(0,\frac{l}{m}\right) + \pi\left[\frac{\pi^2}{12} + (\gamma+\ln 2\pi m)^2\right]\operatorname{ctg}\frac{\pi r}{m}$$
$$- 4\pi(\gamma+\ln 2\pi m)\sum_{l=1}^{m-1}\sin\frac{2\pi r l}{m}\cdot\ln\Gamma\left(\frac{l}{m}\right) \tag{89}$$

In order to obtain a formula for $\gamma_2(r/m)$, we take again expansion (46) and write down its terms up to $O(a-1)^3$. Hence

$$\int\limits_0^\infty \frac{(\operatorname{ch}[(2p-1)x]-1)\ln^2 x}{\operatorname{sh} x}\,dx = \frac{2}{3}C_m(r) - 2B_m(r)\ln\pi m + \left\{2\ln^2\pi m + \frac{\pi^2}{6}\right\}A_m(r) -$$

[39] This formula also appears in an unpublished work sent to the author by Donal Connon.

$$
\begin{aligned}
&- 2\gamma_1(\gamma - \ln 2) + \frac{2}{3}\zeta(3) - \frac{2}{3}\gamma^3 - \gamma_2 + \left(\gamma^2 - \frac{\pi^2}{6}\right)\ln 2 \\
&+ \frac{\pi^2}{12}\ln \pi m + \ln \pi \cdot \ln m \cdot \ln \pi m + \frac{1}{3}\left(\ln^3 \pi + \ln^3 m - \ln^3 2\right)
\end{aligned}
$$

where $p \equiv r/m$ and

$$
\begin{aligned}
C_m(r) \equiv \sum_{l=1}^{m} \cos\frac{2\pi rl}{m} \cdot \zeta'''\left(0, \frac{l}{m}\right) &= \sum_{l=1}^{m-1} \cos\frac{2\pi rl}{m} \cdot \zeta'''\left(0, \frac{l}{m}\right) + \frac{3}{2}\gamma_2 + \gamma^3 - \zeta(3) \\
&+ 3\gamma_1\gamma - \frac{1}{2}\ln^3 2\pi + \left\{3\gamma_1 + \frac{3}{2}\gamma^2 - \frac{\pi^2}{8}\right\}\ln 2\pi
\end{aligned}
$$

Comparing the latter integral to (22) and then using (49), we obtain

$$
\begin{aligned}
\gamma_2\left(\frac{r}{m}\right) + \gamma_2\left(1 - \frac{r}{m}\right) &= 2\gamma_2 - 4\gamma_1 \ln m + 2\gamma^2 \ln 2 + \frac{4}{3}\sum_{l=1}^{m-1} \cos\frac{2\pi rl}{m} \cdot \zeta'''\left(0, \frac{l}{m}\right) \\
&- 4(\gamma + \ln 2\pi m)\sum_{l=1}^{m-1} \cos\frac{2\pi rl}{m} \cdot \zeta''\left(0, \frac{l}{m}\right) + 2\left[\frac{\pi^2}{12} - (\gamma + \ln 2\pi m)^2\right] \\
&\times \sum_{l=1}^{m-1} \cos\frac{2\pi rl}{m} \cdot \ln\sin\frac{\pi l}{m} - \frac{\pi^2}{6}\ln 2 + 2\gamma\left(\ln^2 m + 2\ln^2 2 + 2\ln 2 \cdot \ln \pi m\right) \\
&+ 2\left(\ln^2 2 + \ln^2 m + \ln^2 \pi + 2\ln\pi \ln m + 2\ln 2\ln \pi m\right)\ln 2 + \frac{2}{3}\ln^3 m
\end{aligned}
$$

which, being added to (89), finally yields

$$
\begin{aligned}
\gamma_2\left(\frac{r}{m}\right) = \gamma_2 &+ \frac{2}{3}\sum_{l=1}^{m-1} \cos\frac{2\pi rl}{m} \cdot \zeta'''\left(0, \frac{l}{m}\right) - 2(\gamma + \ln 2\pi m)\sum_{l=1}^{m-1} \cos\frac{2\pi rl}{m} \cdot \zeta''\left(0, \frac{l}{m}\right) \\
&+ \pi\sum_{l=1}^{m-1} \sin\frac{2\pi rl}{m} \cdot \zeta''\left(0, \frac{l}{m}\right) - 2\pi(\gamma + \ln 2\pi m)\sum_{l=1}^{m-1} \sin\frac{2\pi rl}{m} \cdot \ln\Gamma\left(\frac{l}{m}\right) \\
&+ \left[\frac{\pi^2}{12} - (\gamma + \ln 2\pi m)^2\right] \cdot \sum_{l=1}^{m-1} \cos\frac{2\pi rl}{m} \cdot \ln\sin\frac{\pi l}{m} + \gamma^2 \ln 2 - 2\gamma_1 \ln m \\
&+ \left[\frac{\pi^2}{12} + (\gamma + \ln 2\pi m)^2\right] \cdot \frac{\pi}{2}\operatorname{ctg}\frac{\pi r}{m} + \gamma\left(\ln^2 m + 2\ln^2 2 + 2\ln 2 \cdot \ln \pi m\right) \\
&- \frac{\pi^2}{12}\ln 2 + \left(\ln^2 2 + \ln^2 m + \ln^2 \pi + 2\ln\pi \ln m + 2\ln 2\ln \pi m\right)\ln 2 \\
&+ \frac{1}{3}\ln^3 m \qquad (90)
\end{aligned}
$$

This formula is an analog of (50) for the second generalized Stieltjes constant. It can be also reduced to other forms if necessary. For instance, similarly to (53), we may rewrite it in the form containing the Ψ-function

$$\begin{aligned}\gamma_2\left(\frac{r}{m}\right) &= \gamma_2 + \frac{2}{3}\sum_{l=1}^{m-1}\cos\frac{2\pi rl}{m}\cdot\zeta'''\left(0,\frac{l}{m}\right) - 2(\gamma+\ln 2\pi m)\sum_{l=1}^{m-1}\cos\frac{2\pi rl}{m}\cdot\zeta''\left(0,\frac{l}{m}\right)\\ &+ \pi\sum_{l=1}^{m-1}\sin\frac{2\pi rl}{m}\cdot\zeta''\left(0,\frac{l}{m}\right) - 2\pi(\gamma+\ln 2\pi m)\sum_{l=1}^{m-1}\sin\frac{2\pi rl}{m}\cdot\ln\Gamma\left(\frac{l}{m}\right)\\ &- 2\gamma_1\ln m - \gamma^3 - \left[(\gamma+\ln 2\pi m)^2 - \frac{\pi^2}{12}\right]\cdot\Psi\left(\frac{r}{m}\right) - \gamma^2\ln\left(4\pi^2 m^3\right)\\ &+ \frac{\pi^3}{12}\operatorname{ctg}\frac{\pi r}{m} + \frac{\pi^2}{12}(\gamma+\ln m) - \gamma\left(\ln^2 2\pi + 4\ln m\cdot\ln 2\pi + 2\ln^2 m\right)\\ &- \left\{\ln^2 2\pi + 2\ln 2\pi\cdot\ln m + \frac{2}{3}\ln^2 m\right\}\ln m\end{aligned}$$

