On Links Between the Random Matrix and Random Operator Theories

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- ”Corresponding” Families of Random Operators
  - Description
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1 Introduction

Let $\{M_n\}$ be the sequence of $n \times n$ random hermitian matrices with ν_n non-zero entries, all are on the principal and adjacent diagonals and independent (ergodic) modulo symmetry.

Then $\{M_n\}$ determines

- random operator, if $\nu_n/n \to 2p + 1,\ n \to \infty,\ p \in \mathbb{Z}$;
- random matrix, if $\nu_n/n \to \infty,\ n \to \infty$.

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Random Operator Theory (ROT) is mostly on spectral types of ”limiting” selfadjoint ergodic operators in $l^2(\mathbb{Z})$ more generally in $l^2(\mathbb{Z}^d)$, defined by the double infinite ”limit” of the corresponding finite size matrices).

Random Matrix Theory (RMT) is mostly on the eigenvalue distribution as $n \longrightarrow \infty$ (no ”limiting” operators but still well defined limiting and asymptotic spectral characteristics, cf statistical mechanics).

Common topics

- Integrated Density of States, the $n \longrightarrow \infty$ limit of the Normalized Counting Measure of Eigenvalues of M_n.
- Asymptotics of spacing between adjacent eigenvalues.
- Eigenvectors

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Subject of the talk: families of random ergodic operators, possessing certain properties of random matrices in certain asymptotic regimes and vice versa.

2 Most Widely Known Random Matrices

2.1 Description

2.1.1 Gaussian Unitary Ensemble (GUE)

$$M_n = n^{-1/2} W_n,\ W_n = \{W_{jk}\}_{j,k=1}^n,\ \overline{W_{jk}} = W_{kj} \in \mathbb{C}$$

$W_{jk},\ 1 \geq j \geq k \geq n$ are independent complex Gaussian and

$$\mathbf{E}\{W_{jk}\} = \mathbf{E}\{W_{jk}^2\} = 0,\ \mathbf{E}\{|W_{jk}|^2\} = w^2(1 + \delta_{jk})/2.$$

(i) *Band Version*

$$(M_{n,b})_{j,k} = b_n^{-1/2}\varphi(|j-k|/\beta_n)W_{j,k}, \; b_n = 2\beta_n + 1, \; \beta_n \in \mathbb{N},$$

$$\operatorname{supp}\varphi = [0,1], \; \int_{\mathbb{R}} \varphi^2(t)dt = 1.$$

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(ii) *Deformed Version*

$$M_n = M_n^{(0)} + n^{-1/2}W_n,$$

where $M_n^{(0)}$ is either non-random or random and independent of W_n.

2.1.2 "Wishart" Matrices

$$M_n = n^{-1} X^*_{m,n} X_{m,n}, \; X_{m,n} = \{X_{\alpha j}\}_{\alpha,j}^{m,n},$$

where $\{X_{\alpha j}\}_{\alpha,j}^{m,n}$ are i.i.d. complex Gaussian and

$$\mathbf{E}\{X_{\alpha j}\} = \mathbf{E}\{X^2_{\alpha j}\} = 0, \; \mathbf{E}\{|X_{\alpha j}|^2\} = a^2$$

In statistics one calls *white (or null) Wishart matrices* those with real Gaussian X's (sample covariance matrix of Gaussian population). The above case is known in the RMT as the *Laguerre Ensemble*.

Deformed Versions (both additively and multiplicatively):

$$M_n = M_n^{(0)} + n^{-1} X^*_{m,n} T_m X_{m,n}$$

and (signal+noise)

$$M_n = (A_{m,n}^{(0)} + n^{-1/2} X_{m,n})^* T_m (A_{m,n}^{(0)} + X_{m,n}),$$

where $M_n^{(0)}$, $A_{m,n}$, and T_m are either non-random or random and independent of $X_{m,n}$ and one of another.

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where $M_n^{(0)}$, $A_{m,n}$, and T_m are either non-random or random and independent of $X_{m,n}$ and one of another.

2.1.3 Law of Addition (Free Probability)

$$M_n = A_n + U_n^* B_n U_n,$$

where U_n is Haar distributed over $U(n)$ and A_n and B_n are $n \times n$ either non-random or random hermitian and independent of U_n and one of another.

