Superstring compactification and Frobenius manifold structures.

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Introduction.

To obtain the low-energy Lagrangian of String theory compactified on a CY manifold, one needs to know the Special Kähler geometry on the moduli space of CY manifold X.
A way to compute this was proposed in the famous work by Candelas, de la Ossa,Green and Parkes.
Kähler potential of the metric on the moduli space is expressed bilinearly in terms of periods of the CY 3-form Ω.

$$\omega_\mu := \oint_{q_\mu \in H_3(X,\mathbb{Z})} \Omega,$$

Here q_μ is a special basis of cycles in $H_3(X,\mathbb{Z})$.
We present an alternative approach to the computation of Kähler potential for the case when CY manifold is given by a Hypersurface $W_0(x) = 0$ in a weighted projective space.

This approach is based on the fact that the moduli space of CY manifold is a subspace of a Frobenius manifold (FM) which arises on the deformations of the singularity defined by the LG superpotential $W_0(x)$.

This allows to introduce and compute the additional basises of periods called σ_μ.
Since both $\omega^{\pm}_{\mu}(\phi)$ and $\sigma^{\pm}_{\nu}(\phi)$ are periods defined as the integrals over diffent basises of the cycles in the same group $H_3(X,\mathbb{Z})$, they are connected by some constant matrix T^{ν}_{μ}:

$$\omega^{\pm}_{\mu}(\phi) = T^{\nu}_{\mu}\, \sigma^{\pm}_{\nu}(\phi).$$

Kähler potential is then given in terms of the periods σ_μ, the holomorphic FM metric $\eta_{\mu\rho}$ and the matrix T^{ν}_{μ} :

$$e^{-K} = \sigma^{+}_{\mu}\eta^{\mu\rho}M^{\nu}_{\rho}\overline{\sigma^{-}_{\nu}}, \qquad M = T^{-1}\bar{T}.$$

The basis ω_μ has the advantage of being taken over homology cycles q_μ with real coefficients. Therefore Kähler potential and FM metric are expressed in terms of the same intersection matrix

$$e^{-K(\phi)} = \omega_\mu(\phi) C^{\mu\nu} \bar{\omega}_\nu(\phi),$$
$$h_{\alpha\beta} = \omega_{\alpha\mu}(0) C^{\mu\nu} \omega_{\beta\nu}(0),$$

The inverse intersection matrix $(C^{-1})_{\mu\nu} = q_\mu \cap q_\nu$.
$\omega_{\alpha\mu}(\phi)$ are the different bases of periods for the different α defined by integration over the same cycles.
Using these two formulae together with the relation

$$\omega^{\pm}_\mu(\phi) = T^\nu_\mu \, \sigma^{\pm}_\nu(\phi).$$

permit leads to the above result

$$e^{-K} = \sigma^+_\mu \eta^{\mu\rho} M^\nu_\rho \overline{\sigma^-_\nu}, \qquad M = T^{-1} \bar{T}.$$

We demonstrate our method on the famous Quintic hypersurface, Fermat surfaces and special more general CY, which are given by an equation with minimal number of monomials.

Special geometry

Recall the basic facts about the special Kähler geometry and how it arises on the CY moduli space.

Let moduli space $\mathcal{M}$ of complex structures of a given CY manifold is n-dimensional and $z^1 \cdots z^{n+1}$ are the special (projective) coordinates on it.
Then there exists a holomorphic homogeneous function $F(z)$ of degree 2 in z called a prepotential such that the Kähler potential $K(z)$ of the moduli space metric is given by

$$e^{-K(z)} = z^a \cdot \frac{\partial \bar{F}}{\partial \bar{z}^{\bar{a}}} - \bar{z}^{\bar{a}} \cdot \frac{\partial F}{\partial z^a}$$

This metric on the moduli space of complex structures is a metric that naturally arises from deWitt (=Polyakov) metric on a space of metrics on CY manifolds.

Special geometry on moduli spaces

Let X is CY three-fold and y^μ ($\mu = 1, 2, 3$) are complex coordinates on X.

The moduli space of X is the space of metric perturbations of X that preserve Ricci-flatness.

The metric on the complex structure CY moduli space obtained from natural metric for CY metric deformations of type $\delta_a g_{\mu\nu},\ \delta_{\bar{b}} g_{\bar{\mu}\bar{\nu}}$ preserving Ricci-flatness is

$$G_{a\bar{b}} = \int_X \mathrm{d}^6 y\ g^{1/2}\ g^{\mu\bar{\sigma}} g^{\nu\bar{\rho}} \delta_a g_{\mu\nu} \delta_{\bar{b}} g_{\bar{\sigma}\bar{\rho}}.$$

The deformations which leave the metric Ricci flat corresponds to elements in $H^{2,1}(X)$:

$$\delta g_{\bar{\alpha}\bar{\beta}} \to \chi_{\mu\nu\bar{\beta}} \sim \Omega_{\mu\nu\lambda} g^{\lambda\bar{\alpha}} \delta g_{\bar{\alpha}\bar{\beta}}$$

We can then rewrite the above metric as

$$G_{a\bar{b}} = \frac{\int_X \chi_a \wedge \bar{\chi}_{\bar{b}}}{\int_X \Omega \wedge \bar{\Omega}}.$$

$a, \bar{b}$ are indices of complex coordinates in the deformation space.