Thus, corresponding expressions for higher generalized Stieltjes constants at rational points are expected to be quite long and to contain higher derivatives of the Hurwitz zeta-function at zero at rational points $\zeta^{(n)}(0, l/m)$ whose properties are currently little studied.

Acknowledgments

The author would like to thank anonymous reviewers for their suggestions and remarks which improved the quality of the manuscript. The author is also grateful to Donal F. Connon for sending alternative demonstrations of some of the results derived in this paper,[40] as well as to Mark W. Coffey for providing Refs. [88] and [54]. Finally, the author is very grateful to Stefan Krämer for his precious remarks related to series (9), for drawing attention to Mascheroni's work [69] and for providing Refs. [48] and [56].

Appendix A. Closed-form expressions for some Stieltjes constants

In this first supplementary part of our work, we provide some information about particular values of $\gamma_1(v)$ which are free from $\zeta''(0, l/m) + \zeta''(0, 1 - l/m)$ or which contain only one combination of it. The value of $\gamma_1(1/2)$ has been long-time known and may be found in numerous works. The values of $\gamma_1(1/4)$, $\gamma_1(3/4)$ and $\gamma_1(1/3)$ were independently obtained by Donal Connon in [21, pp. 17–18] and by the author in [10, p. 100]. Closed-form expressions for $\gamma_1(2/3)$, $\gamma_1(1/6)$ and $\gamma_1(5/6)$ were given by the author in [10, pp. 100–101]. All these values do not contain the Hurwitz ζ-function. Below, we

[40] Private communication.

provide some further values which may be of interest and which may be reduced to only one transcendent $\zeta''(0,l/m)+\zeta''(0,1-l/m)$.

$$\begin{aligned}
\gamma_1\left(\frac{1}{5}\right) &= \gamma_1 + \frac{\sqrt{5}}{2}\left\{\zeta''\left(0,\frac{1}{5}\right)+\zeta''\left(0,\frac{4}{5}\right)\right\} \\
&\quad + \frac{\pi\sqrt{10+2\sqrt{5}}}{2}\ln\Gamma\left(\frac{1}{5}\right) + \frac{\pi\sqrt{10-2\sqrt{5}}}{2}\ln\Gamma\left(\frac{2}{5}\right) \\
&\quad + \left\{\frac{\sqrt{5}}{2}\ln 2 - \frac{\sqrt{5}}{2}\ln(1+\sqrt{5}) - \frac{5}{4}\ln 5 - \frac{\pi\sqrt{25+10\sqrt{5}}}{10}\right\}\cdot\gamma \\
&\quad - \frac{\sqrt{5}}{2}\left\{\ln 2 + \ln 5 + \ln\pi + \frac{\pi\sqrt{25-10\sqrt{5}}}{10}\right\}\cdot\ln(1+\sqrt{5}) \\
&\quad + \frac{\sqrt{5}}{2}\ln^2 2 + \frac{\sqrt{5}(1-\sqrt{5})}{8}\ln^2 5 + \frac{3\sqrt{5}}{4}\ln 2\cdot\ln 5 + \frac{\sqrt{5}}{2}\ln 2\cdot\ln\pi + \\
&\quad + \frac{\sqrt{5}}{4}\ln 5\cdot\ln\pi - \frac{\pi(2\sqrt{25+10\sqrt{5}}+5\sqrt{25+2\sqrt{5}})}{20}\ln 2 \\
&\quad - \frac{\pi(4\sqrt{25+10\sqrt{5}}-5\sqrt{5+2\sqrt{5}})}{40}\ln 5 \\
&\quad - \frac{\pi(5\sqrt{5+2\sqrt{5}}+\sqrt{25+10\sqrt{5}})}{10}\ln\pi \\
&= -8.030205511\ldots
\end{aligned}$$

Stieltjes constants $\gamma_1(2/5)$, $\gamma_1(3/5)$ and $\gamma_1(4/5)$ may be similarly expressed in terms of $\zeta''(0,1/5)+\zeta''(0,4/5)$, $\Gamma(1/5)$, $\Gamma(2/5)$, γ_1, γ and elementary functions, which, by the way, contain the golden ratio ϕ.

$$\begin{aligned}
\gamma_1\left(\frac{1}{8}\right) &= \gamma_1 + \sqrt{2}\left\{\zeta''\left(0,\frac{1}{8}\right)+\zeta''\left(0,\frac{7}{8}\right)\right\} \\
&\quad + 2\pi\sqrt{2}\ln\Gamma\left(\frac{1}{8}\right) - \pi\sqrt{2}(1-\sqrt{2})\ln\Gamma\left(\frac{1}{4}\right) \\
&\quad - \left\{\frac{1+\sqrt{2}}{2}\pi + 4\ln 2 + \sqrt{2}\ln(1+\sqrt{2})\right\}\cdot\gamma \\
&\quad - \frac{1}{\sqrt{2}}(\pi + 8\ln 2 + 2\ln\pi)\cdot\ln(1+\sqrt{2}) - \frac{7(4-\sqrt{2})}{4}\ln^2 2 \\
&\quad + \frac{1}{\sqrt{2}}\ln 2\cdot\ln\pi - \frac{\pi(10+11\sqrt{2})}{4}\ln 2 - \frac{\pi(3+2\sqrt{2})}{2}\ln\pi \\
&= -16.64171976\ldots
\end{aligned}$$

