Phase transitions in full counting statistics in time-periodic setups

Dmitri Ivanov[1] and Alexander Abanov[2]

[1] EPFL
[2] Stony Brook University

July 2011

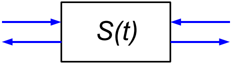
EPL **92**, 37008 (2010)
+ in preparation

Outline

1. "Counting" phase transitions in classical and quantum systems
2. **Example 1:** (classical) "weather model"
3. **Example 2:** (quantum) adiabatic pump for non-interacting fermions
4. **Example 3:** (quantum) statistics of free fermions on a 1D line segment (and generalized Fisher–Hartwig conjecture)

Full counting statistics

- Quantum:



- Classical:

t

10001**1011000101001100**00101

Generating function:

$$\chi_t(\lambda) = \sum_n P_n e^{i\lambda n}$$

Here t – duration of measurement,
P_n – probability of counting n events (transmitting charge n)

Properties of the generating function

- Periodicity: $\chi_t(\lambda) = \chi_t(\lambda + 2\pi)$ – reflects charge quantization
- For any t, the number n is limited
  $\Rightarrow \chi_t(\lambda)$ is analytic on the circle $z = e^{i\lambda}$
- Multiplicativity for independent processes
  ("partition function"): $\chi_{A+B}(\lambda) = \chi_A(\lambda) \cdot \chi_B(\lambda)$
- In a time-periodic setup (with a period τ), define the extensive part of $\chi(\lambda)$:

$$\chi^{(0)}(\lambda) = \lim_{N \to \infty} \left[\chi_{N\tau}(\lambda)\right]^{1/N}$$

“Counting” phase transitions

$\chi^{(0)}(\lambda)$ is periodic in λ (by construction), but:

1. The corresponding “quasiprobabilities”

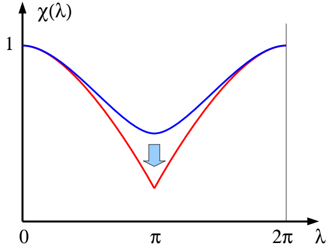
$$\chi^{(0)}(\lambda) = \sum_n P_n^{(0)} e^{i\lambda n}$$

are not necessarily positive (nothing surprising)

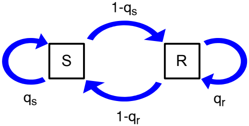
and

2. It may develop a singularity

$\Rightarrow$ PHASE TRANSITION



Example 1 (classical): “weather model”



Recursive relation for the generating function:

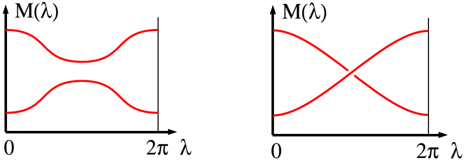
$$\begin{pmatrix} \chi^S_{N+1} \\ \chi^R_{N+1} \end{pmatrix} = M(\lambda) \begin{pmatrix} \chi^S_N \\ \chi^R_N \end{pmatrix}$$

2×2 transfer matrix:

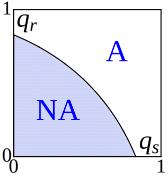
$$M(\lambda) = \begin{pmatrix} q_s e^{i\lambda} & (1-q_r)e^{i\lambda} \\ 1-q_s & q_r \end{pmatrix} = \begin{pmatrix} q_s & 1-q_r \\ 1-q_s & q_r \end{pmatrix} \begin{pmatrix} e^{i\lambda} & 0 \\ 0 & 1 \end{pmatrix}$$

Weather model – continued

$\chi^{(0)}(\lambda)$ is determined by the maximal (complex) eigenvalue of the transfer matrix $M(\lambda)$:

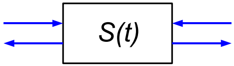


"Analytic" phase "Nonanalytic" phase



Phase diagram as a function of the parameters (q_r, q_s)

Example 2 (quantum): adiabatic pumping of noninteracting fermions



periodic in time scattering matrix

⇒ Factorization of FCS into single-particle processes
[D.I. and A. Abanov '07 and '09]

$$\chi(\lambda) = \det\left(\left[1 + (e^{i\lambda} - 1)\tilde{X}\right] e^{-i\lambda Q}\right)$$

Q – projection onto one of the leads,

$$\tilde{X} = (1 - n_F)Q + n_F^{1/2} S^\dagger Q S n_F^{1/2}$$

– a Hermitian operator with the specrum within $[0, 1]$
(effective transparencies of the contact)

