Hitting Times under Taboo for Markov Chains

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Definitions

Let $\eta = \{\eta(t), t \geq 0\}$ be an irreducible continuous time Markov chain generated by Q-matrix $A = (a(x,y))_{x,y \in S}$.

τ_x is the first exit time from x given that $\eta(0) = x$.

H is the taboo set, $H \subset S$.

The transition probability from x to y in time t under the taboo H is ${}_H p_{x,y}(t)$ equal
$\mathbb{P}(\eta(t) = y, \eta(u) \notin H, min[\tau_x, t] < u < t | \eta(0) = x)$.

The hitting time of y under the taboo H is ${}_H\tau_{x,y}$ defined on the set $\{\eta(0) = x\}$ as $\inf\{t \geq \tau_x : \eta(t) = y,\ \eta(u) \notin H,\ \tau_x < u < t\}$ (as usual, $\inf\{t \in \varnothing\} = \infty$).

$${}_HF_{x,y}(t) := \mathbb{P}({}_H\tau_{x,y} \leq t | \eta(0) = x).$$

Chung K.L. (1962),
Tweedie R.L. (1974),
Kemeny J., Snell L., Knapp A., Griffeath D. (1976),
Zubkov A.M. (1979),
Syski R. (1992), ...

Recall that

$F_{x,y}(\infty) = 1$ if η is recurrent,

$F_{x,y}(\infty) = \frac{P_{x,y}(\infty)}{P_{y,y}(\infty)} \in (0,1)$ if η is transient and $x \neq y$,

$F_{x,x}(\infty) = 1 + \frac{1}{a(x,x)P_{x,x}(\infty)} \in (0,1)$ if η is transient.

Here $F(\infty) = \lim_{t\to\infty} F(t)$, $F_{x,y}(t) := {}_{\varnothing}F_{x,y}(t)$ and ${}_HP_{x,y}(t) = \int_0^t {}_Hp_{x,y}(u)\,du$.

Main results

Theorem (A)

For any nonempty taboo set H and $x, y \in S$, $y \notin H$, one has

$$ {}_HF_{x,y}(\infty) = \frac{{}_HP_{x,y}(\infty)}{{}_HP_{y,y}(\infty)} \in [0,1], \quad x \neq y, $$

$$ {}_HF_{x,x}(\infty) = 1 + \frac{1}{a(x,x)\,{}_HP_{x,x}(\infty)} \in [0,1), \ x \notin H, $$

where $0 \leq {}_HP_{x,y}(\infty) < \infty$ *and* $0 < {}_HP_{y,y}(\infty) < \infty$.

Theorem (B)

If H is nonempty subset of S and $y, z \notin H$, $z \neq y$, then

$$ {}_{z,H}F_{x,y}(\infty) = \frac{{}_{H}F_{x,y}(\infty) - {}_{H}F_{x,z}(\infty)\,{}_{H}F_{z,y}(\infty)}{1 - {}_{H}F_{y,z}(\infty)\,{}_{H}F_{z,y}(\infty)} $$

where ${}_{H}F_{y,z}(\infty)\,{}_{H}F_{z,y}(\infty) < 1$.

Inversely, if H is any subset of S and $x \notin H$, $x \neq y$, then

$$ {}_{H}F_{x,y}(\infty) = \frac{{}_{x,H}F_{x,y}(\infty)}{1 - {}_{y,H}F_{x,x}(\infty)}. $$

Moreover, for any $H \subset S$ and $x, y \in S$ one has

$$ {}_{H}F_{x,y}(\infty) = (\delta_{x,y} - 1)\frac{a(x,y)}{a(x,x)} - \sum_{z \in S, z \neq x, z \neq y, z \notin H} \frac{a(x,z)}{a(x,x)}\,{}_{H}F_{z,y}(\infty). $$

Theorem (C)

