The Li comparison principle and its elaboration

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In memory of Prof. Wenbo Li

The problem of small ball probabilities:

x is random variable;

$$P\{|x| \leqslant \varepsilon\} \sim ?, \qquad \varepsilon \to 0.$$

Basic example: $x = \|X\|_{\mathfrak{X}}$, where X is a Gaussian random vector in a locally convex space $\mathfrak{X}$.

Lifshits M. A., *Gaussian Random Functions*, Kluwer, Dordrecht, 1995, 333 pp.

Li W. V., Shao Q.-M., *Gaussian processes: inequalities, small ball probabilities and applications*, in "Stochastic Processes: Theory and Methods", Handbook of Statist. **19**, 533-597. North-Holland, Amsterdam, 2001.

Lifshits M. A., *Asymptotic behavior of small ball probabilities*, in "Probability Theory and Mathematical Statistics": Proceedings of the Seventh International Vilnius Conference, 453-468, TEV, Vilnius, 1999.

Li W. V., "Comparison results for the lower tail of Gaussian seminorms", *J. Theoret. Probab.* **5** (1992), 1-31.

Let ξ_j, $j \in \mathbb{N}$, be independent standard Gaussian r.v.s, and let $\{\lambda_j\}$ and $\{\tilde\lambda_j\}$ be two non-increasing, summable sequences of positive numbers. Then, as $\varepsilon \to 0$,

$$\mathbf{P}\Big\{\sum_1^\infty \lambda_j \xi_j^2 \leqslant \varepsilon\Big\} \sim \Big(\prod_1^\infty \frac{\tilde\lambda_j}{\lambda_j}\Big)^{\frac{1}{2}} \cdot \mathbf{P}\Big\{\sum_1^\infty \tilde\lambda_j \xi_j^2 \leqslant \varepsilon\Big\},$$

provided

$$\sum_1^\infty \Big|1 - \frac{\lambda_j}{\tilde\lambda_j}\Big| < \infty.$$

Some generalizations:

Gao F., Hannig J., Lee T.-Y., Torcaso F., *Exact L^2 small balls of Gaussian processes,* J. Theor. Prob. **17** (2004), N2, 503-520.

A weaker assumption

$$\prod_{1}^{\infty} \frac{\tilde{\lambda}_j}{\lambda_j} < \infty.$$

Gao F., Hannig J., Torcaso F., *Comparison theorems for small deviations of random series*, Electron. J. Probab. **8** (2003), N21, 1-17.

$$\mathbf{P}\Big\{ \sum_{1}^{\infty} \lambda_j |\xi_j|^p \leqslant \varepsilon \Big\} \sim \Big(\prod_{1}^{\infty} \frac{\tilde{\lambda}_j}{\lambda_j} \Big)^{\frac{1}{p}} \cdot \mathbf{P}\Big\{ \sum_{1}^{\infty} \tilde{\lambda}_j |\xi_j|^p \leqslant \varepsilon \Big\},$$

for any $p > 0$.

(conjectured by W. V. Li (1992))

Why is the case $p = 2$ most important?

Let $\mathfrak{X}$ be a Hilbert space, and let $X \in \mathfrak{X}$ be a zero-mean Gaussian random vector. Then, by the Karhunen-Loève expansion, we have the distributional equality

$$\|X\|_{\mathfrak{X}}^2 = \sum_1^\infty \lambda_j \xi_j^2,$$

where ξ_j, $j \in \mathbb{N}$, are independent standard Gaussian r.v.s while $\lambda_j > 0$ are the eigenvalues of the **covariance operator**

$$\mathcal{G}_X f \equiv \mathbf{E}(f, X)_{\mathfrak{X}} X = \lambda f.$$

Note that we require $\sum_1^\infty \lambda_j < \infty$, otherwise $\|X\|_{\mathfrak{X}} = \infty$ a.s.

This allows us to obtain the small ball probability results from the information on the covariance operator.

Namely, for a sufficiently simple "comparison sequence" $\tilde{\lambda}_j$, the asymptotics of $\mathbf{P}\Big\{\sum\limits_1^\infty \tilde{\lambda}_j \xi_j^2 \leqslant \varepsilon\Big\}$ as $\varepsilon \to 0$ can be obtained by the method of Sytaya (1974) developed by Lifshits (1997) and Dunker-Lifshits-Linde (1998).

On the another hand, if $\tilde{\lambda}_j$ give a good approximation of eigenvalues of $\mathcal{G}_X$ then the Li relation provides the small ball asymptotics for X up to the **distortion constant** $\left(\prod\limits_1^\infty \frac{\tilde{\lambda}_j}{\lambda_j}\right)^{\frac{1}{2}}$.