2.1.4 Wigner Matrices

Replace $W_{jk},\ 1 \leq j \leq k \leq n$ in the GUE and its band and deformed versions by arbitrary random variables (double array) with the same first and second moment.

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2.1.5 Sample Covariance Matrices

Replace $\{X_{\alpha j}\}_{\alpha,j}^{m,n}$ in ”Wishart” and its deformed versions by arbitrary random variables with the same first and second moment.

2.2 Basic Results

Introduce the Normalized Counting Measure (NCM) N_n of eigenvalues $\{\lambda_l^{(n)}\}_{l=1}^n$ of M_n

$$N_n(\Delta) = \sharp\{l = 1, ..., n : \lambda_l^{(n)} \in \Delta\}/n, \; \Delta \subset \mathbb{R}$$

and assume that the NCM's $N_n^{(0)}$ for $H_n^{(0)}, \; \sigma_m$ of $T_m, \; N_{A_n}$ of A_n, and B_n have weak limits (with probability 1 if random) as $m, \; n \to \infty$, $m/n \to c \in [0, \infty)$.

Then in all above cases N_n converges weakly with probability 1 to a non-random limit N. The limit can be found via its Stieltjes transform

$$f(z) = \int \frac{N(d\lambda)}{\lambda - z}, \Im z \neq 0,$$

that solves the functional equations below, and the inversion formula

$$N(\Delta) = \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{\Delta} \Im f(\lambda + i0) d\lambda.$$

2.2.1 Deformed GUE

(i) The Stieltjes transform of the limiting NCM solves the equation

$$f(z) = f^{(0)}(z + w^2 f(z)),$$

that has a unique solution in the class of Nevanlinna functions, i.e., analytic for non-real z and such that

$$\Im f \Im z \geq 0, \quad f(z) = -z^{-1} + o(z^{-1}), \; z \to \infty.$$

The corresponding limiting measure is known as the deformed semicircle (or Wigner) law. N is absolutely continuous and has continuous density ρ.

The same limit is for Wigner matrices (macroscopic universality) *P. 72*.

In particular, if $M_n^{(0)} = 0$ (GUE, Wigner), then we have the *semicircle law* by Wigner

$$
\begin{aligned}
f(z) &= \frac{1}{2w^2}\left(\sqrt{z^2 - 4w^2} - z\right), \\
N(d\lambda) &= \rho(\lambda) d\lambda, \\
\rho(\lambda) &= (2\pi w^2)^{-1/2}\sqrt{4w^2 - \lambda^2}\mathbf{1}_{[-2w,2w]}(\lambda).
\end{aligned}
$$

The same limit is for band matrices if $b_n/n \to 0,\ n \to \infty$.

Khorunzhy, Molchanov, P. 92.

(ii) If λ_0 belongs to the interior (bulk) of the support of N and

$$E_n(s) = \mathbf{P}\{[\lambda_0, \lambda_0 + s/\rho(\lambda_0)] \notin \lambda_l^{(n)},\ l = 1, ..., n\}$$

is the gap probability. Then we have for the deformed GUE the Gaudin law for

$$E(s) = \lim_{n\to\infty} E_n(s) = \det(1 - S(s)),$$

where

$$(S(s)f)(x) = \int_0^s \frac{\sin \pi(x-y)}{\pi(x-y)} f(y)dy.$$

In particular, we have for the limiting probability density $p(s) = E''(s)$ of spacing between adjacent eigenvalues:

$$p(s) = \frac{\pi}{36} s^2 (1 + o(1)), \ s \to 0,$$

i.e., the *eigenvalue (level) repulsion*.

(*Gaudin 61, Brezin-Hikami 96, Johansson 01, 09, T. Shcherbina 09*).

(iii). Eigenvectors of the GUE are uniformly (Haar) distributed over $U(n)$, an analog of pure absolutely continuous spectrum (complete delocalization).