$$\begin{aligned}
\gamma_1\left(\frac{3}{8}\right) &= \gamma_1 - \sqrt{2}\left\{\zeta''\left(0,\frac{1}{8}\right)+\zeta''\left(0,\frac{7}{8}\right)\right\} \\
&\quad + 2\pi\sqrt{2}\ln\Gamma\left(\frac{1}{8}\right) - \pi\sqrt{2}(1+\sqrt{2})\ln\Gamma\left(\frac{1}{4}\right) \\
&\quad + \left\{\frac{1-\sqrt{2}}{2}\pi - 4\ln 2 + \sqrt{2}\ln(1+\sqrt{2})\right\}\cdot\gamma \\
&\quad - \frac{1}{\sqrt{2}}(\pi - 8\ln 2 - 2\ln\pi)\cdot\ln(1+\sqrt{2}) - \frac{7(4+\sqrt{2})}{4}\ln^2 2 \\
&\quad - \frac{1}{\sqrt{2}}\ln 2\cdot\ln\pi + \frac{\pi(10-11\sqrt{2})}{4}\ln 2 + \frac{\pi(3-2\sqrt{2})}{2}\ln\pi \\
&= -2.577714402\ldots
\end{aligned}$$

$$\begin{aligned}
\gamma_1\left(\frac{5}{8}\right) &= \gamma_1 - \sqrt{2}\left\{\zeta''\left(0,\frac{1}{8}\right)+\zeta''\left(0,\frac{7}{8}\right)\right\} \\
&\quad - 2\pi\sqrt{2}\ln\Gamma\left(\frac{1}{8}\right) + \pi\sqrt{2}(1+\sqrt{2})\ln\Gamma\left(\frac{1}{4}\right) \\
&\quad - \left\{\frac{1-\sqrt{2}}{2}\pi + 4\ln 2 - \sqrt{2}\ln(1+\sqrt{2})\right\}\cdot\gamma \\
&\quad + \frac{1}{\sqrt{2}}(\pi + 8\ln 2 + 2\ln\pi)\cdot\ln(1+\sqrt{2}) - \frac{7(4+\sqrt{2})}{4}\ln^2 2 \\
&\quad - \frac{1}{\sqrt{2}}\ln 2\cdot\ln\pi - \frac{\pi(10-11\sqrt{2})}{4}\ln 2 - \frac{\pi(3-2\sqrt{2})}{2}\ln\pi \\
&= -0.7353809459\ldots
\end{aligned}$$

$$\begin{aligned}
\gamma_1\left(\frac{7}{8}\right) &= \gamma_1 + \sqrt{2}\left\{\zeta''\left(0,\frac{1}{8}\right)+\zeta''\left(0,\frac{7}{8}\right)\right\} \\
&\quad - 2\pi\sqrt{2}\ln\Gamma\left(\frac{1}{8}\right) + \pi\sqrt{2}(1-\sqrt{2})\ln\Gamma\left(\frac{1}{4}\right) \\
&\quad + \left\{\frac{1+\sqrt{2}}{2}\pi - 4\ln 2 - \sqrt{2}\ln(1+\sqrt{2})\right\}\cdot\gamma \\
&\quad + \frac{1}{\sqrt{2}}(\pi - 8\ln 2 - 2\ln\pi)\cdot\ln(1+\sqrt{2}) - \frac{7(4-\sqrt{2})}{4}\ln^2 2 \\
&\quad + \frac{1}{\sqrt{2}}\ln 2\cdot\ln\pi + \frac{\pi(10+11\sqrt{2})}{4}\ln 2 + \frac{\pi(3+2\sqrt{2})}{2}\ln\pi \\
&= -0.1906592305\ldots
\end{aligned}$$

$$\begin{aligned}\gamma_1\left(\frac{1}{12}\right) &= \gamma_1 + \sqrt{3}\left\{\zeta''\left(0,\frac{1}{12}\right) + \zeta''\left(0,\frac{11}{12}\right)\right\} + 4\pi\ln\Gamma\left(\frac{1}{4}\right) + 3\pi\sqrt{3}\ln\Gamma\left(\frac{1}{3}\right)\\ &\quad - \left\{\frac{2+\sqrt{3}}{2}\pi + \frac{3}{2}\ln 3 - \sqrt{3}(1-\sqrt{3})\ln 2 + 2\sqrt{3}\ln(1+\sqrt{3})\right\}\cdot\gamma\\ &\quad - 2\sqrt{3}(3\ln 2 + \ln 3 + \ln\pi)\cdot\ln(1+\sqrt{3}) - \frac{7-6\sqrt{3}}{2}\ln^2 2 - \frac{3}{4}\ln^2 3\\ &\quad + \frac{3\sqrt{3}(1-\sqrt{3})}{2}\ln 3\cdot\ln 2 + \sqrt{3}\ln 2\cdot\ln\pi - \frac{\pi(17+8\sqrt{3})}{2\sqrt{3}}\ln 2\\ &\quad + \frac{\pi(1-\sqrt{3})\sqrt{3}}{4}\ln 3 - \pi\sqrt{3}(2+\sqrt{3})\ln\pi = -29.84287823\ldots\end{aligned}$$

$$\begin{aligned}\gamma_1\left(\frac{7}{12}\right) &= \gamma_1 - \sqrt{3}\left\{\zeta''\left(0,\frac{1}{12}\right) + \zeta''\left(0,\frac{11}{12}\right)\right\} - 4\pi\ln\Gamma\left(\frac{1}{4}\right) + 3\pi\sqrt{3}\ln\Gamma\left(\frac{1}{3}\right)\\ &\quad - \left\{\frac{-2+\sqrt{3}}{2}\pi + \frac{3}{2}\ln 3 + \sqrt{3}(1+\sqrt{3})\ln 2 - 2\sqrt{3}\ln(1+\sqrt{3})\right\}\cdot\gamma\\ &\quad + 2\sqrt{3}(3\ln 2 + \ln 3 + \ln\pi)\cdot\ln(1+\sqrt{3}) - \frac{7+6\sqrt{3}}{2}\ln^2 2 - \frac{3}{4}\ln^2 3\\ &\quad - \frac{3\sqrt{3}(1+\sqrt{3})}{2}\ln 3\cdot\ln 2 - \sqrt{3}\ln 2\cdot\ln\pi - \frac{\pi(17-8\sqrt{3})}{2\sqrt{3}}\ln 2\\ &\quad + \frac{\pi(1+\sqrt{3})\sqrt{3}}{4}\ln 3 - \pi\sqrt{3}(2-\sqrt{3})\ln\pi = -0.900932495\ldots\end{aligned}$$

Expressions for Stieltjes constants $\gamma_1(5/12)$ and $\gamma_1(11/12)$ may be similarly written in terms of $\zeta''(0,1/12) + \zeta''(0,11/12)$, $\Gamma(1/3)$, $\Gamma(1/4)$, γ_1, γ and elementary functions, see e.g. (72).