From the Kodaira Lemma:

$$\partial_a \Omega = k_a \Omega + \chi_a,$$

it follows that this metric is a Kähler :

$$G_{a\bar{b}} = -\partial_a \partial_{\bar{b}} \ln \int_X \Omega \wedge \bar{\Omega}$$

To obtain the bilinear formulae written above, define the basis of periods as integrals over Poincare dual symplectic basises $A^a, B_b \in H_3(X, \mathbb{Z})$ and $\alpha_a, \beta^b \in H^3(X, \mathbb{Z})$:

$$A^a \cap B_b = \delta^a_b, \qquad A^a \cap A^b = 0, \quad B_a \cap B_b = 0.$$

$$\int_{A^a} \alpha_b = \delta^a_b, \quad \int_{A^a} \beta^b = 0, \quad \int_{B_a} \alpha_b = 0, \quad \int_{B_a} \beta^b = \delta^b_a,$$

$$\int_X \alpha_a \wedge \beta^b = \delta^b_a, \qquad \int_X \alpha_a \wedge \alpha_b = 0 \qquad \int_X \beta^a \wedge \beta^b = 0.$$

With this, we decompose Ω as

$$\Omega = z^a \alpha_a + F_b \beta^b,$$

$$z^a = \int_{A^a} \Omega, \; F_b = \int_{B_b} \Omega.$$

We obtain

$$e^{-K} = \int_X \Omega \wedge \bar{\Omega} = z^a \cdot \bar{F}_{\bar{a}} - \bar{z}^{\bar{a}} \cdot F_a.$$

From the same lemma we obtain

$$\int_X \Omega \wedge \partial_a \Omega = F_a - z^b \partial_a F_b = 0.$$

It follows

$$F_a(z) = \frac{1}{2} \partial_a F(z),$$

where $F(z) = 1/2 z^b F_b(z)$.
So $G_{a\bar{b}}$ is the special Kähler metric with prepotential $F(z)$ and with special coordinates given by the periods.
Using the notation for the vector of periods,

$$\Pi = \left(F_\alpha, z^b\right)$$

we write the expression for the Kähler potential as

$$e^{-K(z)} = \Pi_a \Sigma^{ab} \bar{\Pi}_b,$$

where Σ is a symplectic unit, which is an inverse intersection matrix for cycles A^a and B_b.

CY as Hypersurface in a weighted projective space

Further, we concentrate on the case where the CY manifold is realized as a zero locus of a single polynomial equation in a weighted projective space.
Let $x_1, \ldots, x_5$ be homogeneous coordinates in a weighted projective space and

$$X = \{x_1, \ldots, x_5 \ \in \mathbb{P}^4_{(k_1,\ldots,k_5)} | \ W_0(x) = 0\}.$$

$W_0(x)$ is some quasi-homogeneous polynomial,

$$W_0(\lambda^{k_i} x_i) = \lambda^d W_0(x_i)$$

and

$$\deg W_0(x) = d = \sum_{i=1}^{5} k_i.$$

The last relation ensures that X is a CY manifold.

$W_0(x)$ defines an isolated singularity in the origin.
The moduli space of complex structures is then given by homogeneous polynomial deformations of this singularity modulo coordinate transformations:

$$W(x,\phi) = W_0(x) + \phi_0 \prod x_i + \sum_{s=0}^{\mu} \phi^s e_s(x),$$

$e_s(x)$ are polynomials of x with the same weight as $W_0(x)$. In this case, the holomorphic 3-form Ω is given as a residue of a 5-form in the underlying affine space $\mathbb{C}^5$:

$$\Omega = \frac{x_5 \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3}{\partial W(x)/\partial x_4} = \mathrm{Res}_{W(x)=0} \frac{x_5 \mathrm{d}x_1 \cdots \mathrm{d}x_4}{W(x)} =$$
$$= \frac{1}{2\pi i} \oint_{|x_5|=\delta} \mathrm{Res}_{W(x)=0} \frac{\mathrm{d}x_1 \cdots \mathrm{d}x_5}{W(x)},$$

where the last equality is due to the homogeneity of the integrand.

A basis of periods $\omega_\mu(\phi)$

Having explicit expression for Ω, we can define and compute a basis of periods $\omega_\mu(\phi)$ as follows.

We take a so-called fundamental cycle q_1, which is a torus in the large complex structure limit $\phi_0 >> 1$ (for simplicity other $\phi^s = 0$):

$$W(x,\phi) = W_0(x) + \phi_0 \prod x_i \ .$$

In this limit, we can define an 5−dimensional torus $Q_1 = |x_i| = \delta_i$ surrounding the hypersurface $W(x) = 0$ in $\mathbb{C}^5$. It corresponds to an 3-dimensional torus $q_1 \subset X$. Then the fundamental period is

$$\omega_1(\phi) := \int_{q_1} \Omega = \int_{Q_1} \frac{\mathrm{d}x^1 \cdots \mathrm{d}x^5}{W(x,\phi)}$$

and is given by a residue in its large ϕ_0 expansion.

More periods ω_μ may be obtained as analytic continuations of ω_1 in ϕ. This can be done by continuing $\omega_1(\phi)$ in a small ϕ_0 region using Barnes' trick and using the symmetry of $W_0(x)$ afterwards.

