Loops and trees: spectral and resonance properties of quantum graphs

Pavel Exner

in collaboration with *Brian Davies* and *Jiři Lipovský*

exner@ujf.cas.cz

Doppler Institute

for Mathematical Physics and Applied Mathematics

Prague

Talk overview

In this talk I am going to present several recent results on spectral and resonance properties of quantum graphs:

- *Geometric perturbations:* resonances due to edge rationality violation in graphs with leads

Talk overview

In this talk I am going to present several recent results on spectral and resonance properties of quantum graphs:

- *Geometric perturbations:* resonances due to edge rationality violation in graphs with leads
- *High-energy asymptotics* of resonances: Weyl and non-Weyl behaviour, and when each of them occurs

Talk overview

In this talk I am going to present several recent results on spectral and resonance properties of quantum graphs:

- *Geometric perturbations:* resonances due to edge rationality violation in graphs with leads
- *High-energy asymptotics* of resonances: Weyl and non-Weyl behaviour, and when each of them occurs
- *Absence of transport at trees:* homogeneous trees which are "sparse" have *generically* empty *ac* spectrum

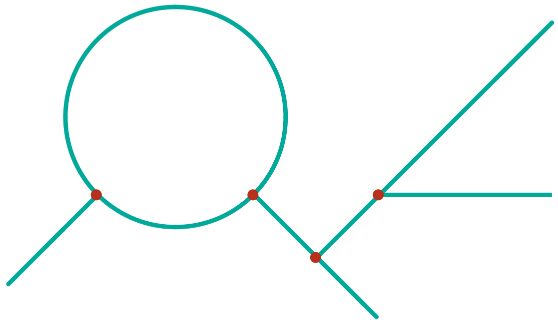
Introduction: the quantum graph concept

The idea of investigating quantum particles confined to a graph was first suggested by L. Pauling in 1936 and worked out by Ruedenberg and Scherr in 1953 in a model of aromatic hydrocarbons

Introduction: the quantum graph concept

The idea of investigating quantum particles confined to a graph was first suggested by L. Pauling in 1936 and worked out by Ruedenberg and Scherr in 1953 in a model of aromatic hydrocarbons

The concept extends, however, to graphs of *arbitrary shape*



Hamiltonian: $-\frac{\partial^2}{\partial x_j^2} + v(x_j)$
on graph edges,
boundary conditions at vertices

and what is important, it became *practically important* after experimentalists learned in the last two decades to fabricate tiny graph-like structure for which this is a good model

Remarks

- There are many graph-like systems based on *semiconductor* or *metallic* materials, *carbon nanotubes*, etc. The dynamics can be also simulated by *microwave network* built of optical cables – see [Hul et al.'04]

Remarks

- There are many graph-like systems based on *semiconductor* or *metallic* materials, *carbon nanotubes*, etc. The dynamics can be also simulated by *microwave network* built of optical cables – see [Hul et al.'04]
- Here we consider *Schrödinger operators* on graphs, most often free, $v_j = 0$. Naturally one can external electric and magnetic fields, spin, etc.

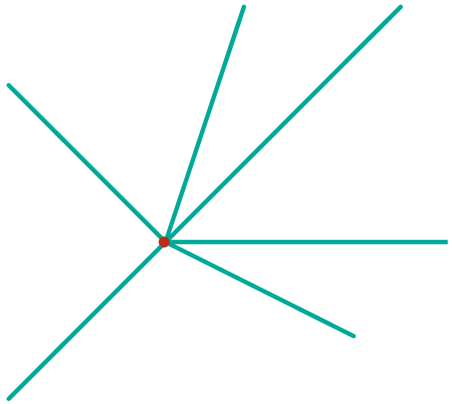
Remarks

- There are many graph-like systems based on *semiconductor* or *metallic* materials, *carbon nanotubes*, etc. The dynamics can be also simulated by *microwave network* built of optical cables – see [Hul et al.'04]
- Here we consider *Schrödinger operators* on graphs, most often free, $v_j = 0$. Naturally one can external electric and magnetic fields, spin, etc.
- Graphs can support also *Dirac operators*, see [Bulla-Trenckler'90], [Bolte-Harrison'03], and many recent applications to *graphene* and its derivates

Remarks

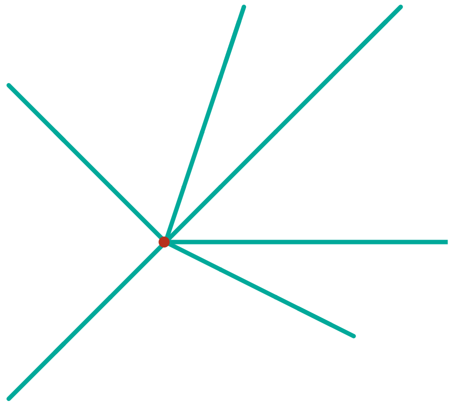
- There are many graph-like systems based on *semiconductor* or *metallic* materials, *carbon nanotubes*, etc. The dynamics can be also simulated by *microwave network* built of optical cables – see [Hul et al.'04]
- Here we consider *Schrödinger operators* on graphs, most often free, $v_j = 0$. Naturally one can external electric and magnetic fields, spin, etc.
- Graphs can support also *Dirac operators*, see [Bulla-Trenckler'90], [Bolte-Harrison'03], and many recent applications to *graphene* and its derivates
- The graph literature is extensive; a good up-to-date reference are proceedings of the recent semester AGA Programme at INI Cambridge

Vertex coupling



The most simple example is a *star graph* with the state Hilbert space $\mathcal{H} = \bigoplus_{j=1}^{n} L^2(\mathbb{R}_+)$ and the particle Hamiltonian acting on $\mathcal{H}$ as $\psi_j \mapsto -\psi_j''$

Vertex coupling



The most simple example is a *star graph* with the state Hilbert space $\mathcal{H} = \bigoplus_{j=1}^{n} L^2(\mathbb{R}_+)$ and the particle Hamiltonian acting on $\mathcal{H}$ as $\psi_j \mapsto -\psi_j''$