This connects the small ball probabilities with the spectral asymptotics of self-adjoint compact (really, **trace class**) operators. In what follows, we suppose that $\mathfrak{X}$ is conventional functional space $L_2(0,1;\mu)$ where μ is a measure. The covariance operator in this case can be represented as integral operator; its kernel is **the covariance function**

$$G_X(s,t) = \mathbf{E}X(s)X(t), \qquad s,t \in [0,1].$$

However, in general case the problem on sharp (for example, two-term) spectral asymptotics of integral operators is quite difficult and sensitive to small perturbations of the kernel.

Nazarov-Nikitin (2004) and then Nazarov (2009) emphasized the concept of the **Green** process, i.e. Gaussian process with covariance being the Green function for a self-adjoint **ordinary differential** operator. This allows us to use a powerful machinery of ODE spectral theory. The approach developed in these papers permits one to obtain the sharp (up to a constant) asymptotics of small deviations in L_2-norm for this class of processes.

In the papers

Nazarov A. I., *On the sharp constant in the small ball asymptotics of some Gaussian processes under L_2-norm*, Probl. Mat. Anal. **26** (2003), 179-214 (in Russian). English transl.: J. Math. Sci. (N.Y.) **117** (2003), 4185-4210.

Nazarov A. I., Pusev R. S., *Exact small deviation asymptotics in L_2-norm for some weighted Gaussian processes*, ZNS POMI **364** (2009), 166-199 (in Russian). English transl.: J. Math. Sci. (N.Y.) **163** (2009), 409-429.

Pusev R. S., *Small deviations asymptotics for Matérn processes and fields under weighted quadratic norm*, Teor. Veroyatn. Primen. **55** (2010), 187-195 (in Russian). English transl.: Theory Probab. Appl. **55** (2011), 164-172.

Pusev R. S., *Asymptotics of small deviations of the Bogoliubov processes with respect to a quadratic norm*, Teoret. Mat. Fiz. **165** (2010), 134-144 (in Russian). English transl.: Theoret. Math. Phys. **165** (2010), 1349-1358.

using this approach, we have calculated the sharp asymptotics of small ball probabilities for a large class of particular processes and for various (absolutely continuous) measures. The distortion constants were obtained by the complex variable methods.

Nazarov A. I., Pusev R. S., *Comparison theorems for the small ball probabilities of Green Gaussian processes in weighted L_2-norms*, Algebra and Analysis **25** (2013), N3, to appear (in Russian). English transl. is available at http://arxiv.org/abs/1211.2344

Let the covariance of a zero-mean Gaussian process $X(t)$, $0 \leqslant t \leqslant 1$, be the Green function of a regular self-adjoint ODO L of order $2n$ with sufficiently smooth coefficients. Suppose that the weight functions ψ_1, ψ_2 are sufficiently smooth, bounded away from zero, and

$$\int_0^1 \psi_1^{\frac{1}{2n}}(x)dx = \int_0^1 \psi_2^{\frac{1}{2n}}(x)dx = \vartheta. \qquad (1)$$

Then

$$\lim_{\varepsilon\to 0} \frac{\mathbf{P}(\|X\|_{L_2(0,1;\psi_1)} \le \varepsilon)}{\mathbf{P}(\|X\|_{L_2(0,1;\psi_2)} \le \varepsilon)} = \left|\frac{\theta(\psi_2)}{\theta(\psi_1)}\right|^{1/2}, \qquad (2)$$

where $\theta(\psi)$ is a determinant with entries depending on the boundary conditions of L and on the values $\psi(0)$ and $\psi(1)$.

If boundary conditions of the operator L are **separated** then formula (2) can be significantly simplified:

$$\lim_{\varepsilon\to 0}\frac{\mathbf{P}(\|X\|_{L_2(0,1;\psi_1)}\le\varepsilon)}{\mathbf{P}(\|X\|_{L_2(0,1;\psi_2)}\le\varepsilon)}= \\ =\left(\frac{\psi_2(0)}{\psi_1(0)}\right)^{-\frac{n}{4}+\frac{1}{8}+\frac{\varkappa_0}{4n}}\left(\frac{\psi_2(1)}{\psi_1(1)}\right)^{-\frac{n}{4}+\frac{1}{8}+\frac{\varkappa_1}{4n}},$$

where $\varkappa_0$ and $\varkappa_1$ stand for the sums of orders of boundary conditions at zero and one, respectively.