Appendix B. Some results from the theory of finite Fourier series. Applications to certain summations involving the Ψ-function and the Hurwitz ζ-function

B.1. Theoretical part

Finite Fourier series are well-known and widely used in discrete mathematics, numerical analysis, engineering sciences (especially in signal and image processing) and in a lot of related disciplines. Unlike usual Fourier series, which are essentially variants or particular cases of the same formula, finite Fourier series may take quite different forms and expressions. For instance, in engineering sciences, one usually deals with the following $2m$-points Fourier series

$$f_m(r) = \frac{a_m(0)}{2} + \sum_{l=1}^{m-1}\left(a_m(l)\cos\frac{\pi r l}{m} + b_m(l)\sin\frac{\pi r l}{m}\right) + (-1)^r\frac{a_m(m)}{2}$$

with $r = 0, 1, 2, \dots, 2m-1$ and $m \in \mathbb{N}$. Thanks to the orthogonality properties of circular functions, one may determine the coefficients in this expansion:

$$\begin{cases} a_m(k) = \dfrac{1}{m} \displaystyle\sum_{r=1}^{2m-1} f_m(r) \cos \dfrac{\pi r k}{m}, & k = 0, 1, 2, \dots, m \\ b_m(k) = \dfrac{1}{m} \displaystyle\sum_{r=1}^{2m-1} f_m(r) \sin \dfrac{\pi r k}{m}, & k = 1, 2, 3, \dots, m-1 \end{cases}$$

as well as derive Parseval's theorem

$$\frac{1}{m} \sum_{r=1}^{2m-1} f_m^2(r) = \frac{a_m^2(0)}{2} + \sum_{l=1}^{m-1} \big(a_m^2(l) + b_m^2(l)\big) + \frac{a_m^2(m)}{2},$$

see for more details [41, Chapter 6].

In contrast, in our researches, we encounter the following $(m-1)$-points finite Fourier series

$$f_m(r) = a_m(0) + \sum_{l=1}^{m-1} \left(a_m(l) \cos \frac{2\pi r l}{m} + b_m(l) \sin \frac{2\pi r l}{m} \right) \tag{B.1}$$

$r = 1, 2, 3, \dots, m-1$, $m \in \mathbb{N}$, for which inversion formulae and Parseval's theorem are quite different. Let, first, derive the inversion formulae for the coefficients of this series. Multiplying both sides by $\cos(2\pi rk/m)$, where $k = 1, 2, 3, \dots, m-1$, and summing over $r \in [1, m-1]$, gives

$$\begin{aligned} \sum_{r=1}^{m-1} f_m(r) \cos \frac{2\pi r k}{m} &= \sum_{r=1}^{m-1} \left[a_m(0) + \sum_{l=1}^{m-1} a_m(l) \cos \frac{2\pi r l}{m} + \sum_{l=1}^{m-1} b_m(l) \sin \frac{2\pi r l}{m} \right] \cos \frac{2\pi r k}{m} \\ &= a_m(0) \underbrace{\sum_{r=1}^{m-1} \cos \frac{2\pi r k}{m}}_{-1} + \sum_{l=1}^{m-1} a_m(l) \underbrace{\sum_{r=1}^{m-1} \cos \frac{2\pi r l}{m} \cdot \cos \frac{2\pi r k}{m}}_{\frac{1}{2} m(\delta_{l,k} + \delta_{l,m-k}) - 1} \\ &\quad + \sum_{l=1}^{m-1} b_m(l) \underbrace{\sum_{r=1}^{m-1} \sin \frac{2\pi r l}{m} \cdot \cos \frac{2\pi r k}{m}}_{0} \\ &= -a_m(0) - \sum_{l=1}^{m-1} a_m(l) + \frac{m}{2} \big\{ a_m(k) + a_m(m-k) \big\} \end{aligned} \tag{B.2}$$

Similarly, multiplying both sides of (B.1) by $\sin(2\pi rk/m)$, where $k = 1, 2, 3, \dots, m-1$, and summing over $r \in [1, m-1]$, yields

$$\sum_{r=1}^{m-1} f_m(r)\sin\frac{2\pi rk}{m} = \sum_{r=1}^{m-1}\left[a_m(0)+\sum_{l=1}^{m-1}a_m(l)\cos\frac{2\pi rl}{m}+\sum_{l=1}^{m-1}b_m(l)\sin\frac{2\pi rl}{m}\right]\sin\frac{2\pi rk}{m}$$

$$= a_m(0)\underbrace{\sum_{r=1}^{m-1}\sin\frac{2\pi rk}{m}}_{0}+\sum_{l=1}^{m-1}a_m(l)\underbrace{\sum_{r=1}^{m-1}\cos\frac{2\pi rl}{m}\cdot\sin\frac{2\pi rk}{m}}_{0}$$

$$+\sum_{l=1}^{m-1}b_m(l)\underbrace{\sum_{r=1}^{m-1}\sin\frac{2\pi rl}{m}\cdot\sin\frac{2\pi rk}{m}}_{\frac{1}{2}m(\delta_{l,k}-\delta_{l,m-k})} = \frac{m}{2}\big\{b_m(k)-b_m(m-k)\big\} \tag{B.3}$$

Finally, Parseval's equality for the finite series (B.1) reads:

$$\sum_{r=1}^{m-1} f_m^2(r) = \sum_{r=1}^{m-1}\left[a_m(0)+\sum_{l=1}^{m-1}a_m(l)\cos\frac{2\pi rl}{m}+\sum_{l=1}^{m-1}b_m(l)\sin\frac{2\pi rl}{m}\right]^2 =$$

$$= \sum_{r=1}^{m-1}a_m^2(0)+2a_m(0)\sum_{l=1}^{m-1}a_m(l)\underbrace{\sum_{r=1}^{m-1}\cos\frac{2\pi rl}{m}}_{-1}+2a_m(0)\sum_{l=1}^{m-1}b_m(l)\underbrace{\sum_{r=1}^{m-1}\sin\frac{2\pi rl}{m}}_{0}$$

$$+2\sum_{l=1}^{m-1}\sum_{n=1}^{m-1}a_m(l)b_m(n)\underbrace{\sum_{r=1}^{m-1}\cos\frac{2\pi rl}{m}\cdot\sin\frac{2\pi rn}{m}}_{0}$$