Since it is second-order, the boundary condition involve $\Psi(0) := \{\psi_j(0)\}$ and $\Psi'(0) := \{\psi_j'(0)\}$ being of the form

$$A\Psi(0) + B\Psi'(0) = 0\,;$$

by [Kostrykin-Schrader'99] the $n \times n$ matrices A, B give rise to a self-adjoint operator if they satisfy the conditions

- $\text{rank}\,(A, B) = n$
- AB^* is self-adjoint

Unique boundary conditions

The non-uniqueness of the above b.c. can be removed:

Proposition [Harmer'00, K-S'00]: Vertex couplings are uniquely characterized by unitary $n \times n$ matrices U such that

$$A = U - I\,, \quad B = i(U + I)$$

Unique boundary conditions

The non-uniqueness of the above b.c. can be removed:

Proposition [Harmer'00, K-S'00]: Vertex couplings are uniquely characterized by unitary $n \times n$ matrices U such that

$$A = U - I\,, \quad B = i(U + I)$$

One can derive them modifying the argument used in [Fülöp-Tsutsui'00] for generalized point interactions, $n = 2$

Self-adjointness requires vanishing of the boundary form,

$$\sum_{j=1}^{n} (\bar\psi_j \psi_j' - \bar\psi_j' \psi_j)(0) = 0\,,$$

which occurs *iff* the norms $\|\Psi(0) \pm i\ell\Psi'(0)\|_{\mathbb{C}^n}$ with a fixed $\ell \neq 0$ coincide, so the vectors must be related by an $n \times n$ unitary matrix; this gives $(U - I)\Psi(0) + i\ell(U + I)\Psi'(0) = 0$

Examples of vertex coupling

- Denote by $\mathcal{J}$ the $n \times n$ matrix whose all entries are equal to one; then $U = \frac{2}{n+i\alpha}\mathcal{J} - I$ corresponds to the standard δ *coupling*,

$$\psi_j(0) = \psi_k(0) =: \psi(0)\,, \;\; j,k = 1,\dots,n\,, \;\; \sum_{j=1}^{n} \psi_j'(0) = \alpha\psi(0)$$

with "coupling strength" $\alpha \in \mathbb{R}$; $\alpha = \infty$ gives $U = -I$

Examples of vertex coupling

- Denote by $\mathcal{J}$ the $n \times n$ matrix whose all entries are equal to one; then $U = \frac{2}{n+i\alpha}\mathcal{J} - I$ corresponds to the standard δ *coupling*,

$$\psi_j(0) = \psi_k(0) =: \psi(0)\,,\; j,k = 1,\dots,n\,, \; \sum_{j=1}^{n} \psi_j'(0) = \alpha\psi(0)$$

  with "coupling strength" $\alpha \in \mathbb{R}$; $\alpha = \infty$ gives $U = -I$

- $\alpha = 0$ corresponds to the "free motion", the so-called *free boundary conditions* (better name than Kirchhoff)

Examples of vertex coupling

- Denote by $\mathcal{J}$ the $n \times n$ matrix whose all entries are equal to one; then $U = \frac{2}{n+i\alpha}\mathcal{J} - I$ corresponds to the standard δ *coupling*,

$$\psi_j(0) = \psi_k(0) =: \psi(0)\,,\;\; j,k = 1,\dots,n\,,\;\; \sum_{j=1}^n \psi_j'(0) = \alpha\psi(0)$$

with "coupling strength" $\alpha \in \mathbb{R}$; $\alpha = \infty$ gives $U = -I$

- $\alpha = 0$ corresponds to the "free motion", the so-called *free boundary conditions* (better name than Kirchhoff)

- Similarly, $U = I - \frac{2}{n-i\beta}\mathcal{J}$ describes the δ_s' *coupling*

$$\psi_j'(0) = \psi_k'(0) =: \psi'(0)\,,\;\; j,k = 1,\dots,n\,,\;\; \sum_{j=1}^n \psi_j(0) = \beta\psi'(0)$$

with $\beta \in \mathbb{R}$; for $\beta = \infty$ we get *Neumann* decoupling, etc.

A lot is known about graph spectra

- many particular examples

A lot is known about graph spectra

- many particular examples
- a spectral *duality* mapping the problem on a difference equation: originally by Alexander and de Gennes in the early 80’s, mathematically rigorous [E’97], [Cattaneo’97]

A lot is known about graph spectra

- many particular examples
- a spectral *duality* mapping the problem on a difference equation: originally by Alexander and de Gennes in the early 80's, mathematically rigorous [E'97], [Cattaneo'97]
- *trace formulæ* expressing spectral properties a compact graph Hamiltonian in terms of closed orbits on the graph – [Kottos-Smilansky'97], [Bolte-Endres'09]

A lot is known about graph spectra

- many particular examples
- a spectral *duality* mapping the problem on a difference equation: originally by Alexander and de Gennes in the early 80's, mathematically rigorous [E'97], [Cattaneo'97]
- *trace formulæ* expressing spectral properties a compact graph Hamiltonian in terms of closed orbits on the graph – [Kottos-Smilansky'97], [Bolte-Endres'09]
- *inverse problems* like "Can one hear the shape of a graph?" [Gutkin-Smilansky'01] and many others

A lot is known about graph spectra

- many particular examples
- a spectral *duality* mapping the problem on a difference equation: originally by Alexander and de Gennes in the early 80's, mathematically rigorous [E'97], [Cattaneo'97]
- *trace formulæ* expressing spectral properties a compact graph Hamiltonian in terms of closed orbits on the graph – [Kottos-Smilansky'97], [Bolte-Endres'09]
- *inverse problems* like "Can one hear the shape of a graph?" [Gutkin-Smilansky'01] and many others
- *Anderson localization* on graphs [Aizenman-Sims -Warzel'06], [E-Helm-Stollmann'07], [Hislop-Post'08]

A lot is known about graph spectra

- many particular examples
- a spectral *duality* mapping the problem on a difference equation: originally by Alexander and de Gennes in the early 80's, mathematically rigorous [E'97], [Cattaneo'97]
- *trace formulæ* expressing spectral properties a compact graph Hamiltonian in terms of closed orbits on the graph – [Kottos-Smilansky'97], [Bolte-Endres'09]
- *inverse problems* like "Can one hear the shape of a graph?" [Gutkin-Smilansky'01] and many others
- *Anderson localization* on graphs [Aizenman-Sims -Warzel'06], [E-Helm-Stollmann'07], [Hislop-Post'08]
- *gaps by decoration* [Aizenman-Schenker'01] and others

