Stark-Wannier resonances and cubic exponential sums

Alexander Fedotov
Saint Petersburg State University, Russia
fedotov.s@mail.ru

The talk is based on a joint work with Frederic Klopp from the University of Pierre and Marie Curie (Paris VII). We discuss the Schrödinger operator $H = -\frac{\partial^2}{\partial x^2} + v(x) - \epsilon x$ acting in $L^2(\mathbb{R})$. Here, v is an entire 1-periodic function, and ϵ is a positive constant. This operator is a model describing an electron in a crystal placed in a constant electric field. The parameter ϵ is proportional to the value of the electric field. The spectrum of H is absolutely continuous and fills the real axis.

The operator attracted attention of both physicists and mathematicians after the discovery of the Stark-Wannier ladders. These are ϵ-periodic sequences of resonances, i.e., of the poles of the meromorphic continuation of the resolvent from the upper half-plane of the complex plane across the spectrum, see [1, 3]. A series of papers was devoted to the description of the ladders for small ϵ, see, i.g., [2]. The complexity of this problem is related to the fact that, as $\epsilon \to 0$, there are ladders exponentially close to the real axis. Actually, only the case of finite gap potentials v was understood relatively well: for these potentials, there is only a finite number of ladders that are close to the real axis. It appeared that the ladders non-trivially "interact" as ϵ changes, and physicists conjectured that the behavior of the resonances strongly depends on the arithmetic nature of ϵ, see, for example, [3].

We assume that $v(x) = 2\cos(2\pi x)$ and study the reflection coefficient $r(E)$ in the lower half-plane of the complex plane of the spectral parameter E. There, the function $E \mapsto 1/r(E)$ is an analytic ϵ-periodic function. Its zeros are resonances (and vice versa). Represent $1/r$ by its Fourier series, $1/r(E) = \sum_{m\in\mathbb{Z}} p(m) e^{2\pi i m E/\epsilon}$. Let $a(\epsilon) = \sqrt{2/\epsilon}\,\pi e^{i\pi/4}$. We prove that, as $m \to \infty$,

$$p(m) = a(\epsilon)\sqrt{m}\, e^{-2\pi i \omega m^3 - 2m\log(2\pi m/e) + \delta(m)}, \quad \omega = \left\{\frac{\pi^2}{3\epsilon}\right\}, \tag{1}$$

where $\{x\}$ denotes the fractional part of $x \in \mathbb{R}$, and $\delta(m) = O(\log^2 m/m)$, the estimate being locally uniform in $\epsilon > 0$.

Obviously, the asymptotic behavior of $1/r(E)$ as $\operatorname{Im} E \to -\infty$, is determined by the Fourier series terms with large positive m, and so, roughly, as $\operatorname{Im} E \to -\infty$,

$$\frac{1}{r(E)} \approx a(\epsilon)\,\mathcal{P}(E/\epsilon), \quad \mathcal{P}(s) = \sum_{m\geq 1} \sqrt{m}\, e^{-2\pi i \omega m^3 - 2m\log(2\pi m/e) + 2\pi i m s}. \tag{2}$$

It is worth to compare the function $1/r$ with the cubic exponential sums $\sum_{n=1}^{N} e^{-2\pi i \omega n^3}$. They were extensively studied for large N in the analytic number theory, see [4], and proved to depend strongly on the arithmetic nature of ω. This appears to be true in our case too. In the talk, for rational ω, we describe in detail behavior of the resonances far from the real axis. In particular, we show that, if $\omega = p/q$ with coprime integers $0 < p < q$, then their asymptotics are determined by beautiful and nontrivial properties of the complete rational exponential sums $\sum_{l=0}^{q-1} e^{-2\pi i \frac{pl^3 - ml}{q}}$, $m \in \mathbb{Z}$.

References

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