2.2.2 Deformed Wishart and Sample Covariance Matrices

The Stieltjes transform solves

$$f(z) = f^{(0)}\left(z - a^2 c \int \frac{\tau\sigma(d\tau)}{1 + a^2\tau f(z)}\right),$$

where $c = \lim_{n\to\infty} m/n$. *Marchenko, P. 67*

In particular, for $M_n^{(0)} = 0,\ T_m = Id$

$$N_{MP}(d\lambda) = (1-c)_+\delta(\lambda)d\lambda + \rho_{MP}(\lambda)d\lambda,$$

$$\rho_{MP}(\lambda) = (2\pi a^2\lambda)^{-1/2}\sqrt{(a_+ - \lambda)(\lambda - a_-)}\mathbf{1}_{[a_+,a_-]}$$

and $a_\pm = a^2(1 \pm \sqrt{c})^2$, $x_+ = \max\{x, 0\}$.

2.2.3 Law of Addition

The Stieltjes transform of the limiting NCM solves the system, determined by the Stieltjes transforms f_A and f_B of limiting NCM of $\{A_n\}$ and $\{B_n\}$:

$$\begin{aligned} f(z) &= f_A(h_B(z)) \\ f(z) &= f_B(h_A(z)) \\ f^{-1}(z) &= z - h_A(z) - h_B(z), \end{aligned}$$

and uniquely soluble in the Nevanlinna class for f and h_A, h_B analytic for non-real z and satisfying

$$h_{A,B}(z) = z + O(1), \; z \to \infty.$$

P., Vasilchuk, 00, 07

3 "Corresponding" Random Operators.

3.1 Description

Define symmetric random operators:

(i). H_{R_G} in $l_2(\mathbb{Z}^d)$, $d \geq 1$ by matrix $\{H_{R_G}(x,y)\}_{x,y\in\mathbb{Z}^d}$ as

$$H_{R_G}(x,y) = h(x-y) + R_G^{-d/2}\varphi((x-y)/R_G)W(x,y), \quad x,y \in \mathbb{Z}^d,$$

where $h : \mathbb{Z}^d \to \mathbb{C}$,

$$h(-x) = \overline{h(x)}, \quad \sum_{x\in\mathbb{Z}^d} |h(x)| < \infty,$$

$R_G > 0$, $\varphi : \mathbb{R}^d \to \mathbb{R}$ is piece-wise continuous,

$$\max_{t\in\mathbb{R}} |\varphi(t)| \le \varphi_0 < \infty, \quad \varphi(t) = 0,\ |t| > 1, \quad \int_{\mathbb{R}^d} \varphi^2(t)dt = 1,$$

and

$$W(x,y) = \overline{W(y,x)}, \quad x,y \in \mathbb{Z}^d,$$

are independent (modulo the above symmetry condition) complex Gaussian random variables:

$$\mathbf{E}\{W(x,y)\} = \mathbf{E}\{W(x,y)^2\} = 0, \quad \mathbf{E}\{|W(x,y)|^2\} = 1,$$

In the case $d = 1$ the random part of H_{R_G} is an infinite matrix having nonzero entries only inside the band of width $(2R_G + 1)$ around the principal diagonal, i.e., an analog of band matrix.

(ii) $H_d = \{H_d(x,y)\}_{x,y\in\mathbb{Z}^d}$ in $l_2(\mathbb{Z}^d)$ by

$$H_d(x,y) = h_d(x-y) + (2d)^{-1/2} W_1(x,y),$$

$$h_d(x) = d^{-1/2} \sum_{j=1}^{d} h_1(x_j) \prod_{k\neq i} \delta(x_k), \quad h(0) = 0,\ x = (x_1, \ldots, x_d),$$

δ is the Kronecker symbol, $h_1 : \mathbb{Z} \to \mathbb{C}$ is as in H_{R_G} for $d = 1$ (e.g. the discrete Laplacian) and

$$W_1(x,y) = \begin{cases} W(x,y), & |x-y| = 1, \\ 0, & |x-y| \neq 1, \end{cases}$$

and $W(x,y)$ are as in H_{R_G}.