A lot is known about graph spectra

- many particular examples
- a spectral *duality* mapping the problem on a difference equation: originally by Alexander and de Gennes in the early 80's, mathematically rigorous [E'97], [Cattaneo'97]
- *trace formulæ* expressing spectral properties a compact graph Hamiltonian in terms of closed orbits on the graph – [Kottos-Smilansky'97], [Bolte-Endres'09]
- *inverse problems* like "Can one hear the shape of a graph?" [Gutkin-Smilansky'01] and many others
- *Anderson localization* on graphs [Aizenman-Sims -Warzel'06], [E-Helm-Stollmann'07], [Hislop-Post'08]
- *gaps by decoration* [Aizenman-Schenker'01] and others
- and more

First problem concerning resonances

- A typical resonances situation arises for *finite graphs with semiinfinite leads*

First problem concerning resonances

- A typical resonances situation arises for *finite graphs with semiinfinite leads*
- *Different resonances definitions:* poles of continued resolvent, singularities of on-shell S matrix

First problem concerning resonances

- A typical resonances situation arises for *finite graphs with semiinfinite leads*
- *Different resonances definitions:* poles of continued resolvent, singularities of on-shell S matrix
- Graphs may exhibit embedded eigenvalues due to *invalidity of uniform continuation*

First problem concerning resonances

- A typical resonances situation arises for *finite graphs with semiinfinite leads*
- *Different resonances definitions:* poles of continued resolvent, singularities of on-shell S matrix
- Graphs may exhibit embedded eigenvalues due to *invalidity of uniform continuation*
- Geometric perturbations of such graphs may turn the embedded eigenvalues into resonances

Preliminaries

Consider a graph Γ with vertices $\mathcal{V} = \{\mathcal{X}_j : j \in I\}$, finite edges $\mathcal{L} = \{\mathcal{L}_{jn} : (\mathcal{X}_j, \mathcal{X}_n) \in I_{\mathcal{L}} \subset I \times I\}$ and infinite edges $\mathcal{L}_\infty = \{\mathcal{L}_{j\infty} : \mathcal{X}_j \in I_{\mathcal{C}}\}$. The state Hilbert space is

$$\mathcal{H} = \bigoplus_{L_j \in \mathcal{L}} L^2([0, l_j]) \oplus \bigoplus_{\mathcal{L}_{j\infty} \in \mathcal{L}_\infty} L^2([0, \infty)),$$

its elements are columns $\psi = (f_j : \mathcal{L}_j \in \mathcal{L},\, g_j : \mathcal{L}_{j\infty} \in \mathcal{L}_\infty)^T$.

Preliminaries

Consider a graph Γ with vertices $\mathcal{V} = \{\mathcal{X}_j : j \in I\}$, finite edges $\mathcal{L} = \{\mathcal{L}_{jn} : (\mathcal{X}_j, \mathcal{X}_n) \in I_{\mathcal{L}} \subset I \times I\}$ and infinite edges $\mathcal{L}_\infty = \{\mathcal{L}_{j\infty} : \mathcal{X}_j \in I_{\mathcal{C}}\}$. The state Hilbert space is

$$\mathcal{H} = \bigoplus_{L_j \in \mathcal{L}} L^2([0, l_j]) \oplus \bigoplus_{\mathcal{L}_{j\infty} \in \mathcal{L}_\infty} L^2([0, \infty)),$$

its elements are columns $\psi = (f_j : \mathcal{L}_j \in \mathcal{L},\, g_j : \mathcal{L}_{j\infty} \in \mathcal{L}_\infty)^T$.

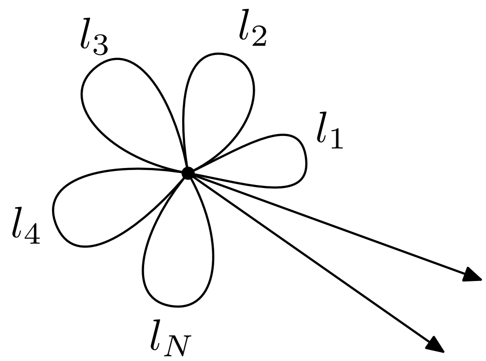
The Hamiltonian acts as $-\mathrm{d}^2/\mathrm{d}x^2$ on each link satisfying the boundary conditions

$$(U_j - I)\Psi_j + i(U_j + I)\Psi_j' = 0$$

characterized by unitary matrices U_j at the vertices $\mathcal{X}_j$.

A universal setting for graphs with leads

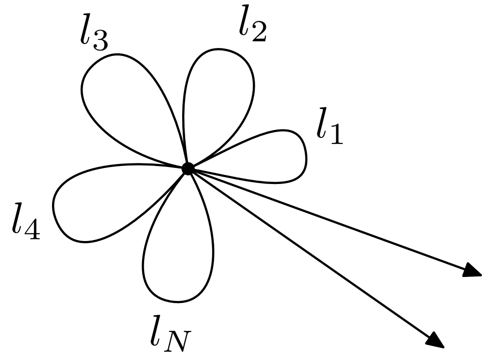
A useful trick is to replace Γ "flower-like" graph with one vertex by putting all the vertices to a single point,



Its degree is $2N + M$ where $N := \mathrm{card}\,\mathcal{L}$ and $M := \mathrm{card}\,\mathcal{L}_\infty$

A universal setting for graphs with leads

A useful trick is to replace Γ "flower-like" graph with one vertex by putting all the vertices to a single point,



Its degree is $2N + M$ where $N := \mathrm{card}\,\mathcal{L}$ and $M := \mathrm{card}\,\mathcal{L}_\infty$

The coupling is described by "big", $(2N+M)\times(2N+M)$ unitary block diagonal matrix U consisting of blocks U_j as follows,

$$(U-I)\Psi + i(U+I)\Psi' = 0\,;$$

the block structure of U encodes the original topology of Γ.