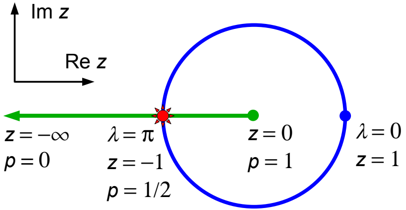
Adiabatic pumping of noninteracting fermions (continued)

Spectral density: $\mu(p) = \mathrm{tr}\, \delta(p - \tilde{X})$

$\chi^{(0)}(z{=}e^{i\lambda})$ may only have singularities at $z \in \mathbb{R}_-$ (negative real axis):

$$p = \frac{1}{1-z}$$

– effective probabilities (transparencies) of single-particle processes

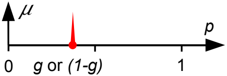


Phase (A or NA) depends on the gap in the spectral density $\mu(p)$ at $p = 1/2$

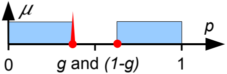
Adiabatic pumping (continued): examples

Example: Voltage applied to a single-channel contact with the transparency g at the temperature T.

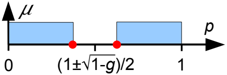
(a) $T = 0$, $V = \text{const}$:



(b) $T = 0$, any $V(t)$:

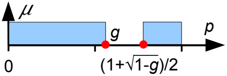


(c) $T > 0$, $V = 0$:



analytic phase

(d) $T > 0$, $V = \text{const}$:



(e) $T > 0$, any $V(t)$:

$\mu(p)$ continuous and asymmetric
$\rightarrow$ see theorem

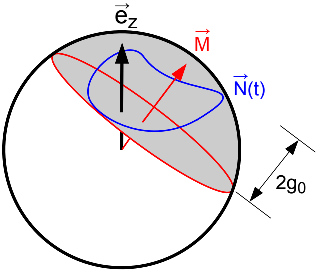
analytic if $g < 1/2$ and nonanalytic if $g > 1/2$

Adiabatic pumping: theorem about analytic phase

Consider: one-channel
contact with arbitrary $S(t)$
[bias-voltage case included]

the trajectory of the unit vector
[as in Makhlin and Mirlin '01]

$$\vec{N}(t) = \frac{1}{2}\,\mathrm{Tr}\,\vec{\sigma}\,S^{\dagger}(t)\,\sigma_z\,S(t)$$

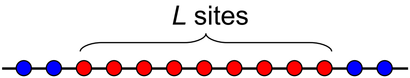


Theorem:
the statistics is in the analytic phase, if this trajectory can be covered by a semisphere (including the north pole, if $T > 0$).
The gap (in the units of p) is given by $1/2 - g_0$.

$\rightarrow$ explains all the examples above

Example 3 (quantum): counting free fermions on a 1D line segment

Free fermions on a 1D chain:



Ground state: states with $|k| < k_F$ filled:

$$|\Psi\rangle = \prod_{|k|<k_F} a_k^+ \, |0\rangle$$

Generating function for counting statistics:

$$\chi_L(\lambda) = \langle\Psi| \, e^{i\lambda N_L} \, |\Psi\rangle \; , \qquad N_L = \sum_{i=1}^{L} a^+(i) \, a(i)$$

Counting free fermions in 1D (continued)

Average and fluctuations:

$$\langle N_L \rangle = \frac{k_F}{\pi} L \qquad \langle\langle N_L^2 \rangle\rangle = \frac{1}{\pi^2} \ln\left(\frac{L}{l_0}\right)$$

(either by bosonization or via Wick theorem)

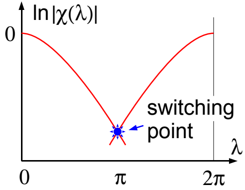
$$l_0^{-1} = 2e^{\gamma_E + 1} \sin k_F, \qquad \gamma_E = 0.57721\ldots \quad \text{Euler constant}$$

At the bosonization level:

$$\chi_L(\lambda) = \exp\left(i\lambda\langle N \rangle - \frac{\lambda^2}{2}\langle\langle N_L^2 \rangle\rangle\right)$$

– NOT periodic in λ (neglects charge discreteness)

Counting free fermions in 1D and Toeplitz determinants



Restoring periodicity:

- bosonization + backscattering [Aristov '98]
- more accurately: theory of Toeplitz determinants

$$\chi_L(\lambda) = \langle e^{i\lambda N_L} \rangle = \left\langle \prod_{i=1}^{L} [1 + (e^{i\lambda} - 1) a^+(i) a(i)] \right\rangle =$$