Namely, there is a group of *phase* symmetries Π_X acting diagonally on x_i and preserving $W_0(x)$.
When $W_0(x)$ is deformed, this group acts on a parameter space with an action $\mathcal{A}$ such that

$$W(g \cdot x,\ \mathcal{A}(g) \cdot \phi_0) = W(x, \phi).$$

The moduli space is then at most a factor of the parameter space $\{\phi^s\}/\mathcal{A}$.
This allows defining a set of other periods by analytic continuation,

$$\omega_{\mu_g}(\phi) = \omega_1(\mathcal{A}(g) \cdot \phi_0), \quad g \in G_X$$

In many cases this construction gives the whole basis of periods for the manifold X.

Periods as oscillatory integrals

The next important step is to transform the integrals for the periods $\int_{q_\mu} \Omega$ to the complex oscillatory form. First we have

$$\omega_\mu(\phi) := \int_{q_\mu} \Omega = \int_{Q_\mu} \frac{\mathrm{d}^5 x}{W(x)},$$

$q_\mu \in H_3(X),\ Q_\mu \in H_5(\mathbb{C}^5 \backslash W(x) = 0)$, and Q_μ is given by a tubular neighbourhood of q_μ. We can present them in the form

$$\int_{Q_\mu} \frac{\mathrm{d}^5 x}{W(x)} = \int_{Q_\mu^\pm} e^{\mp W(x)} \mathrm{d}^5 x$$

where $Q_\mu^\pm \in H_5(\mathbb{C}^5,\ \mathrm{Re} W_0(x) = \pm\infty)$. The map $Q_\mu \to Q_\mu^\pm$ is given by a contour deformation.

This deformation is performed due to the existence of a natural isomorphism

$$H_{r3}(X) \to H_5(\mathbb{C}^5 \backslash W(x) = 0) = H_5(\mathbb{C}^5,\ \mathrm{Re} W_0(x) = \pm\infty)_{w \in d \cdot \mathbb{Z}}$$

such that, indeed

$$Q_\mu^\pm \in H_5(\mathbb{C}^5,\ \mathrm{Re} W_0(x) = \pm\infty)_{w \in d \cdot \mathbb{Z}}$$

is a subgroup of $H_5(\mathbb{C}^5,\ \mathrm{Re} W_0(x) = \pm\infty)$ defined below.
The presentation as oscilatory integrals

$$\omega_\mu = \omega_\mu^\pm = \int_{Q_\mu^\pm} e^{\mp W(x)} \mathrm{d}^5 x$$

makes the computation of periods convenient. In terms of them

$$e^{-K} = \omega_\mu^+ C^{\mu\nu} \bar{\omega}_\nu^-,$$

where $C^{\mu\nu} = q_\mu \cap q_\nu = Q_\mu^+ \cap Q_\nu^-$ and $Q_\mu^\pm \cap Q_\nu^\pm = 0$.
So we need only to know the matrix $C^{\mu\nu}$.
To find $C^{\mu\nu}$ we use the fact that the CY moduli space is the marginal subspace of FM which arises on the deformations of the singularity $W_0(x)$.

Connection with Frobenius manifold structure.

The polynomial $W_0(x)$ in $\mathbb{C}^5$, which defines CY hypersurface in 4-dimensional weighted projective space, is a quasi-homogeneous polynomial:

$$W_0(\lambda^{k_i} x_i) = \lambda^d W_0(x)$$

with an isolated singularity in the origin.
Consider the Milnor ring of this singularity

$$R_0 = \frac{\mathbb{C}[x_1, \ldots, x_5]}{\partial_1 W_0(x) \cdot \ldots \cdot \partial_5 W_0(x)}.$$

Let $e_\mu(x)$ be a basis of this ring that consists of homogeneous monomials. There is a natural multiplication in R_0, and there is also a metric, turning the space of $e_\mu(x)$ into a Frobenius algebra.

The metric and struture constants are given by

$$\eta_{\mu\nu} = \mathrm{Res} \frac{e_\mu \cdot e_\nu}{\partial_1 W_0(x) \cdots \partial_5 W_0(x)},$$

$$C_{\mu\nu\lambda} = C^{\sigma}_{\mu\nu} \eta_{\sigma\lambda} = \mathrm{Res} \frac{e_\mu \cdot e_\nu \cdot e_\lambda}{\partial_1 W_0(x) \cdots \partial_5 W_0(x)}.$$

Consider the space of deformations of this singularity

$$W(x) = W_0(x) + \sum t^\mu e_\mu(x).$$

On the space with parameters t^μ arises the structure of Frobenius manifold $\mathcal{M}_F$ with the multiplication structure constants $C^{\rho}_{\mu\nu}(t)$ for the ring R defined by the deformed singularity $W(x)$

$$R = \frac{\mathbb{C}[x_1, \ldots, x_5]}{\partial_1 W(x) \cdot \ldots \cdot \partial_5 W(x)}.$$

and a Riemanian flat metric $h_{\mu\nu}(t)$. The metric $h_{\mu\nu}(t = 0)$ equal to $\eta_{\mu\nu}$. The structure constants are derivatives of Frobenius potential $F(t)$,

$$C^{\rho}_{\mu\nu}(t) h_{\rho\sigma} = \nabla_\mu \nabla_\nu \nabla_\sigma F(t),$$

where ∇_μ is Levi-Civita connection for $h_{\mu\nu}(t)$.