Let η be a transient Markov chain and $x, y, z \in S$. Then ${}_zF_{x,y}(\infty) \in [0,1)$ and

$${}_zF_{x,y}(\infty) = \frac{P_{x,y}(\infty)P_{z,z}(\infty) - P_{x,z}(\infty)P_{z,y}(\infty)}{P_{z,z}(\infty)P_{y,y}(\infty) - P_{y,z}(\infty)P_{z,y}(\infty)}$$

for $x \neq y, \quad x \neq z, \quad y \neq z,$

$${}_zF_{y,y}(\infty) = 1 + \frac{P_{z,z}(\infty)}{a(y,y)(P_{y,y}(\infty)P_{z,z}(\infty) - P_{y,z}(\infty)P_{z,y}(\infty))}$$

for $y \neq z,$

$${}_zF_{z,y}(\infty) = -\frac{P_{z,y}(\infty)}{a(z,z)(P_{y,y}(\infty)P_{z,z}(\infty) - P_{y,z}(\infty)P_{z,y}(\infty))}$$

for $y \neq z.$

Put $\rho(x,y) := \lim\limits_{t\to\infty} \int_0^t (p_{y,y}(u) - p_{x,y}(u))\,du$ whenever the limit exists.

Theorem (D)

Let η be a symmetric, space-homogeneous random walk on $\mathbb{Z}^d$, $d = 1$ or $d = 2$, having a finite variance of jump sizes. Then for any $x, y, z \in \mathbb{Z}^d$ such that $y \neq z$, one has

$$\rho(y,z) = \frac{1}{(2\pi)^d} \int\limits_{[-\pi,\pi]^d} \frac{\cos(y-z,\theta)-1}{\sum_{x\in\mathbb{Z}^d} a(o,x)\cos(x,\theta)}\,d\theta \in (0,\infty),$$

$${}_zF_{x,y}(\infty) = \frac{1}{2} + \frac{\rho(x,z)(1-\delta_{x,z}) - \rho(x,y)(1-\delta_{x,y}) + a(o,o)^{-1}(\delta_{x,z}-\delta_{x,y})}{2\rho(y,z)}.$$

Note that in the theory of random walks $\rho(o,x)$, $x \in \mathbb{Z}^d$, is called a potential kernel.

Spitzer F. (1964), Lawler G.F., Limic V. (2010)

For a simple random walk on $\mathbb{Z}$,
the potential kernel has a very nice form,
namely $\rho(o,x) = |x|a(o,o)^{-1}$
whereas for a simple random walk on $\mathbb{Z}^2$
the values $\rho(o,x)$ can be computed
due to a certain iterative scheme.

Spitzer F. (1964)

Basic equations

Convolution-type integral equations

$$_{H}p_{x,y}(t) = {}_{z,H}\, p_{x,y}(t) + \int_0^t {}_{H}p_{z,y}(t-u)\, d\, {}_{H}F_{x,z}(u),$$

$$_{H}F_{x,y}(t) = {}_{z,H}\, F_{x,y}(t) + \int_0^t {}_{H}F_{z,y}(t-u)\, d\, {}_{y,H}F_{x,z}(u)$$

for $z \neq y$,

Laplace-Stieltjes transforms, ...

Motivation and Applications

- L. Doering, M. Roberts (2012)
  catalytic branching process with a single catalyst
  many-to-few lemmas, renewal theory
- E.B. Yarovaya (2012)
  branching random walk with finitely many sources of particle generation
  spectral theory of operators
- E.Vl. Bulinskaya (2013)
  catalytic branching process with finitely many catalysts (CBP)
  auxiliary multi-type Bellman-Harris process

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Description of CBP

- particles move according to a Markov chain having the state space S and generated by Q-matrix A
- they branch at the presence of catalysts, $W = \{w_1, \dots, w_N\} \subset S$ is the catalysts set
- hitting w_k a particle either produces a random number of offspring ξ_k or leaves w_k with probabilities α_k and $1 - \alpha_k$, respectively
- newly born particles behave as independent copies of their parent

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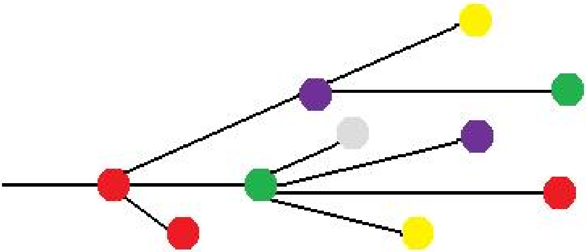
Multi-type Bellman-Harris Processes

- A particle of type i has a life-length with distribution G_i
- Just before the death the particle of type i produces a random number of offsprings according to a generating function g_i, $i = 1, \ldots, L$



Methods of CBP study

The particles located at time t at w_i in CBP are the particles of type i in BHP.