(iii) H_{n_W} in $l_2(\mathbb{Z}^d) \otimes \mathbb{C}^{n_W}$ by

$$H_{n_W} = \{H_{n_W}(x, \alpha; y, \beta)\}_{x,y \in \mathbb{Z}^d, \alpha,\beta=1,\dots n_W}$$

$$H_{n_W}(x, \alpha; y, \beta) = h(x-y)\delta_{\alpha\beta} + n_W^{-1/2}\delta(x-y)W_{\alpha\beta}(x)$$

where $x, y \in \mathbb{Z}^d$, $\alpha, \beta \in \mathbb{N}$, h is the same as H_{R_G}, and

$$W_{\alpha\beta}(x) = \overline{W_{\beta\alpha}(x)}, \quad x \in \mathbb{Z}^d, \ \alpha, \beta = 1, \dots, n_W,$$

are independent (modulo symmetry) complex Gaussians:

$$\mathbf{E}\{W_{\alpha\beta}(x)\} = \mathbf{E}\{W^2_{\alpha\beta}(x)\} = 0, \quad \mathbf{E}\{|W_{\alpha\beta}(x)|^2\} = 1.$$

Wegner 80. It is a n_W-component analog of Anderson model (or the Hamiltonian of a disordered system of dimension $d + n_W$, in which the random potential in n_W ”transverse” dimensions has a ”mean field” form).

(iv). H_{R_L} in $l^2(\mathbb{Z}^d)$ by $\{H_{R_L}(x,y)\}_{x,y\in\mathbb{Z}^d}$:

$$H_{R_L}(x,y) = h(x-y) + R_L^{-d}\varphi((x-y)/R_L)\sum_{\alpha=1}^{m}\overline{X_\alpha(x)}X_\alpha(y),$$

where h is as in H_{R_G},

$$\{X_\alpha(x)\}_{\alpha\in\mathbb{N},x\in\mathbb{Z}^d}$$

are i.i.d. complex Gaussian random variables such that

$$\mathbf{E}\{X_\alpha(x)\} = \mathbf{E}\{X_\alpha^2(x)\} = 0, \quad \mathbf{E}\{|X_\alpha(x)|^2\} = 1,$$

and φ is positive definite, decaying sufficiently fast at infinity.

Random parts of H_{R_G} and H_d resemble the GUE while the random part of H_{R_L} resembles the Laguerre matrices.

(v) H_{n_V} in $l^2(\mathbb{Z}^d) \otimes \mathbb{C}^{n_V}$:

$$H_{n_V}(x, \alpha; y, \beta) = h(x-y)\delta_{\alpha\beta} + \delta(x-y)(U^*_{n_V}(x) B_{n_V} U_{n_V}(x))_{\alpha\beta},$$

where h is as above, $x, y \in \mathbb{Z}^d, \alpha, \beta = 1, \ldots, n_V$, $\{U_{n_V}(x)\}_{x \in \mathbb{Z}^d}$ are i.i.d. $n_V \times n_V$ unitary matrices whose common probability law is the normalized Haar measure on $U(n_V)$, and B_{n_V} is $n_V \times n_V$ hermitian matrix.

Random part of H_{n_V} (matrix valued ”potential”) is reminiscent of that in the law of addition of random matrices.

All the above operators have the form of a non-random translation invariant part and a random part explicitly containing the parameters R, R_L, d, n_W, n_V that we are going to let to infinity. The random parts are such that the larger these parameters are, the more ”extended” and smaller the randomness is. Similar scaling of the interaction is widely used in the mean field and the spherical approximations of statistical mechanics.

What about to extend results and techniques developed for the Schrodinger operator with random potential to the above operators and to see how this will depend on $a \longrightarrow \infty$?

3.2 Integrated Density of States.

Integrated Density of States (IDS) of H_a can be defined either via the "finite box" versions of above operators, and since all of them are ergodic, we have *P. - Figotin 92*

$$N_a(\Delta) = \mathbf{E}\{\mathcal{E}_a(0,0;\Delta)\}, \quad a = R_G, d, R_L$$

where $\{\mathcal{E}_a(x,y;\Delta)\}_{x,y\in\mathbb{Z}^d}$ is the resolution of identity of H_a for $a = R_G, d, R_L$, and

$$N_a(\Delta) = \mathbf{E}\{a^{-1}\sum_{\alpha=1}^{a}\mathcal{E}_a(\alpha,0;\alpha,0;\Delta)\}, \quad a = n_W, n_V,$$

$\mathcal{E}_a(\Delta) = \{\mathcal{E}_a(x,\alpha;y,\beta;\Delta)\}_{x,y\in\mathbb{Z}^d,\alpha,\beta\in[1,a]}$ is the matrix of the resolution of identity of the operators H_a for $a = n_W, n_V$.