$$+\sum_{l=1}^{m-1}\sum_{n=1}^{m-1}a_m(l)a_m(n)\underbrace{\sum_{r=1}^{m-1}\cos\frac{2\pi rl}{m}\cdot\cos\frac{2\pi rn}{m}}_{\frac{1}{2}m(\delta_{l,n}+\delta_{l,m-n})-1}$$

$$+\sum_{l=1}^{m-1}\sum_{n=1}^{m-1}b_m(l)b_m(n)\underbrace{\sum_{r=1}^{m-1}\sin\frac{2\pi rl}{m}\cdot\sin\frac{2\pi rn}{m}}_{\frac{1}{2}m(\delta_{l,n}-\delta_{l,m-n})}$$

$$= (m-1)a_m^2(0)-2a_m(0)\sum_{l=1}^{m-1}a_m(l)-\left[\sum_{l=1}^{m-1}a_m(l)\right]^2$$

$$+\frac{m}{2}\sum_{l=1}^{m-1}\big[a_m^2(l)+a_m(l)a_m(m-l)+b_m^2(l)-b_m(l)b_m(m-l)\big]$$

B.2. Some applications

The finite Fourier series may be successfully used for the finite-length summations in a variety of problems and contexts. Consider, for example, the Gauss' Digamma theorem, which is usually written in one of three equivalent forms

$$\begin{cases} \Psi\left(\dfrac{r}{m}\right) = -\gamma - \ln 2m - \dfrac{\pi}{2}\operatorname{ctg}\dfrac{\pi r}{m} + 2\displaystyle\sum_{l=1}^{\lfloor\frac{1}{2}(m-1)\rfloor}\cos\dfrac{2\pi rl}{m}\cdot\ln\sin\dfrac{\pi l}{m} & \text{(a)} \\ \Psi\left(\dfrac{r}{m}\right) = -\gamma - \ln 2m - \dfrac{\pi}{2}\operatorname{ctg}\dfrac{\pi r}{m} + \displaystyle\sum_{l=1}^{m-1}\cos\dfrac{2\pi rl}{m}\cdot\ln\sin\dfrac{\pi l}{m} & \text{(b)} \\ \Psi\left(\dfrac{r}{m}\right) = -\gamma - \ln 2\pi m - \dfrac{\pi}{2}\operatorname{ctg}\dfrac{\pi r}{m} - 2\displaystyle\sum_{l=1}^{m-1}\cos\dfrac{2\pi rl}{m}\cdot\ln\Gamma\left(\dfrac{l}{m}\right) & \text{(c)} \end{cases} \tag{B.4}$$

$r = 1, 2, \ldots, m-1$, $m \in \mathbb{N}_{\geqslant 2}$, first and second of which are due to Gauss[41] [12, pp. 35–38], [7, vol. I, p. 19, §1.7.3], while the third one is due to Malmsten [66, p. 57, Eq. (70)], [10, p. 37, Eq. (23)]. Remarking that the cotangent may be represented by (54), two latter equations take the form

$$\begin{cases} \Psi\left(\dfrac{r}{m}\right) = -\gamma - \ln 2m + \dfrac{\pi}{m}\displaystyle\sum_{l=1}^{m-1}\sin\dfrac{2\pi rl}{m}\cdot l + \displaystyle\sum_{l=1}^{m-1}\cos\dfrac{2\pi rl}{m}\cdot\ln\sin\dfrac{\pi l}{m} \\ \Psi\left(\dfrac{r}{m}\right) = -\gamma - \ln 2\pi m + \dfrac{\pi}{m}\displaystyle\sum_{l=1}^{m-1}\sin\dfrac{2\pi rl}{m}\cdot l - 2\displaystyle\sum_{l=1}^{m-1}\cos\dfrac{2\pi rl}{m}\cdot\ln\Gamma\left(\dfrac{l}{m}\right) \end{cases} \tag{B.5}$$

$r = 1, 2, \ldots, m-1$, $m \in \mathbb{N}_{\geqslant 2}$, which represent complete finite Fourier series of the same type as (B.1). Hence, the application of (B.2)–(B.4) straightforwardly yields the following important summation formulae

$$\begin{cases} \displaystyle\sum_{r=1}^{m-1}\Psi\left(\dfrac{r}{m}\right)\cdot\cos\dfrac{2\pi rk}{m} = m\ln\left(2\sin\dfrac{k\pi}{m}\right) + \gamma, \quad k = 1, 2, \ldots, m-1 \\ \displaystyle\sum_{r=1}^{m-1}\Psi\left(\dfrac{r}{m}\right)\cdot\sin\dfrac{2\pi rk}{m} = \dfrac{\pi}{2}(2k-m), \quad k = 1, 2, \ldots, m-1 \\ \displaystyle\sum_{r=1}^{m-1}\Psi^2\left(\dfrac{r}{m}\right) = (m-1)\gamma^2 + m(2\gamma + \ln 4m)\ln m - m(m-1)\ln^2 2 \\ \qquad\qquad + \dfrac{\pi^2(m^2-3m+2)}{12} + m\displaystyle\sum_{l=1}^{m-1}\ln^2\sin\dfrac{\pi l}{m} \end{cases} \tag{B.6}$$

[41] Strictly speaking, Gauss wrote them in a slightly different manner, see [34, p. 39].

where the last sum, due to the symmetry of $\ln\sin(\pi l/m)$ about $l = m/2$, may be also written as

$$\sum_{l=1}^{m-1} \ln^2 \sin\frac{\pi l}{m} = 2\sum_{l=1}^{\lfloor\frac{1}{2}(m-1)\rfloor} \ln^2 \sin\frac{\pi l}{m}$$

For the purpose of demonstration, we take Malmsten's representation for the Ψ-function.[42] Inserting expressions for coefficients $a_m(0) = -\gamma - \ln 2\pi m$, $a_m(l) = -2\ln\Gamma(l/m)$ and $b_m(l) = \pi l/m$ into (B.2), yields for the first sum:

$$\sum_{r=1}^{m-1} \Psi\left(\frac{r}{m}\right)\cdot\cos\frac{2\pi rk}{m} = \gamma + \ln 2\pi m - m\underbrace{\left[\ln\Gamma\left(\frac{k}{m}\right) + \ln\Gamma\left(1-\frac{k}{m}\right)\right]}_{\ln\pi - \ln\sin(\pi k/m)} + 2\sum_{l=1}^{m-1}\ln\Gamma\left(\frac{l}{m}\right) = \gamma + m\ln\left(2\sin\frac{k\pi}{m}\right)$$

where the final simplification is performed with the help of the reflection formula and Gauss' multiplication theorem for the logarithm of the Γ-function (77). Analogously, using (B.3) yields for the second sum:

$$\sum_{r=1}^{m-1} \Psi\left(\frac{r}{m}\right)\cdot\sin\frac{2\pi rk}{m} = \frac{m}{2}\left[\frac{\pi k}{m} - \frac{\pi(m-k)}{m}\right] = \frac{\pi}{2}(2k-m)$$