CY Moduli space as a subspace of the Frobenius manifold.

Consider the differentials D^+ and D^-

$$D^\pm = D^\pm_{W_0} = \mathrm{d} \pm \mathrm{d} W_0 \wedge .$$

The fifth cohomology groups $H^5_{D^\pm}(\mathbb{C}^5)$ of this differentials as linear spaces are isomorphic to the Milnor ring R

$$e_\mu(x) \to e_\mu(x)\mathrm{d}^5 x.$$

Cohomology group $H^5_{D^\mp}(\mathbb{C}^5)$ is dual to the homology group $H_5(\mathbb{C}^5, \mathrm{Re} W_0(x) = \mp\infty)$ if pairing between the groups defined as

$$\langle \Gamma^\pm_\mu,\ e_\nu \mathrm{d}^5 x\rangle = \int_{\Gamma^\pm_\mu} e_\nu \cdot e^{\mp W_0(x)} \mathrm{d}^5 x.$$

Then $H_5(\mathbb{C}^5,\ \mathrm{Re} W_0(x) = \pm\infty)_{w \in d\cdot\mathbb{Z}} \subset H_5(\mathbb{C}^5,\ \mathrm{Re} W_0(x) = \pm\infty)$, is a subgroup which consists of elements of dual to $e_\mu(x)\mathrm{d}^5 x$ $\in H^5_{D^\pm}(\mathbb{C}^5,\ \mathrm{Re} W_0(x) = \pm\infty)_{w\in d\cdot\mathbb{Z}}$ such that weight of $e_\mu(x)$ is divisible by weight of the singularity: $[e_\mu(x)] \in d \cdot \mathbb{Z}$.

This is precisely the subgroup invariant under $x_i \to e^{2\pi i k_i/d} x_i$.
Using this duality, we define a set of cycles $\Gamma^{\pm}_{\mu}$ in the group $H_5(\mathbb{C}^5,\ \mathrm{Re} W_0(x) = \pm\infty)_{w\in d\cdot\mathbb{Z}}$ by requiring that

$$\int_{\Gamma^{\pm}_{\mu}} e_\nu \cdot e^{\mp W_0(x)} \mathrm{d}^5 x = \delta^{\mu}_{\nu}.$$

The convenient computation technique in $H_{D^\pm}(\mathbb{C}^5)$ can be used to compute the integrals

$$\int_{\Gamma^{\pm}_{\mu}} e_\nu \cdot e^{\mp W(x,\psi)} \mathrm{d}^5 x.$$

This technique is based on the fact that

$$\int_{\Gamma^{\pm}_{\mu}} P(x) e^{-W_0(x)} \mathrm{d}^5 x = \int_{\Gamma^{\pm}_{\mu}} \tilde{P}(x) e^{-W_0(x)} \mathrm{d}^5 x$$

if the differential forms are equivalient in $D^{\pm}$ cohomology

$$\left(P(x) - \tilde{P}(x)\right) \mathrm{d}^5 x = D^{\pm} U.$$

This reduces the problem to a system of linear equations.

For a generic deformation $W(x) = W_0(x) + \sum t^{\mu} e_{\mu}(x) = 0$ does not define a surface in a projective space.
This only occurs when $W(x)$ is quasihomogeneous, i.e. in a case of marginal deformations or deformations that have the same scaling property as $W_0(x)$.

We let $\{\phi^s\} \subset \{t^{\alpha}\}$ to denote the marginal deformation parameters.
Thus, the marginal deformations $W_0(x) + \sum \phi^s e_s(x)$ define a subspace of a total Frobenius manifold connected with W_0.
This subspace of the FM coincides with the moduli space of the CY manifold.

Computing the Kähler potential

We use the connection of the CY moduli space to the corresponding FM to find the inverse intersection matrix of the cycles $C^{\mu\nu}$, $q_\mu \cap q_\nu = Q^+_\mu \cap Q^-_\nu$.

To do this, introduce a few additional basises of periods $\omega^{\pm}_{\alpha,\mu}(\phi)$ as integrals of $e_\alpha(x)\mathrm{d}^5 x \in H^5_{D^\pm}(\mathbb{C}^5,\ \mathrm{Re} W_0(x) = \pm\infty)_{w\in d\cdot\mathbb{Z}}$ over the cycles $Q^{\pm}_\mu \in H_5(\mathbb{C}^5,\ \mathrm{Re} W_0(x) = \pm\infty)_{w\in d\cdot\mathbb{Z}}$ that have been defined earlier:

$$\omega^{\pm}_{\alpha\mu}(\phi) = \int_{Q^{\pm}_\mu} e_\alpha(x)\, e^{\mp W(x,\phi)} \mathrm{d}^5 x.$$

In particular, the periods $\omega^{\pm}_{1\mu}(\phi)$ coincide with the periods $\omega^{\pm}_{\mu}(\phi)$ defined above since we assume that $e_1(x) = 1$ denotes the unity in the ring R.