Equivalence of resonance definitions

Resonances as poles of analytically continued resolvent, $(H - \lambda\,\mathrm{id})^{-1}$. One way to reveal the poles is to use *exterior complex scaling*. Looking for complex eigenvalues of the scaled operator we do not change the compact-graph part: we set $f_j(x) = a_j \sin kx + b_j \cos kx$ on the internal edges

Equivalence of resonance definitions

Resonances as poles of analytically continued resolvent, $(H - \lambda \,\mathrm{id})^{-1}$. One way to reveal the poles is to use *exterior complex scaling*. Looking for complex eigenvalues of the scaled operator we do not change the compact-graph part: we set $f_j(x) = a_j \sin kx + b_j \cos kx$ on the internal edges

On the semi-infinite edges are scaled by $g_{j\theta}(x) = \mathrm{e}^{\theta/2} g_j(x\mathrm{e}^{\theta})$ with an imaginary θ rotating the essential spectrum into the lower complex halfplane so that the poles of the resolvent on the second sheet become "uncovered" for θ large enough. The "exterior" boundary values are thus equal to

$$g_j(0) = \mathrm{e}^{-\theta/2} g_{j\theta}, \quad g_j'(0) = ik\mathrm{e}^{-\theta/2} g_{j\theta}$$

Resolvent resonances

Substituting into the boundary conditions we get

$$(U-I)C_1(k)\begin{pmatrix} a_1 \\ b_1 \\ a_2 \\ \vdots \\ b_N \\ \mathrm{e}^{-\theta/2}g_{1\theta} \\ \vdots \\ \mathrm{e}^{-\theta/2}g_{M\theta} \end{pmatrix} + ik(U+I)C_2(k)\begin{pmatrix} a_1 \\ b_1 \\ a_2 \\ \vdots \\ b_N \\ \mathrm{e}^{-\theta/2}g_{1\theta} \\ \vdots \\ \mathrm{e}^{-\theta/2}g_{M\theta} \end{pmatrix} = 0,$$

where $C_j := \mathrm{diag}\,(C_j^{(1)}(k), C_j^{(2)}(k), \dots, C_j^{(N)}(k), i^{j-1}I_{M\times M})$, with

$$C_1^{(j)}(k) = \begin{pmatrix} 0 & 1 \\ \sin kl_j & \cos kl_j \end{pmatrix}, \qquad C_2^{(j)}(k) = \begin{pmatrix} 1 & 0 \\ -\cos kl_j & \sin kl_j \end{pmatrix}$$

Scattering resonances

In this case we choose a combination of two planar waves, $g_j = c_j \mathrm{e}^{-ikx} + d_j \mathrm{e}^{ikx}$, as an Ansatz on the external edges; we ask about poles of the matrix $S = S(k)$ which maps the amplitudes of the incoming waves $c = \{c_n\}$ into amplitudes of the outgoing waves $d = \{d_n\}$ by $d = Sc$.

Scattering resonances

In this case we choose a combination of two planar waves, $g_j = c_j \mathrm{e}^{-ikx} + d_j \mathrm{e}^{ikx}$, as an Ansatz on the external edges; we ask about poles of the matrix $S = S(k)$ which maps the amplitudes of the incoming waves $c = \{c_n\}$ into amplitudes of the outgoing waves $d = \{d_n\}$ by $d = Sc$. The b.c. give

$$(U-I)C_1(k)\begin{pmatrix} a_1 \\ b_1 \\ a_2 \\ \vdots \\ b_N \\ c_1+d_1 \\ \vdots \\ c_M+d_M \end{pmatrix} + ik(U+I)C_2(k)\begin{pmatrix} a_1 \\ b_1 \\ a_2 \\ \vdots \\ b_N \\ d_1-c_1 \\ \vdots \\ d_M-c_M \end{pmatrix} = 0$$

Equivalence of resonance definitions, cont'd

Since we are interested in zeros of $\det S^{-1}$, we regard the above relation as an equation for variables a_j, b_j and d_j while c_j are just parameters. Eliminating the variables a_j, b_j one derives from here a system of M equations expressing the map $S^{-1}d = c$. It is *not* solvable, $\det S^{-1} = 0$, if

$$\det\left[(U-I)\,C_1(k) + ik(U+I)\,C_2(k)\right] = 0$$

Equivalence of resonance definitions, cont'd

Since we are interested in zeros of $\det S^{-1}$, we regard the above relation as an equation for variables a_j, b_j and d_j while c_j are just parameters. Eliminating the variables a_j, b_j one derives from here a system of M equations expressing the map $S^{-1}d = c$. It is *not* solvable, $\det S^{-1} = 0$, if

$$\det\left[(U-I)\,C_1(k) + ik(U+I)\,C_2(k)\right] = 0$$

This is the same condition as for the previous system of equations, hence we are able to conclude:

Proposition [E-Lipovský'10]: The two above resonance notions, the resolvent and scattering one, are equivalent for quantum graphs.

Effective coupling on the finite graph

The problem can be reduced to the compact subgraph only.

We write U in the block form, $U = \begin{pmatrix} U_1 & U_2 \\ U_3 & U_4 \end{pmatrix}$, where U_1 is the $2N \times 2N$ refers to the compact subgraph, U_4 is the $M \times M$ matrix related to the exterior part, and U_2 and U_3 are rectangular matrices connecting the two.

Effective coupling on the finite graph

The problem can be reduced to the compact subgraph only.

We write U in the block form, $U = \begin{pmatrix} U_1 & U_2 \\ U_3 & U_4 \end{pmatrix}$, where U_1 is the $2N \times 2N$ refers to the compact subgraph, U_4 is the $M \times M$ matrix related to the exterior part, and U_2 and U_3 are rectangular matrices connecting the two.