By taking advantage of this opportunity, we would like to remark that a formula of the similar nature appears also in [12, p. 39] and [80, p. 19, Eq. (49)]. Sadly, the formula given in the former source contains two errors; the correct variant of the formula is

$$\sum_{r=1}^{m} \Psi\left(\frac{r}{m}\right)\cdot\exp\frac{2\pi rki}{m} = m\ln\left(1-\exp\frac{2\pi ki}{m}\right), \qquad k\in\mathbb{Z},\ m\in\mathbb{N},\ k\neq m.$$

Finally, by (B.4), we derive Parseval's theorem for the Ψ-function of a discrete argument

$$\sum_{r=1}^{m-1} \Psi^2\left(\frac{r}{m}\right) = (m-1)(\gamma+\ln 2\pi m)^2 - 4(\gamma+\ln 2\pi m)\underbrace{\sum_{l=1}^{m-1}\ln\Gamma\left(\frac{l}{m}\right)}_{\frac{1}{2}(m-1)\ln 2\pi - \frac{1}{2}\ln m} - 4\left[\sum_{l=1}^{m-1}\ln\Gamma\left(\frac{l}{m}\right)\right]^2 + 2m\sum_{l=1}^{m-1}\ln\Gamma\left(\frac{l}{m}\right)\cdot\left[\ln\Gamma\left(\frac{l}{m}\right)+\ln\Gamma\left(1-\frac{l}{m}\right)\right] +$$

[42] The reader may perform the same procedure with the more usual Gauss' representation as an exercise.

$$+\frac{\pi^2}{m}\cdot\sum_{l=1}^{m-1} l^2-\frac{\pi^2}{2}\cdot\sum_{l=1}^{m-1} l=(m-1)\gamma^2+m(2\gamma+\ln 4m)\ln m$$

$$-m(m-1)\ln^2 2+\frac{\pi^2(m^2-3m+2)}{12}+2m\sum_{l=1}^{\lfloor\frac{1}{2}(m-1)\rfloor}\ln^2\sin\frac{\pi l}{m} \tag{B.7}$$

where the sum from the third line, thanks to the symmetry of $\ln\sin(\pi l/m)$ about $l=m/2$ and to the fact that $\ln\sin(\pi l/m)=0$ for $l=m/2$, could be simplified as follows

$$\sum_{l=1}^{m-1}\ln\Gamma\left(\frac{l}{m}\right)\cdot\left[\ln\Gamma\left(\frac{l}{m}\right)+\ln\Gamma\left(1-\frac{l}{m}\right)\right]=\sum_{l=1}^{m-1}\ln\Gamma\left(\frac{l}{m}\right)\cdot\left[\ln\pi-\ln\sin\frac{\pi l}{m}\right]$$

$$=\frac{\ln\pi}{2}\big[(m-1)\ln 2\pi-\ln m\big]-\sum_{l=1}^{m-1}\ln\Gamma\left(\frac{l}{m}\right)\cdot\ln\sin\frac{\pi l}{m}$$

$$=\frac{\ln\pi}{2}\big[(m-1)\ln 2\pi-\ln m\big]-\sum_{l=1}^{\lfloor\frac{1}{2}(m-1)\rfloor}\left[\ln\pi-\ln\sin\frac{\pi l}{m}\right]\ln\sin\frac{\pi l}{m}$$

$$=\frac{\ln\pi}{2}\big[(m-1)\ln 2\pi-\ln m\big]-\ln\pi\cdot\sum_{l=1}^{\lfloor\frac{1}{2}(m-1)\rfloor}\ln\sin\frac{\pi l}{m}$$

$$+\sum_{l=1}^{\lfloor\frac{1}{2}(m-1)\rfloor}\ln^2\sin\frac{\pi l}{m}=\frac{\ln\pi}{2}\big[(m-1)\ln 4\pi-2\ln m\big]+\sum_{l=1}^{\lfloor\frac{1}{2}(m-1)\rfloor}\ln^2\sin\frac{\pi l}{m} \tag{B.8}$$

because

$$\sum_{l=1}^{\lfloor\frac{1}{2}(m-1)\rfloor}\ln\sin\frac{\pi l}{m}=\ln\prod_{l=1}^{\lfloor\frac{1}{2}(m-1)\rfloor}\sin\frac{\pi l}{m}=\frac{1-m}{2}\ln 2+\frac{1}{2}\ln m$$

and where

$$\sum_{l=1}^{m-1} l^2=\frac{m(m-1)(2m-1)}{6}\quad\text{and}\quad\sum_{l=1}^{m-1} l=\frac{m(m-1)}{2} \tag{B.9}$$

respectively, which completes the evaluation of the third formula in (B.6).

In like manner, we may also derive similar summation formulae for the Hurwitz ζ-function. Rewriting Hurwitz's functional equation (32) in the form analogous to (B.1)

$$\zeta\left(a,\frac{r}{m}\right)=m^{a-1}\zeta(a)+\frac{2\Gamma(1-a)}{(2\pi m)^{1-a}}\left[\sin\frac{\pi a}{2}\sum_{l=1}^{m-1}\cos\frac{2\pi rl}{m}\cdot\zeta\left(1-a,\frac{l}{m}\right)\right.$$

$$\left.+\cos\frac{\pi a}{2}\sum_{l=1}^{m-1}\sin\frac{2\pi rl}{m}\cdot\zeta\left(1-a,\frac{l}{m}\right)\right]$$

yields

$$\sum_{r=1}^{m-1}\zeta\left(a,\frac{r}{m}\right)\cdot\cos\frac{2\pi rk}{m}=\frac{m\Gamma(1-a)}{(2\pi m)^{1-a}}\sin\frac{\pi a}{2}\cdot\left\{\zeta\left(1-a,\frac{k}{m}\right)+\zeta\left(1-a,1-\frac{k}{m}\right)\right\}$$
$$-\zeta(a)$$