The crucial fact for possibility to compute $C^{\mu\nu}$ is its connection with the FM metric $h_{\alpha\beta}(t=0)$ as:

$$\eta_{\alpha\beta} = \omega^{+}_{\alpha,\mu}(t=0)\, C^{\mu\nu} \omega^{-}_{\beta,\nu}(t=0)$$

So we need to prove the relation

$$h_{ab}(t=0) = \mathrm{Res}\frac{e_a \cdot e_b \, \mathrm{d}^n x}{\partial_1 W_0 \cdots \partial_n W_0} =$$
$$= \int_{Q_\mu^+} e_a \; e^{-W_0} \mathrm{d}^n x \; C^{\mu\nu} \int_{Q_\nu^-} e_b \; e^{W_0} \mathrm{d}^n x$$

To do this consider a small perturbation $W(x,t) = W_0(x) + t_a e_a$, so that 0 - critical point of W becomes a set of Morse points $p_1, \ldots, p_\mu$ and consider a bilinear form

$$h_{ab}(t,z) = \int_{Q_\mu^+} e_a \; e^{-W(x,t)/z} \mathrm{d}^n x \; C^{\mu\nu} \int_{Q_\nu^-} e_b \; e^{W(x,t)/z} \mathrm{d}^n x$$

Notice, that

$$h_{ab}(t=0,z) = z^k \cdot h_{ab}(t=0, z=1),$$

because if $t = 0$, we can absorb z by coordinate transform $x_i \to z^{k_i/d} x_i$.

We can choose basis of cycles : $L_i^{\pm}$ to start from p_i and go along the gradient of $\mathrm{Re}(W(x,t))$ in positive/negative direction and their intersections $L_i^+ \cap L_j^- = \delta_{ij}$.
In this basis rhs becomes:

$$\sum_{i=1}^{\mu} \int_{L_i^+} e_a \; e^{-W(x,t)/z} \mathrm{d}^n x \; \int_{L_i^-} e_b \; e^{W(x,t)/z} \mathrm{d}^n x$$

Using stationary phase expansion as $z \to 0$ we obtain for a period:

$$\int_{L_i^+} e_a(x) \; e^{-W(x,t)/z} \mathrm{d}^n x = \pm \frac{(2\pi z)^{N/2}}{\sqrt{\mathrm{Hess}\, W(p_i, t)}} (e_a(p_i) + O(z))$$

From this we get

$$h_{ab}(t,z) = \pm \sum_{i=1}^{\mu} (2\pi i z)^N \frac{e_a(p_i) \cdot e_b(p_i)}{\mathrm{Hess}(W(p_i,t))} (1 + O(z)) =$$

$$= (2\pi i z)^N \left(\mathrm{Res} \frac{e_a \cdot e_b \mathrm{d}^n x}{\partial_1 W \cdots \partial_N W} + O(z) \right)$$

By analytic continuation it holds for $t = 0$. Also we have $h_{ab}(0,z) = z^k \cdot h_{ab}(0,1)$. The above equality now follows from the previous formula.

This formula helps to obtain the expression for $C^{\mu\nu}$ if we know values of $\omega^+_{\alpha,\mu}(t=0)$ for all α.
From the their definition

$$\omega^{\pm}_{\alpha\mu}(\phi) = \int_{Q^{\pm}_{\mu}} e_{\alpha}(x)\, e^{\mp W(x,\phi)} \mathrm{d}^5 x.$$

we can see that $\omega^+_{\alpha,\mu}(t=0)$ is expressed in terms of a few first derivatives over ϕ of the periods $\omega^{\pm}_{\mu}(\phi)$ for $\phi = 0$. Denote

$$\omega^{\pm}_{\alpha,\mu}(\phi=0) := (T^{\pm})^{\alpha}_{\mu}.$$

From the eq-n above we have

$$\eta^{\mu\nu} = (T^+)^{\mu}_{\rho}\; C^{\rho\sigma}\; (T^-)^{\nu}_{\sigma}.$$

Expressing the intersection matrix $C^{\rho\sigma}$ in terms Frobenius metric $\eta^{\mu\nu}$ and matrix T we insert it to the Kahler potential formula

$$e^{-K(\phi)} = \omega_{\mu}(\phi) C^{\mu\nu} \bar{\omega}_{\nu}(\phi)$$

to obtain the explicit expression for $K(\phi)$.

To get more covenient expression for $K(\phi)$ we define one more basis of periods $\sigma^{\pm}_{\mu}(\phi)$ as integrals over the cycles $\Gamma^{\pm}_{\mu} \in H_5(\mathbb{C}^5,\ \mathrm{Re} W_0(x) = \pm\infty)_{w \in d\cdot\mathbb{Z}}$ defined above:

$$\sigma^{\pm}_{\mu}(\phi) = \int_{\Gamma^{\pm}_{\mu}} e^{\mp W(x,\phi)} \mathrm{d}^5 x,$$

Once we have an oscillatory representation for the periods $\sigma^{\pm}_{\mu}(\phi)$ over the corresponding cycles $\Gamma^{\pm}_{\mu}$, we can define additional integrals $\sigma^{\pm}_{\alpha,\mu}(\phi)$ over the same cycles as

$$\sigma^{\pm}_{\alpha,\mu}(\phi) = \int_{\Gamma^{\pm}_{\mu}} e_{\alpha}(x)\, e^{\mp W(x,\phi)} \mathrm{d}^5 x$$

It follows from $e_1(x) = 1$ that $\sigma^{\pm}_{1\mu} = \sigma^{\pm}_{\mu}$. Due to our choice of the cycles $\Gamma^{\pm}_{\mu}$ we also have $\sigma^{\pm}_{\alpha,\mu}(t = 0) = \delta_{\alpha,\mu}$.