Eliminating the external part leads to an effective coupling on the compact subgraph expressed by the condition

$$(\tilde{U}(k) - I)(f_1, \dots, f_{2N})^{\mathrm{T}} + i(\tilde{U}(k) + I)(f_1', \dots, f_{2N}')^{\mathrm{T}} = 0\,,$$

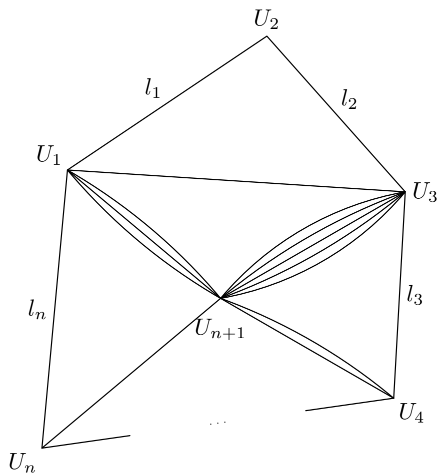
where the corresponding coupling matrix

$$\tilde{U}(k) := U_1 - (1-k)U_2[(1-k)U_4 - (k+1)I]^{-1}U_3$$

is obviously *energy-dependent* and, in general, *non-unitary*

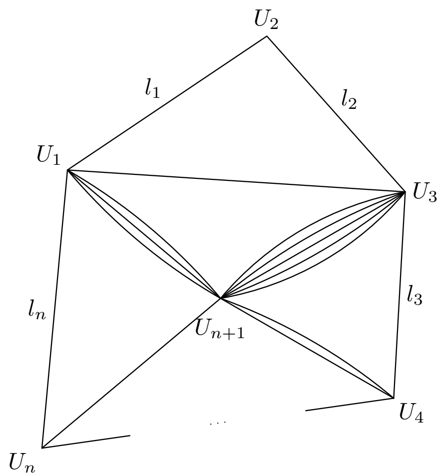
Embedded ev's for commensurate edges

Suppose that the compact part contains a loop consisting of rationally related edges



Embedded ev's for commensurate edges

Suppose that the compact part contains a loop consisting of rationally related edges



Then the graph Hamiltonian can have eigenvalues *with compactly supported eigenfunctions*; they are embedded in the continuum corresponding to external semiinfinite edges

Embedded eigenvalues

Theorem [E-Lipovský'10]: Let Γ consist of a single vertex and N finite edges emanating from this vertex and ending at it, with the coupling described by a $2N \times 2N$ unitary matrix U. Let the lengths of the first n edges be integer multiples of a positive real number l_0. If the rectangular $2N \times 2n$ matrix

$$M_{\mathrm{even}} = \begin{pmatrix} u_{11} & u_{12}-1 & u_{13} & u_{14} & \cdots & u_{1,2n-1} & u_{1,2n} \\ u_{21}-1 & u_{22} & u_{23} & u_{24} & \cdots & u_{2,2n-1} & u_{2,2n} \\ u_{31} & u_{32} & u_{33} & u_{34}-1 & \cdots & u_{3,2n-1} & u_{3,2n} \\ u_{41} & u_{42} & u_{43}-1 & u_{44} & \cdots & u_{4,2n-1} & u_{4,2n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ u_{2N-1,1} & u_{2N-1,2} & u_{2N-1,3} & u_{2N-1,4} & \cdots & u_{2N-1,2n-1} & u_{2N-1,2n} \\ u_{2N,1} & u_{2N,2} & u_{2N,3} & u_{2N,4} & \cdots & u_{2N,2n-1} & u_{2N,2n} \end{pmatrix}$$

has rank smaller than $2n$ then the spectrum of the corresponding Hamiltonian $H = H_U$ contains eigenvalues of the form $\epsilon = 4m^2\pi^2/l_0^2$ with $m \in \mathbb{N}$ and the multiplicity of these eigenvalues is at least the difference between $2n$ and the rank of M_{even}.

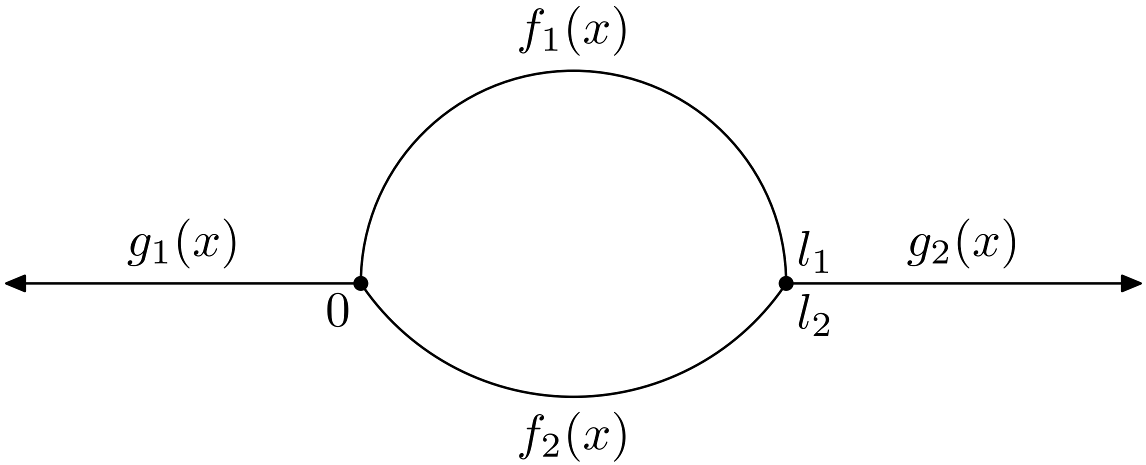
Embedded eigenvalues

Theorem [E-Lipovský'10]: Let Γ consist of a single vertex and N finite edges emanating from this vertex and ending at it, with the coupling described by a $2N \times 2N$ unitary matrix U. Let the lengths of the first n edges be integer multiples of a positive real number l_0. If the rectangular $2N \times 2n$ matrix