$$\sum_{r=1}^{m-1}\zeta\left(a,\frac{r}{m}\right)\cdot\sin\frac{2\pi rk}{m}=\frac{m\Gamma(1-a)}{(2\pi m)^{1-a}}\cos\frac{\pi a}{2}\cdot\left\{\zeta\left(1-a,\frac{k}{m}\right)-\zeta\left(1-a,1-\frac{k}{m}\right)\right\}$$

$$\sum_{r=1}^{m-1}\zeta^2\left(a,\frac{r}{m}\right)=\left(m^{2a-1}-1\right)\zeta^2(a)+$$
$$+\frac{2m\Gamma^2(1-a)}{(2\pi m)^{2-2a}}\sum_{l=1}^{m-1}\left\{\zeta\left(1-a,\frac{l}{m}\right)-\cos\pi a\cdot\zeta\left(1-a,1-\frac{l}{m}\right)\right\}\cdot\zeta\left(1-a,\frac{l}{m}\right)$$

which hold for any $r=1,2,3,\dots,m-1$ and $k=1,2,3,\dots,m-1$, where m is positive integer.

By the way, there are many other functions which are orthogonal or semi-orthogonal over some discrete interval. For instance, by considering another set of circular functions and their properties

$$\sum_{r=0}^{m-1}\cos\frac{(2r+1)k\pi}{m}=\sum_{r=0}^{m-1}\sin\frac{(2r+1)k\pi}{m}=0,\qquad k=1,2,\dots,m-1$$

$$\sum_{r=0}^{m-1}\cos\frac{(2r+1)k\pi}{m}\cdot\sin\frac{(2r+1)l\pi}{m}=0,\qquad k,l=1,2,\dots,m-1$$

$$\sum_{r=0}^{m-1}\cos\frac{(2r+1)k\pi}{m}\cdot\cos\frac{(2r+1)l\pi}{m}=\frac{n}{2}(\delta_{k,l}-\delta_{k,m-l}-\delta_{k,m+l}+\delta_{k,2m-l})$$

$$\sum_{r=0}^{m-1}\sin\frac{(2r+1)k\pi}{m}\cdot\sin\frac{(2r+1)l\pi}{m}=\frac{n}{2}(\delta_{k,l}+\delta_{k,m-l}-\delta_{k,m+l}-\delta_{k,2m-l})\qquad\text{(B.10)}$$

where in last two formulae $k,l=1,2,\dots,2m-1$, as well as (B.5), one may easily prove that

$$\begin{cases}\displaystyle\sum_{r=0}^{m-1}\Psi\left(\frac{2r+1}{2m}\right)\cdot\cos\frac{(2r+1)k\pi}{m}=m\ln\operatorname{tg}\frac{\pi k}{2m},\qquad k=1,2,\dots,m-1\\[2ex]\displaystyle\sum_{r=0}^{m-1}\Psi\left(\frac{2r+1}{2m}\right)\cdot\sin\frac{(2r+1)k\pi}{m}=-\frac{\pi m}{2},\qquad k=1,2,\dots,m-1\end{cases}$$

By a similar line of reasoning, we also derive

$$\sum_{r=1}^{2m-1}(-1)^r\cdot\Psi\left(\frac{r}{2m}\right)=2m\ln 2+\gamma$$

$$\sum_{r=0}^{2m-1}(-1)^r\cdot\Psi\left(\frac{2r+1}{4m}\right)=-\pi m$$

$$\sum_{r=1}^{m-1}\operatorname{ctg}\frac{\pi r}{m}\cdot\Psi\left(\frac{r}{m}\right)=-\frac{\pi(m-1)(m-2)}{6}$$

$$\sum_{r=1}^{m-1}\frac{r}{m}\cdot\Psi\left(\frac{r}{m}\right)=-\frac{\gamma}{2}(m-1)-\frac{m}{2}\ln m-\frac{\pi}{2}\sum_{r=1}^{m-1}\frac{r}{m}\cdot\operatorname{ctg}\frac{\pi r}{m}$$

$$\sum_{r=1}^{m-1}\cos\frac{(2l+1)\pi r}{m}\cdot\Psi\left(\frac{r}{m}\right)=-\frac{\pi}{m}\cdot\sum_{r=1}^{m-1}\frac{r\cdot\sin\frac{2\pi r}{m}}{\cos\frac{2\pi r}{m}-\cos\frac{(2l+1)\pi}{m}}$$

$$\sum_{r=1}^{m-1}\sin\frac{(2l+1)\pi r}{m}\cdot\Psi\left(\frac{r}{m}\right)=-(\gamma+\ln 2m)\operatorname{ctg}\frac{(2l+1)\pi}{2m}$$
$$+\sin\frac{(2l+1)\pi}{m}\cdot\sum_{r=1}^{m-1}\frac{\ln\sin\frac{\pi r}{m}}{\cos\frac{2\pi r}{m}-\cos\frac{(2l+1)\pi}{m}} \qquad \text{(B.11)}$$

where the last two formulae remain valid for any $l\in\mathbb{Z}$.

Appendix C. An integral formula for the logarithm of the Γ-function at rational arguments

In this part, we evaluate integral (82) for $p=k/n$ and show that it reduces to the logarithm of the Γ-function at rational argument, Euler's constant γ and elementary functions.

From a simple algebraic argument, it follows that

$$\sum_{r=1}^{n-1}\operatorname{sh} rx\cdot\sin\frac{2\pi rk}{n}=-\frac{1}{2}\cdot\frac{\operatorname{sh} nx\cdot\sin\frac{2\pi k}{n}}{\operatorname{ch} x-\cos\frac{2\pi k}{n}},\qquad x\in\mathbb{C},\ k\in\mathbb{Z}.$$

Then, for $p=k/n$, where k and n are positive integers such that k does not exceed n, the denominator of integrand (82) may be replaced by the above identity and hence

$$\int_0^\infty\frac{e^{-nx}\cdot\ln x}{\operatorname{ch} x-\cos\frac{2\pi k}{n}}\,dx=-4\csc\frac{2\pi k}{n}\sum_{r=1}^{n-1}\sin\frac{2\pi rk}{n}\cdot\int_0^\infty\frac{\operatorname{sh} rx\cdot\ln x}{e^{2nx}-1}\,dx \qquad \text{(C.1)}$$