The construction

Since both $\omega_\mu^\pm(\phi)$ and $\sigma_\nu^\pm(\phi)$ are basises of periods defined as the integrals over the cycles in $H_5(\mathbb{C}^5,\ \mathrm{Re}W_0(x)=\pm\infty)_{w\in d\cdot\mathbb{Z}}$, they are connected by some constant matrix $(T^\pm)_\mu^\nu$:

$$\omega_\mu^\pm(\phi)=(T^\pm)_\mu^\nu\,\sigma_\nu^\pm(\phi).$$

To find T, it suffices to take a few first terms of the expansion over ϕ of the periods $\omega_\mu^\pm(\phi)$ and $\sigma_\mu^\pm(\phi)$. The same relation connects periods $\omega_{\alpha\mu}^\pm(\phi)$ and $\sigma_{\alpha\nu}^\pm(\phi)$ for each α.
Knowing that $\sigma_{\alpha,\mu}^\pm(\phi=0)=\delta_{\alpha,\mu}$, we obtain

$$\omega_{\alpha,\mu}^\pm(\phi=0)=(T^\pm)_\mu^\alpha.$$

From above eq-n we then obtain

$$\eta^{\mu\nu}=(T^+)_\rho^\mu\ C^{\rho\sigma}\ (T^-)_\sigma^\nu.$$

So we express the intersection matrix $C^{\rho\sigma}$ in terms of the known Frobenius metric $\eta^{\mu\nu}$ and the also known matrix T.

Main statement.

Thus we arrive to the main statement that

$$e^{-K(\phi)} = \sigma_\mu(\phi)\, \eta^{\mu\nu}\ M_\nu^\lambda\, \overline{\sigma_{\lambda}^{-},(\phi)}$$

where the matrix $M_b^a = (T^{-1})_c^a\ \bar{T}_b^c$.

It gives an explicit expression for the Kähler potential K in terms of the periods $\sigma_\mu(\phi)$, FM metric $\eta_{\mu\nu}$ and matrix T_ν^μ.
All these data can be computed exactly as it has been explained above.
It makes sense to stress that having the exact expression for $\omega_\nu^\pm(\phi)$, we can obtain the exact and explicit expressions for the periods $\sigma_\mu^\pm(\phi)$:

$$\sigma_\mu^\pm(\phi) = \left((T^\pm)^{-1}\right)_\mu^\nu\ \omega_\nu^\pm(\phi).$$

In terms of the periods $\sigma_\mu^\pm(\phi)$ expression for the Kähler potential has a convenient form for calculating the metric on the CY moduli space.

Example 1: Quintic

The one-parameter family of CY manifold is defined as

$$X_\psi = \{x_i \in \mathbb{P}^4 \mid W_\psi(x) = x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 - \\ - 5\psi x_1 x_2 x_3 x_4 x_5 = 0\}.$$

In this case, the phase symmetry is $\mathbb{Z}_5^5$ and the induced action $\mathcal{A}$ on the one-dimensional space $\{\psi\}$ is $\mathbb{Z}_5 : \psi \to e^{2\pi i/5}\psi$.

That is the whole complex structure moduli space of the quotient $X/\mathbb{Z}_5^3 =: \hat{X}$, that is the mirror manifold of the original quintic. In particular, $h_{1,1}(\hat{X}) = 101,\ h_{2,1}(\hat{X}) = 1$.

We choose cycles $\Gamma_\mu^\pm$ dual to the cohomology classes $\mathrm{d}^5 x,\ \prod x_i \cdot \mathrm{d}^5 x,\ \prod x_i^2 \cdot \mathrm{d}^5 x,\ \prod x_i^3 \cdot \mathrm{d}^5 x$, a basis in the cohomology subgroup invariant under the $\mathbb{Z}_5^3$.

For the periods, the recursion procedure gives:

$$\sigma_\mu^\pm(\psi) = \frac{(\pm 1)^{\mu-1}}{\Gamma(\mu/5)^5 5^\mu \psi} \sum_{n=0}^{\infty} \frac{\Gamma^5(n+\mu/5)}{\Gamma(5n+\mu)} (5\psi)^{5n+\mu} = \\ = \frac{(\pm\psi)^{\mu-1}}{\Gamma(\mu)} + O(\psi^{\mu+3})$$

The fundamental period for the quintic is defined as a residue of a holomorphic three-form Ω

$$\frac{x_5 \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3}{\partial P_\psi / \partial x_4},$$

and given by an integral over a cycle q_1, which is three-dimensional torus. Its analytic continuations as explained give the whole basis of periods in a basis of cycles with integral coefficients:

$$\omega_\mu(\psi) = \sum_{m=1}^{\infty} \frac{e^{4\pi i m/5} \Gamma(m/5)(5e^{2\pi i(\mu-1)/5}\psi)^{m-1}}{\Gamma(m)\Gamma^4(1-m/5)}, \qquad |\psi| < 1,$$

Taking the first four terms of the expansion of the periods above we obtain

$$T^\mu_\nu = \frac{5^{\nu-1} e^{2\pi i((\nu-1)(\mu-1)+2\nu)/5} \Gamma(\nu/5)}{\Gamma^4(1-\nu/5)},$$