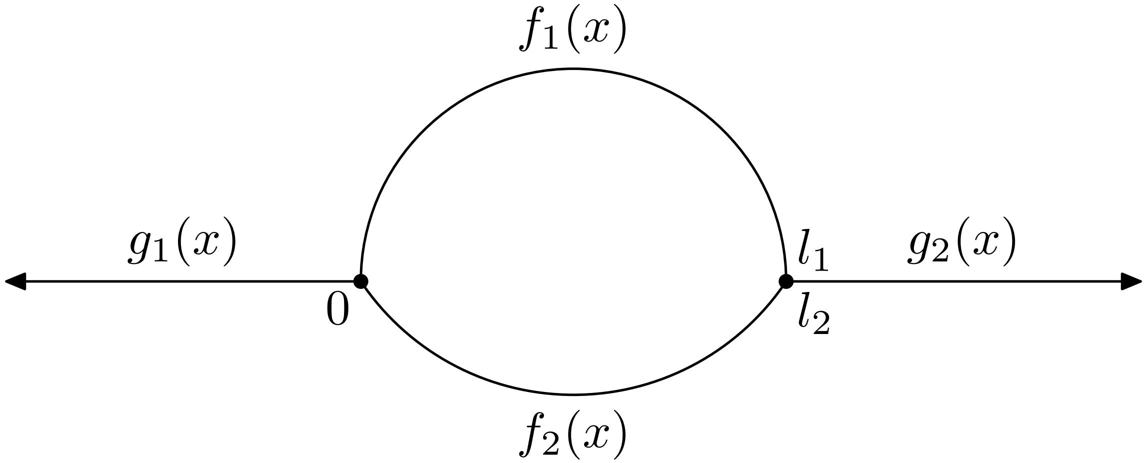
$$M_{\text{even}} = \begin{pmatrix} u_{11} & u_{12}-1 & u_{13} & u_{14} & \cdots & u_{1,2n-1} & u_{1,2n} \\ u_{21}-1 & u_{22} & u_{23} & u_{24} & \cdots & u_{2,2n-1} & u_{2,2n} \\ u_{31} & u_{32} & u_{33} & u_{34}-1 & \cdots & u_{3,2n-1} & u_{3,2n} \\ u_{41} & u_{42} & u_{43}-1 & u_{44} & \cdots & u_{4,2n-1} & u_{4,2n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ u_{2N-1,1} & u_{2N-1,2} & u_{2N-1,3} & u_{2N-1,4} & \cdots & u_{2N-1,2n-1} & u_{2N-1,2n} \\ u_{2N,1} & u_{2N,2} & u_{2N,3} & u_{2N,4} & \cdots & u_{2N,2n-1} & u_{2N,2n} \end{pmatrix}$$

has rank smaller than $2n$ then the spectrum of the corresponding Hamiltonian $H = H_U$ contains eigenvalues of the form $\epsilon = 4m^2\pi^2/l_0^2$ with $m \in \mathbb{N}$ and the multiplicity of these eigenvalues is at least the difference between $2n$ and the rank of M_{even}. This result corresponds to $\sin kl_0/2 = 0$, an analogous claim is valid in the odd case, $\cos kl_0/2 = 0$.

Example: a loop with two leads



Example: a loop with two leads



The setting is as above, the b.c. at the nodes are

$$
\begin{aligned}
f_1(0) &= f_2(0)\,, \qquad f_1(l_1) = f_2(l_2)\,, \\
f_1(0) &= \alpha_1^{-1}(f_1'(0) + f_2'(0)) + \gamma_1 g_1'(0)\,, \\
f_1(l_1) &= -\alpha_2^{-1}(f_1'(l_1) + f_2'(l_2)) + \gamma_2 g_2'(0)\,, \\
g_1(0) &= \bar\gamma_1(f_1'(0) + f_2'(0)) + \tilde\alpha_1^{-1} g_1'(0)\,, \\
g_2(0) &= -\bar\gamma_2(f_1'(l_1) + f_2'(l_2)) + \tilde\alpha_2^{-1} g_2'(0)
\end{aligned}
$$

Resonance condition

Writing the loop edges as $l_1 = l(1-\lambda)$, $l_2 = l(1+\lambda)$, $\lambda \in [0,1]$ — which effectively means shifting one of the connections points around the loop as λ is changing – one arrives at the final resonance condition

$$\sin kl(1-\lambda)\sin kl(1+\lambda) - 4k^2\beta_1^{-1}(k)\beta_2^{-1}(k)\sin^2 kl$$

$$+k[\beta_1^{-1}(k)+\beta_2^{-1}(k)]\sin 2kl = 0\,,$$

where $\beta_i^{-1}(k) := \alpha_i^{-1} + \frac{ik|\gamma_i|^2}{1-ik\tilde{\alpha}_i^{-1}}$.

Resonance condition

Writing the loop edges as $l_1 = l(1-\lambda)$, $l_2 = l(1+\lambda)$, $\lambda \in [0,1]$ — which effectively means shifting one of the connections points around the loop as λ is changing – one arrives at the final resonance condition