The latter integral was already evaluated in our previous work, see [10, p. 73, n° 25]. By setting in exercise n° 25-a $b=n$, $m=r$, and then by rewriting the result for $2n$ instead of n, we get

$$\int_0^\infty \frac{\operatorname{sh} rx \cdot \ln x}{e^{2nx}-1}\,dx = -\frac{\pi}{4n}\operatorname{ctg}\frac{r\pi}{2n}\cdot\ln 2\pi - \frac{\gamma+\ln r}{2r} + \frac{\pi}{2n}\sum_{l=1}^{2n-1}\sin\frac{\pi r l}{n}\cdot\ln\Gamma\!\left(\frac{l}{2n}\right) \tag{C.2}$$

By inserting the above formula into (C.1) and by recalling that for $k = 1, 2, 3, \ldots, n-1$ and $l = 1, 2, 3, \ldots, 2n-1$

$$\begin{cases} \displaystyle\sum_{r=1}^{n-1} \sin\frac{2\pi rk}{n}\cdot\operatorname{ctg}\frac{\pi r}{2n} = n-2k \\ \displaystyle\sum_{r=1}^{n-1} \sin\frac{2\pi rk}{n}\cdot\sin\frac{\pi rl}{n} = \frac{n}{2}\{\delta_{k,\frac{l}{2}} - \delta_{k,n-\frac{l}{2}}\} \end{cases}$$

the expression for integral (82) at $p = k/n$ takes its final form

$$\begin{aligned} \int_0^\infty \frac{e^{-nx}\cdot\ln x}{\operatorname{ch} x - \cos\frac{2\pi k}{n}}\,dx &= 2\int_0^1 \frac{x^n \ln\ln\frac{1}{x}}{x^2 - 2x\cos\frac{2\pi k}{n}+1}\,dx = 2\int_1^\infty \frac{\ln\ln x}{x^n(x^2-2x\cos\frac{2\pi k}{n}+1)}\,dx \\ &= \left\{\frac{\pi(n-2k)\ln 2\pi}{n} - 2\pi\ln\Gamma\!\left(\frac{k}{n}\right) + \pi\ln\pi - \pi\ln\sin\frac{\pi k}{n} + 2\sum_{r=1}^{n-1}\frac{\gamma+\ln r}{r}\cdot\sin\frac{2\pi rk}{n}\right\} \\ &\quad\times\csc\frac{2\pi k}{n}, \qquad k = 1, 2, 3, \ldots, n-1, \quad k \neq \frac{n}{2} \end{aligned} \tag{C.3}$$

Whence

$$\begin{aligned} \ln\Gamma\!\left(\frac{k}{n}\right) &= \frac{(n-2k)\ln 2\pi}{2n} + \frac{1}{2}\left\{\ln\pi - \ln\sin\frac{\pi k}{n}\right\} + \frac{1}{\pi}\sum_{r=1}^{n-1}\frac{\gamma+\ln r}{r}\cdot\sin\frac{2\pi rk}{n} \\ &\quad - \frac{1}{2\pi}\sin\frac{2\pi k}{n}\cdot\int_0^\infty \frac{e^{-nx}\cdot\ln x}{\operatorname{ch} x - \cos\frac{2\pi k}{n}}\,dx, \qquad k = 1, 2, 3, \ldots, n-1, \end{aligned} \tag{C.4}$$

$k \neq n/2$. By the way, (C.3)–(C.4) may be proven by other methods as well. For instance, one may directly employ (81) because $a_0 = 0$ for $p = k/n$ and all remaining integrals in the right-hand side are known. Yet, (C.3)–(C.4) may be also obtained with the aid of previously derived results in exercises n° 60 and 58 in [10, Sect. 4], as well as Malmsten's representation for the logarithm of the Γ-function

$$\ln\Gamma(z) = \frac{1}{2}\ln\pi - \frac{1}{2}\ln\sin\pi z - \frac{2z-1}{2}\ln 2\pi - \frac{\sin 2\pi z}{2\pi}\int_0^\infty \frac{\ln x}{\operatorname{ch}x - \cos 2\pi z}\,dx \tag{C.5}$$

where $0 < \operatorname{Re} z < 1$, see exercises n° 2, 29-h, 30 [10, Sect. 4].

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Journal of Number Theory 151 (2015) 276–277

Contents lists available at ScienceDirect

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www.elsevier.com/locate/jnt

Erratum

Erratum to "A theorem for the closed-form evaluation of the first generalized Stieltjes constant at rational arguments and some related summations" [J. Number Theory 148 (2015) 537–592]

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ARTICLE INFO

Article history:
Received 28 January 2015
Accepted 29 January 2015
Available online 3 February 2015
Communicated by David Goss

The publisher regrets the following list of errors in the published article.

Place		Should read
Page 541, line 5	[8,46,58,60,81]	[8,46,58,60,81,72,96,97,94,89,98,99]
Page 541, line 8	[4,37,57,64]	[4,37,57,64,90,93,91,92]
Page 541, footnote 8	$\beta_k = 1$	$\beta_k = (-1)^k$
Page 543, footnote 13	$\gamma(1/4), \gamma(3/4)$ and $\gamma(1/3)$	$\gamma_1(1/4), \gamma_1(3/4)$ and $\gamma_1(1/3)$
Page 552, line 8	result^{22}	result^{21}
Page 552, footnote 22	^{22} The value	^{21} The value
Page 562, footnote 27	Conon	Connon
Page 590, Ref. [28]	Dwigth	Dwight

DOI of original article: http://dx.doi.org/10.1016/j.jnt.2014.08.009.
E-mail address: iaroslav.blagouchine@univ-tln.fr.

http://dx.doi.org/10.1016/j.jnt.2015.01.001

Note that references [96], [97], [94], [89], [98], [99], [90], [93], [91], [92] are new and were not mentioned in the article. We decided to add them, because they represent important contributions in the their fields. At the same time, we would like to remark that in a small article it is, of course, difficult to encompass all works devoted to Stieltjes constants. Moreover, some authors do not call them *Stieltjes constants* nor *generalized Euler's constants*; instead, they simply call them *Laurent series coefficients of* $\zeta(s)$ or *MacLaurin coefficients of* $(s-1)\zeta(s)$ or $\zeta(s)-(s-1)^{-1}$, and such works may be very hard to find. Finally, there are numerous general works in which Stieltjes constants are simply mentioned or briefly discussed, e.g. [39, n° 388, p. 48], [56], [95].

Acknowledgments

The author is grateful to Vladimir V. Reshetnikov for pointing out several misprints.

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