Denote $N^{(0)}$ the IDS of the non-random (unperturbed) parts:

$$N^{(0)}(d\lambda) = \text{mes}\{k \in \mathbb{T}^d : \widehat{h}(k) \in d\lambda\},$$

where $\mathbb{T}^d = [0,1]^d$ is d-dimensional torus and

$$\widehat{h}(k) = \sum_{x \in \mathbb{Z}^d} h(x) e^{2\pi i (k,x)}$$

Note that for H_d the non-random part and its IDS depend also on d and the limit $d \to \infty$ affects also the unperturbed IDS, yielding

$$N^{(0)}(d\lambda) = (2\pi h_2)^{-1/2} e^{-\lambda^2/2h_2} d\lambda,$$

$$h_2 = \sum_{x \in \mathbb{Z}} h_1^2(x).$$

No limiting operator but still well defined IDS!

3.3 Asymptotic Results on the IDS

Theorem *Let H_a, $a = R_G, d, n_W, R_L, n_V$ be the above ergodic operators, N_a be their IDS, and $N^{(0)}$ be the IDS of their non-random parts. Then*

(i) for $a = R_G, d, n_W$ N_a converges weakly as $a \to \infty$, to the probability measure N_{dsc} (the deformed semicircle law);

(ii) for $a = R_L$ N_{R_L} converges to the limiting NCM of the deformed Laguerre ensembles, and the role of σ plays

$$\sigma(\Delta) = \mathrm{mes}\{k \in \mathrm{supp}\, \widehat{\varphi} : \widehat{\varphi}(k) \in \Delta\},$$

where $\widehat{\varphi}$ is the Fourier transform of positive definite φ of compact support;

(iii) for $a = n_V$*, the NCM* $N_{B_{n_V}}$ *of* B_{n_V} *satisfying the condition*

$$\sup_{n_V} \int |\lambda|^4 N_{B_{n_V}}(d\lambda) < \infty,$$

and converging weakly to N_B *as* $n_V \to \infty$ *the IDS of* H_{n_V} *converges weakly as* $n_V \to \infty$ *to the measure, corresponding to the law of addition of random matrices, in which* N_A *is as for* $a = R_G$

The proofs of the theorems use (recent) tools from the RMT. Two main ingredients are: the Poincaré - Nash bound for the variance of functions of Gaussian and classical group random variables and versions of integrating by parts for them (*Khorunzhy-P 93, P.-Vasilchuk 07, P. 09*).

4 Comments

4.1 Generalities

Operators $H_a \; a = R_G, d, R_L, n_W, n_V$ are analogs of Hamiltonians of lattice models of statistical mechanics, where the limits of infinite interaction radius, dimensionality or the number of spin components lead to the mean field or the spherical versions of the models.

On the other hand, the studies of elementary excitations and wave propagation in disordered media are essentially based on the spectral properties of the Schrodinger operator with random potential.

Spectral analysis of this and other finite difference and differential operators with random coefficients are among the main objectives of the ROT and of condensed matter theory (theory of disordered systems (DST)).

In particular, the DST uses approximation schemes, analogous to the mean field approximations in statistical mechanics (see e.g. *Lifshitz, Gredeskul, P. 92*). One may ask then about the meaning of above results in the context of the ROT and the DST.

It can be shown that the result for the limiting IDS of H_{R_L} with $\widehat{\varphi} = a\mathbf{1}_A,\ a > 0,\ A \subset \mathbb{R}^d$, i.e., σ, having the atoms at zero and a, the latter of the mass mes$A$, corresponds to the so called modified propagator approximation, and the result for H_{n_V} corresponds to the so called coherent potential approximation.

It is widely believed in physics literature that these ”approximations”, applied for the first and second moments of Green function, describe, at least qualitatively, the delocalized regime (e.g. transport).

4.2 Supports, Questions

- No Lifshitz tails (it is widely believed that they are intimately related to the localization), thus

  What about the Wegner lemma?

  more generally,

  How does the multiscale analysis regime disappears outside of support of the semicircle law?

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  What about densities of the measures and their pointwise limits inside the support of the semicircle law?

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- *Erdos et al, 09, Tao-Vu:09*: complete delocalization and (eigenvalue repulsion) for the Wigner matrices

via the local semicircle law ($\Im z$ down to n^{-1}!)

and analogous results for sample covariance matrices;