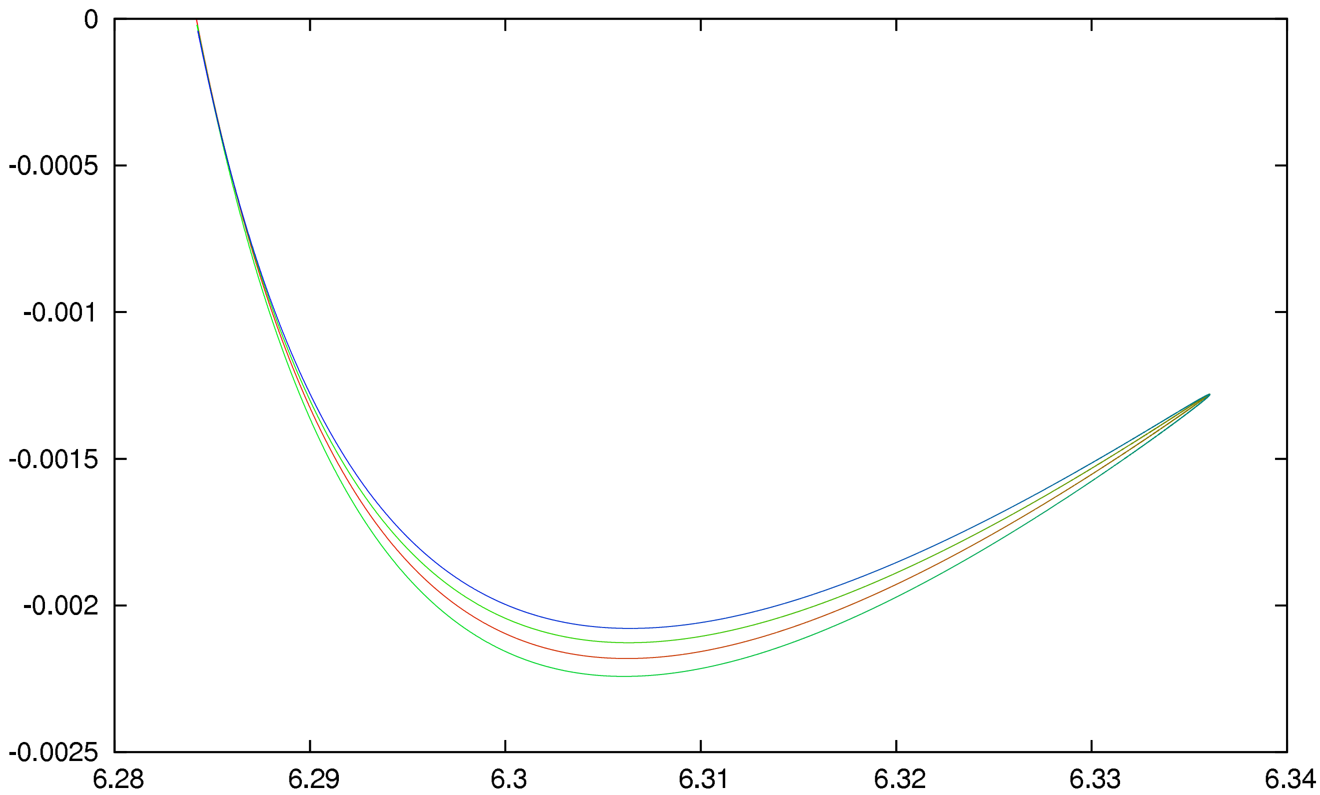
$$\sin kl(1-\lambda)\sin kl(1+\lambda) - 4k^2\beta_1^{-1}(k)\beta_2^{-1}(k)\sin^2 kl$$

$$+k[\beta_1^{-1}(k)+\beta_2^{-1}(k)]\sin 2kl = 0\,,$$

where $\beta_i^{-1}(k) := \alpha_i^{-1} + \frac{ik|\gamma_i|^2}{1-ik\tilde{\alpha}_i^{-1}}$.

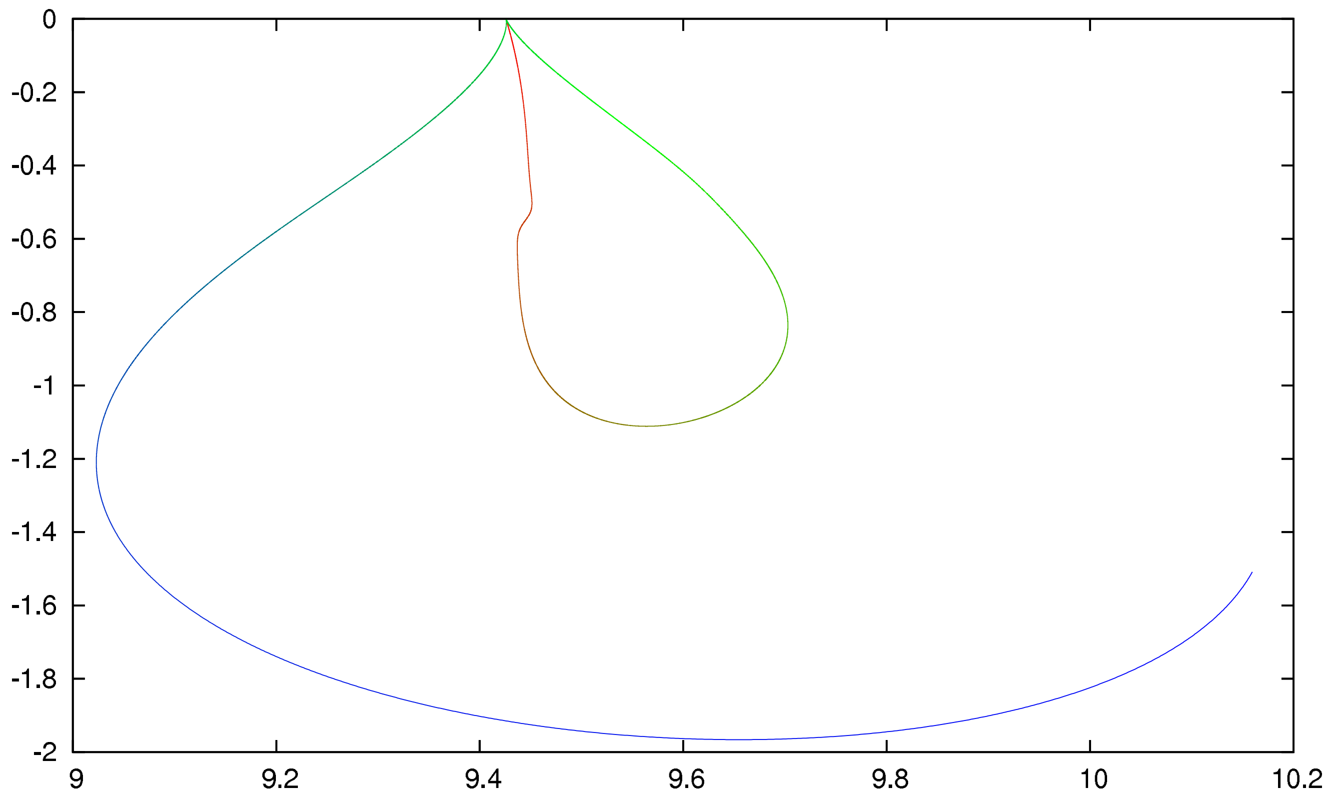
The condition can be solved numerically to find the resonance trajectories with respect to the variable λ.