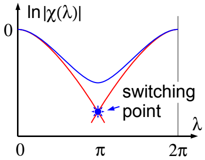
$$= \det \Big(\delta_{ij} + (e^{i\lambda} - 1) g_{ij} \Big)_{L\times L}$$

— Toeplitz determinant with the generating function

$$\sigma(k) = \begin{cases} e^{i\lambda}\,, & |k| < k_F \\ 1\,, & |k| > k_F \end{cases}$$

(two Fisher-Hartwig singularitites)

1D free fermions and generalized Fisher–Hartwig formula



Generalized Fisher–Hartwig conjecture
– recently proven [Deift, Its, Krasovsky '09]:

$$\chi_L(\lambda) \approx \tilde{\chi}_L(\lambda) + \tilde{\chi}_L(\lambda - 2\pi)$$

$$\tilde{\chi}_L(\lambda) = \exp\left[i\lambda \frac{k_F}{\pi} L - \frac{\lambda^2}{2\pi} \ln(2L \sin k_F) \;\; + F_0(\lambda) \right]$$

where

$$F_0(\lambda) = 2 \ln \left| G\left(1 + \frac{\lambda}{2\pi}\right) G\left(1 - \frac{\lambda}{2\pi}\right) \right|$$

and G is the Barnes G function

$$G(1+z) = (2\pi)^{z/2} e^{-[z+(\gamma_E+1)z^2]/2} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^n e^{-z+\frac{z^2}{2n}}$$

Rigorously proven up to a relative $o(1)$. Can we do better?

Exploring the precision of the generalized Fisher–Hartwig formula

Based on numerical calculations of the Toeplitz determinant up to $L = 5000$ [with Yachao Qian (Stony Brook)]:

$$\chi_L(\lambda) = \tilde{\chi}_L(\lambda) + \tilde{\chi}_L(\lambda - 2\pi) + \varepsilon$$

$$\tilde{\chi}_L(\lambda) = \exp\left[i\lambda \frac{k_F}{\pi} L - \frac{\lambda^2}{2\pi} \ln(2L \sin k_F) \; + F_0(\lambda) + F_1(\lambda, k_F) L^{-1} \right]$$

$$\epsilon = \left(\left| \tilde{\chi}_L(\lambda) \right| + \left| \tilde{\chi}_L(\lambda - 2\pi) \right| \right) \cdot O(L^{-2})$$

– numerically verified and $F_1(\lambda, k_F)$ extracted.

Within numerical precision (and in agreement with $\langle\langle N_L^3 \rangle\rangle$),

$$F_1(\lambda, k_F) = -\frac{i}{4} \left(\frac{\lambda}{\pi} \right)^3 \cot k_F$$

Conclusions, interpretations, conjectures: 1

1. "Counting" phase transition (analytic $\leftrightarrow$ nonanalytic) may occur at various values of λ.
   In our examples: always $\lambda = \pi$ (the only possible value for noninteracting fermions), but other λ are also possible

   [see, e.g., Karzig and von Oppen '09:
   phase transition at $\lambda = 0$ in a quantum-dot chain].

2. The signatures of the phase transition are subtle.
   Possible test: "staggered cumulant"

   $$\langle\langle n^2 \rangle\rangle_\pi = \langle n^2 \rangle_\pi - \langle n \rangle_\pi^2 = (-i\partial_\lambda)^2 \ln \chi(\lambda)|_{\lambda=\pi}$$

   where $\langle A \rangle_\pi := \langle (-1)^n A \rangle / \langle (-1)^n \rangle$

   $\rightarrow$ grows as t in the analytic phase and as t^2 in the nonanalytic phase.

Conclusions, interpretations, conjectures: 2

3. Conjecture on long-time behavior in the nonanalytic phase:

$$\chi(\lambda) = C_+ \left[\chi_0(\lambda - 0)\right]^N + C_- \left[\chi_0(\lambda + 0)\right]^N$$

   where $C_\pm$ depend on N subexponentially.

   - obvious in the classical example
   - can be proven in some free-fermion problems using the generalized Fisher-Hartwig formula

4. Conjecture on improving the generalized Fisher–Hartwig formula (based on numerics)

5. Possible implications of "counting" phase transitions for non-equilibrium bosonization [Gutman, Gefen, Mirlin '09, '10]?