Each particle in CBP that has left w_j at least once within time interval $[0, t]$, upon the last leaving w_j has not yet reached W by time t but eventually will hit w_k before possible hitting $W \setminus \{w_k\}$ is of the $(jN + k)$-th type in BHP.

We have constructed a Bellman-Harris process with $\leq N(N+1)+1$ types of particles.

Classification of CBP

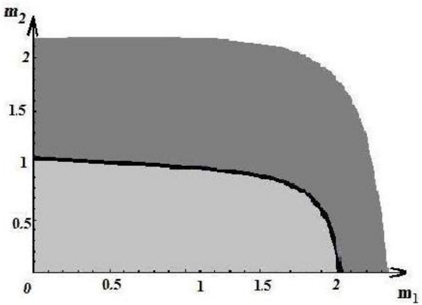
Let $M = (m_{ij})$ be the mean matrix of BHP, i.e.
m_{ij} is the mean number of the offsprings of type j produced by a particle of type i.

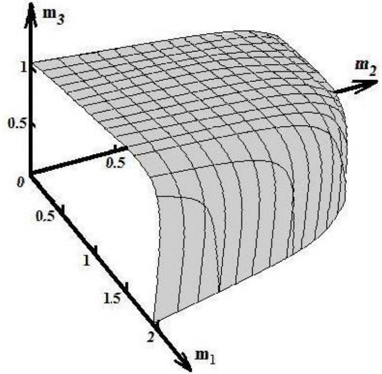
M is an irreducible matrix. Therefore, according to the Perron-Frobenius theorem M has a positive eigenvalue $\rho(M)$ with maximal modulus which is called the Perron root.

Definition by E.Vl. Bulinskaya (2013)
CBP is called supercritical, critical or subcritical if $\rho(M) > 1$, $\rho(M) = 1$ or $\rho(M) < 1$, respectively.

Structure of the Criticality Set

For $\mathbb{E}\xi_i = m_i$, $i = 1, \ldots, N$,
the criticality set is
$C = \{(m_1, \ldots, m_N) \in \mathbb{R}_+^N : \rho(M) = 1\}$.





Notation

Let $\mu(t;y)$ be the number of particles at site y at time t in CBP. In other words,
$\mu(t;y)$ is the local particles number.

Set $\mu(t) = \sum_{y\in S}\mu(t;y)$, i.e.
$\mu(t)$ is the total number of particles at time t.

Put also

$$m(t;x,y) = \mathbb{E}_x\mu(t;y), \quad M(t;x) = \mathbb{E}_x\mu(t).$$

Theorem (E.Vl. Bulinskaya (2013))

If $\rho(M) > 1$ *then the following relation holds true*

$$m(t;x,y) \sim C(x,y)e^{\lambda t}, \quad t \to \infty,$$

for any $x, y \in S$ *and some* $\lambda > 0$, $C(x,y) > 0$. *Under specific additional conditions one has*

$$\mu(t;y)e^{-\lambda t} \to \nu(y) \quad a.s., \quad t \to \infty,$$

where $\nu(y)$, $y \in S$, *are certain non-trivial random variables.*

Comments

A similar result is true for
the total particles numbers as well.

Note that for the subcritical case ($\rho(M) < 1$)
one can prove that $m(t; x, y) \to 0$
for $t \to \infty$ and any $x, y \in S$.

To prove the Theorem we employ
results for the Bellman-Harris processes
and hitting times under taboo
established by E.Vl. Bulinskaya (2013).