Remark. One can see that the small ball probabilities of the Green Gaussian process in a weighted L_2-norm under normalization assumption (1) do not depend on the values of the weight function at interior points of the interval $(0,1)$. This phenomenon is well known in the spectral theory of ODOs. It is quite interesting to explain it in terms of random processes.

If the assumption (1) does not hold then the probabilities $\mathbf{P}(\|X\|_{L_2(0,1;\psi_{1,2})}\le\varepsilon)$ have different **logarithmic** asymptotics.

Nazarov A. I., *On a set of transformations of Gaussian random functions*, Teor. Ver. Primen. **54** (2009), N2, 209-225 (in Russian). English transl.: Theor. Probab. Appl. **54** (2010), N2, 203-216.

We return to a general case. Let X be a zero-mean Gaussian random vector in a Hilbert space $\mathfrak{X}$. Let φ be a linear measurable functional of X. Consider a set of one-dimensional linear perturbations of X:

$$\mathcal{X}_{\varphi,\alpha} = X - \alpha\psi\langle\varphi, X\rangle,$$

where $\psi = \mathbf{E}\, X\langle\varphi, X\rangle$, $\alpha \in \mathbb{R}$.

Set $q = \langle\varphi, \psi\rangle$. If $\alpha \neq 1/q$, then, as $\varepsilon \to 0$,

$$\mathbf{P}\left\{\|\mathcal{X}_{\varphi,\alpha}\|_{\mathfrak{X}} \leqslant \varepsilon\right\} \sim \frac{1}{|1-\alpha q|} \cdot \mathbf{P}\left\{\|X\|_{\mathfrak{X}} \leqslant \varepsilon\right\}.$$

For $\widehat{\alpha} = 1/q$, the result is more complicated. If $\varphi \in \mathfrak{X}$, then, as $\varepsilon \to 0$,

$$\mathbf{P}\left\{\|\mathcal{X}_{\varphi,\widehat{\alpha}}\|_{\mathfrak{X}} \leqslant \varepsilon\right\} \sim$$

$$\sim \frac{\sqrt{q}}{\|\varphi\|_{\mathfrak{X}}}\sqrt{\frac{2}{\pi}}\int_0^{\varepsilon^2} \frac{d}{dt}\mathbf{P}\left\{\|X\|_{\mathfrak{X}} \leqslant t\right\} \frac{dt}{\sqrt{\varepsilon^2 - t^2}}.$$

Nazarov A. I., *Log-level comparison principle for small ball probabilities*, Stat. and Prob. Letters **79** (2009), N4, 481-486.

If we cannot derive the exact asymptotics, we restrict ourselves to **logarithmic** asymptotics, that is the asymptotics of $\ln \mathbf{P}\{||X||_{\mathfrak{X}} \leq \varepsilon\}$ as $\varepsilon \to 0$.

We introduce the **counting function**

$$\mathcal{N}(\lambda) = \#\{n \ : \ \lambda_n > \lambda\}.$$

Similarly, we define $\widetilde{\mathcal{N}}(\lambda)$ for the sequence $(\widetilde{\lambda}_n)$.

Suppose $\mathcal{N}$ satisfies

$$\liminf_{x\to 0} \frac{\int\limits_0^{hx} \mathcal{N}(\lambda)\ d\lambda}{\int\limits_0^{x} \mathcal{N}(\lambda)\ d\lambda} > 1 \qquad \text{for any} \qquad h > 1.$$

If $\mathcal{N}(\lambda) \sim \widetilde{\mathcal{N}}(\lambda)$ as $\lambda \to 0$, then, as $r \to 0$,

$$\ln \mathbf{P}\Big\{ \sum_{n=1}^{\infty} \widetilde{\lambda}_n \xi_n^2 \leqslant r \Big\} \sim \ln \mathbf{P}\Big\{ \sum_{n=1}^{\infty} \lambda_n \xi_n^2 \leqslant r \Big\}.$$

Note that the asymptotics of eigenvalues counting function is quite stable with respect to perturbations of the kernel and can be calculated explicitely in many cases.

Previous particular results:

Nazarov-Nikitin (2004b) for fBm and similar processes

Nazarov (2004) for the Green processes and singular self-similar measure μ

Karol-Nazarov-Nikitin (2008) for fractional Brownian sheet and other Gaussian fields with "tensor product" structure

Further applications:

Karol-Nazarov (preprint 2010) for the tensor products of smooth Gaussian fields

Nazarov-Sheipak (2012) for the Green processes and discrete self-similar measure