The FM holomorphic metric in this case

$$\eta = \mathrm{antidiag}(1,1,1,1).$$

Finally we obtain $\hat{\eta} = \eta T^{-1} \bar{T}$ and Kähler potential for the metric:

$$e^{-K(\psi)} = \frac{\Gamma^5(1/5)}{125\Gamma^5(4/5)}\sigma_{11}^{+}\overline{\sigma_{11}^{-}} + \frac{\Gamma^5(2/5)}{5\Gamma^5(3/5)}\sigma_{12}^{+}\overline{\sigma_{12}^{-}} + \\ + \frac{5\Gamma^5(3/5)}{\Gamma^5(2/5)}\sigma_{13}^{+}\overline{\sigma_{13}^{-}} + \frac{125\Gamma^5(4/5)}{\Gamma^5(1/5)}\sigma_{14}^{+}\overline{\sigma_{14}^{-}}.$$

In particular,

$$G_{\psi\overline{\psi}}(0) = 25\frac{\Gamma^5(4/5)\Gamma^5(2/5)}{\Gamma^5(1/5)\Gamma^5(3/5)}$$

that coincides with the famous result by Candelas et al.

Example 2: Fermat hypersurface

The direct generalization of the quintic is a Fermat hypersurface, which is the one given by the equation

$$W_0(x) = \sum_{i=1}^{5} x_i^{n_i}, \qquad n_i = d/k_i, \quad \sum k_i = d,$$

and the degree d is equal to the least common multiple of $\{k_i\}$. As in the case above, we consider a one-dimensional deformation $W(x,\phi_0) = W_0(x) + \phi_0 \prod_{i=1}^{5} x_i$. The phase symmetry group is $\Pi_X = \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_5}$. The lifted action on ϕ_0 is $\mathbb{Z}_d : \ \phi_0 \to \zeta\phi_0, \ \zeta = e^{2\pi i/d}$. We take the expression for the fundamental period the known result by Berglund et al:

$$\omega_1(\phi_0) == \sum_{\mu=1}^{d-1} A(\mu) \frac{\phi_0^{\mu-1}}{\Gamma(\mu)} + O(\phi_0^{d-1}).$$

and

$$A(\mu) = \frac{(-1)^{\mu-1} e^{\frac{-\pi i \mu}{d}}}{\sin\frac{\mu\pi}{d} \prod_{i=1}^{5} \Gamma(1 - \frac{k_i\mu}{d})}$$

We note that $A(\mu)$ vanishes if $k_i\mu/d \in \mathbb{Z}$, i.e. $\mu/n_i \in \mathbb{Z}$. According to the general analytic continuation procedure

$$\omega_\mu(\phi_0) = \sum \zeta^{(\nu-1)(\mu-1)} A(\nu) \frac{\phi_0^{\nu-1}}{\Gamma(\nu)} + O(\phi_0^{d-1}).$$

Using the definitions for $\sigma_\mu^+(\phi_0)$ we obtain

$$\sigma_\mu^+(\phi_0) = \frac{\phi_0^{\mu-1}}{\Gamma(\mu)} + O(\phi_0^{\mu+d-2}), \qquad \mu/n_i \notin \mathbb{Z}, \text{ otherwise } 0$$

This latter condition implies that ω_μ form a basis in the periods of Ω deformed by ϕ_0. We obtain the transition matrix

$$T_\nu^\mu = \zeta^{(\mu-1)(\nu-1)} A(\mu), \qquad \mu/n_i \notin \mathbb{Z}, \qquad \nu/n_i \notin \mathbb{Z}$$
$$(T^{-1})_\mu^\lambda = \frac{\bar{\zeta}^{(\lambda-1)(\nu-1)}}{\tilde{d}-1} \frac{1}{A(\mu)}$$

and the real structure

$$M_\nu^\mu = \frac{\bar{A}(\mu)}{A(d-\mu)} \delta_{\mu+\nu,d} \; .$$

In this case, $\eta_{\mu,\nu} = \delta_{\mu+\nu,d}$ and therefore

$$e^{-K(\phi_0)} = \sum_{\mu=1,\ \mu/n_i \notin \mathbb{Z}}^{d-1} \prod_{i=1}^{5} \gamma\left(\frac{k_i \mu}{d}\right) \sigma_\mu^+(\phi_0)\overline{\sigma_\mu^-(\phi_0)}$$

where $\gamma(x) = \Gamma(x)/\Gamma(1-x)$ and

$$\sigma_\mu^{\pm}(\phi_0) = \pm \sum_{R=0}^{\infty} \frac{\phi_0^{\mu-1+dR}}{\Gamma(dR+\mu)} \prod_{j=1}^{5} \frac{\Gamma\left(k_j(R+\frac{\mu}{d})\right)}{\Gamma(\frac{k_j\mu}{d})}$$

From this we get a formula for the metric itself

$$G_{\phi_0\overline{\phi_0}} = \prod_{i=1}^{5}\left(\gamma\left(\frac{k_i\mu_0}{d}\right)\gamma\left(1-\frac{k_i}{d}\right)\right)\frac{|\phi_0|^{2(\mu_0-1)}}{\Gamma(\mu_0)^2} + O(|\phi_0|^{2\mu_0}),$$

μ_0 is the least integer $1 \le \mu_0 < d$ such that $(\mu_0+1)/n_j \neq \mathbb{Z}$.
The last formula reproduces the known results for CY manifolds $\mathbb{P}^4_{(2,1,1,1,1)}[6]$, $\mathbb{P}^4_{(4,1,1,1,1)}[8]$ and $\mathbb{P}^4_{(5,2,1,1,1)}[10]$ obtained by Klemm and Theisen.

Example 3: Sums of 5 monomials

We assume that the above approach is applicable to the case of CY manifold defined in terms of the hypersurface in weighted projective spaces defining polynomial is

$$W_0(x) = \sum_{j=1}^{5} \prod_{i=1}^{5} x_i^{a_{ij}}, \qquad \sum k_i a_{ij} = d,$$

and

$$\sum k_i = d.$$

In this case periods are given in terms of the *mirror* CY manifold $\hat{X}$. The polynimial $W_0(x)$ has a group Π_X of phase symmetries represented as

$$\Pi_X = Q_X \times G_X,$$

where Q_x, a *quantum symmetry* group $\simeq (\mathbb{Z}_d : k_1, \cdots, k_5)$, acts as $x_i \to e^{2\pi i k_i/d}$. We note that action of the quantum symmetries on X is trivial. The complement to Q_X in Π_X is called a geometric symmetry group G_X.

For mirror manifolds the total phase symmetry is unchanged whereas roles of quantum and geometric symmetries switch:

$$G_X = Q_{\hat{X}}, \qquad Q_X = G_{\hat{X}}.$$

To build such a mirror, we must first to consider a polynomial $\hat{W}_0(x)$ with a transposed matrix of exponents $\hat{a}_{ij} = a_{ji}$,

$$\hat{W}_0(x) = \sum_{j=1}^{5} \prod_{i=1}^{5} x_i^{\hat{a}_{ij}}, \qquad \sum \hat{k}_i a_{ji} = \hat{d},$$

and

$$\sum \hat{k}_i = \hat{d}.$$

Here $\hat{k}_i$ and $\hat{d}$ are uniquely defined by the reqirement that the equalities above are satisfied.
This polynomial has the same group of phase symmetries, however generically the needed condition is not fulfiled, i.e. its quantum symmetry is smaller, than geometric symmetry of the original hypersurface.

To get a mirror we need to enlarge quantum symmetry of $\{\hat{W}_0(x)=0\}$. For this purpose we take a quotient of the hypersurface $\{\hat{W}_0(x)=0\}/H$, where H is some subgroup of phase symmetries which is to be found in each case.
Thus, computing complex moduli space for the manifold X (or $\hat{X}$) we compute also a complexified Kähler moduli space metric for the mirror CY through the mirror map.
The periods $\omega_\mu(\phi)$ in this case were computed earlier and, if we set all parameters ϕ^s (but not ϕ_0) equal to zero for simplicity , then we have:

$$\omega_1(\phi_0)=\sum_{r=1}^{\hat{d}-1} A(r)\frac{\phi_0^{r-1}}{\Gamma(r)}+O(\phi_0^{\hat{d}-1})$$

$$A(\mu)=(-1)^\mu\frac{\pi}{\hat{d}\sin\frac{\pi\mu}{\hat{d}}}\prod_{j=1}^{5}\frac{1}{\Gamma(1-\frac{\hat{k}_j\mu}{\hat{d}})}.$$

For our general method to work, this must give all relevant periods. Basically we must check that all possible periods are obtained from this one (with all $\phi^s\neq 0$) by phase-symmetry analytic continuations.

In other words it is necessary to verify the relation

$$\dim\langle\omega_0(\mathcal{A}(g)\cdot\phi)\rangle_{g\in G_X} = \dim H_3(X).$$

This was certainly the case in the preceding examples, but not in this case, we are not aware of this fact in general (it is so in all examples). As in the previous example, in the one-modulus case we obtain

$$e^{-K(\phi_0)} = \sum_{\mu=1,\ \mu\hat{k}_i/\hat{d}\notin\mathbb{Z}}^{\hat{d}-1} \eta^{\mu,\hat{d}-\nu}\prod_{j=1}^{5}\gamma\left(\frac{\hat{k}_j\mu}{\hat{d}}\right)\sigma_\mu^+(\phi_0)\overline{\sigma_\nu^-(\phi_0)}.$$

For this formula to hold the number of linearly independent elements $\prod_{i=1}^5 x_i^n \mathrm{d}^5x \in H^5_{D^\pm}(\mathbb{C}^5)$ should be equal to the number of $1 \le \mu < \hat{d},\ \mu k_i/d \neq \mathbb{Z}$.

Conclusion

A new method for computing the metric of CY moduli space is proposed.This method does not demand using of Picard–Fuchs equations. Instead, the cohomology technique for computing periods can be applied. It can be used for the computations of the CY moduli space geometry in cases when the dimesion of the moduli space more than one.

The FM structure naturally arising from an N=2 SCFT plays a significant role. The result is given in terms of the topological metric on FM and two basises of periods, both of which we are able to compute avoiding the complicated direct computation of the symplectic basis of periods.

The method was used here for CY manifolds, given by one polynomial equation, such as the case of Fermat hypersurfaces. We suppose the same approach can be applied to CY manifolds of a more general type.