Pole trajectory



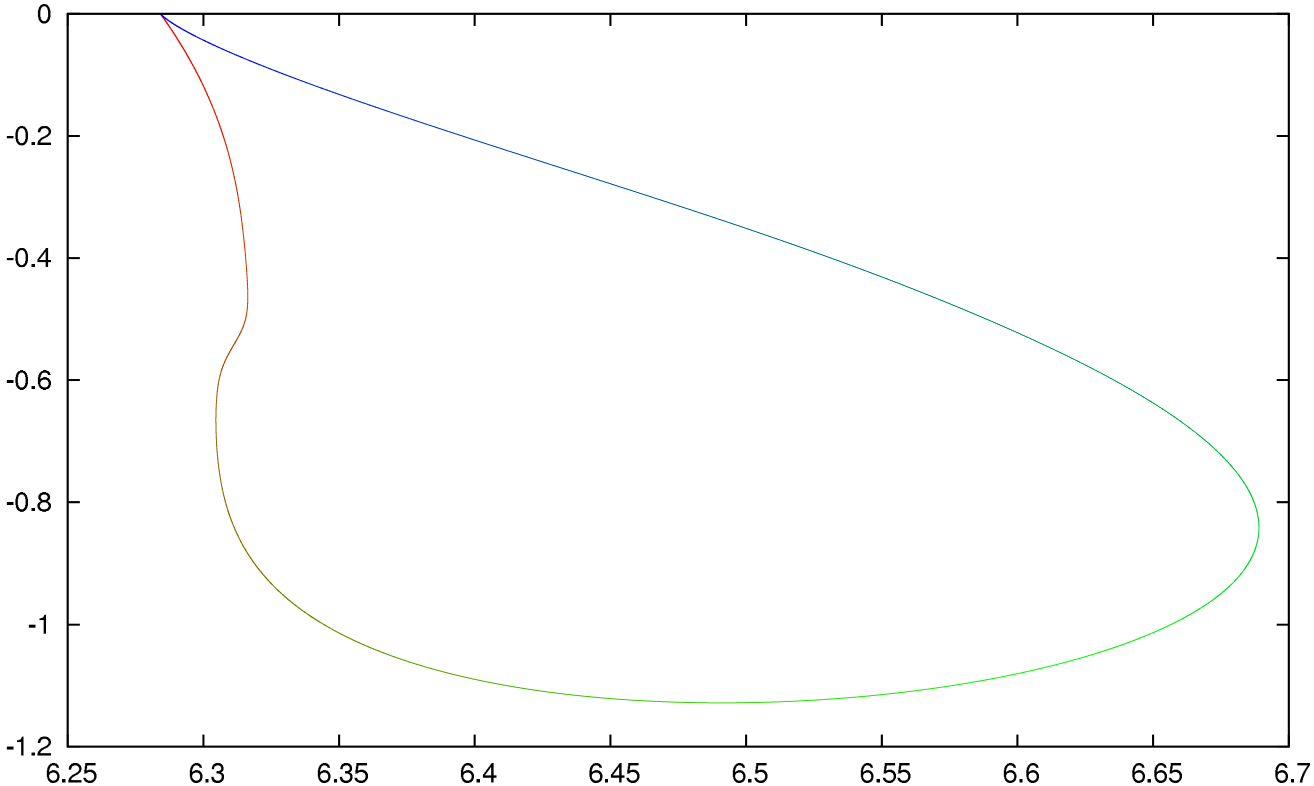
The trajectory of the resonance pole in the lower complex halfplane starting from $k_0 = 2\pi$ for the coefficients values $\alpha_1^{-1} = 1$, $\tilde{\alpha}_1^{-1} = -2$, $|\gamma_1|^2 = 1$, $\alpha_2^{-1} = 0$, $\tilde{\alpha}_2^{-1} = 1$, $|\gamma_2|^2 = 1$, $n = 2$. The colour coding shows the dependence on λ changing from red ($\lambda = 0$) to blue ($\lambda = 1$).

Pole trajectory



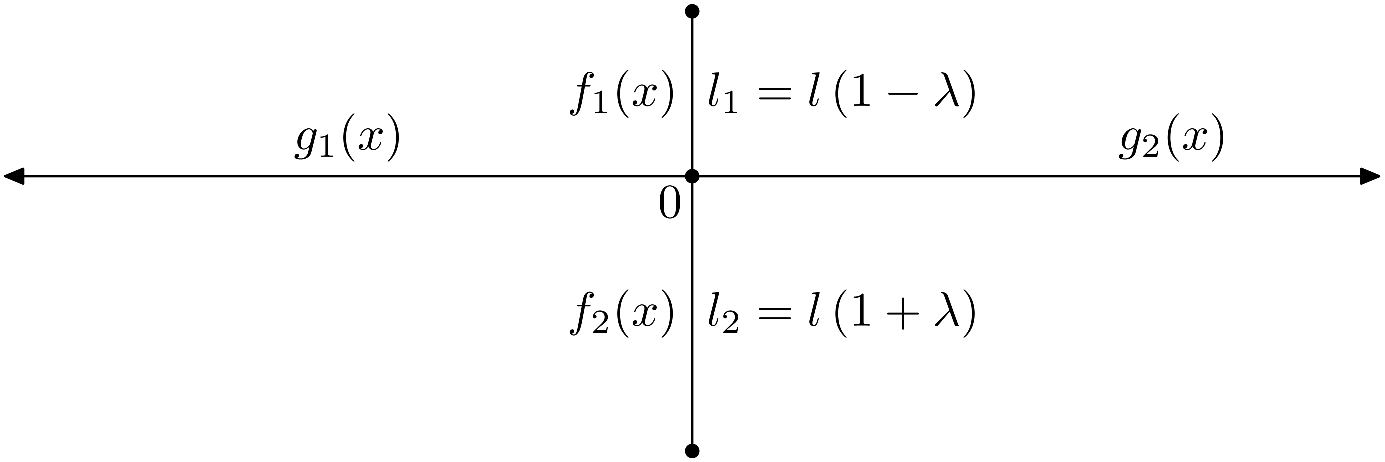
The trajectory of the resonance pole starting at $k_0 = 3\pi$ for the coefficients values $\alpha_1^{-1} = 1$, $\alpha_2^{-1} = 1$, $\tilde{\alpha}_1^{-1} = 1$, $\tilde{\alpha}_2^{-1} = 1$, $|\gamma_1|^2 = |\gamma_2|^2 = 1$, $n = 3$. The colour coding is the same as in the previous picture.

Pole trajectory

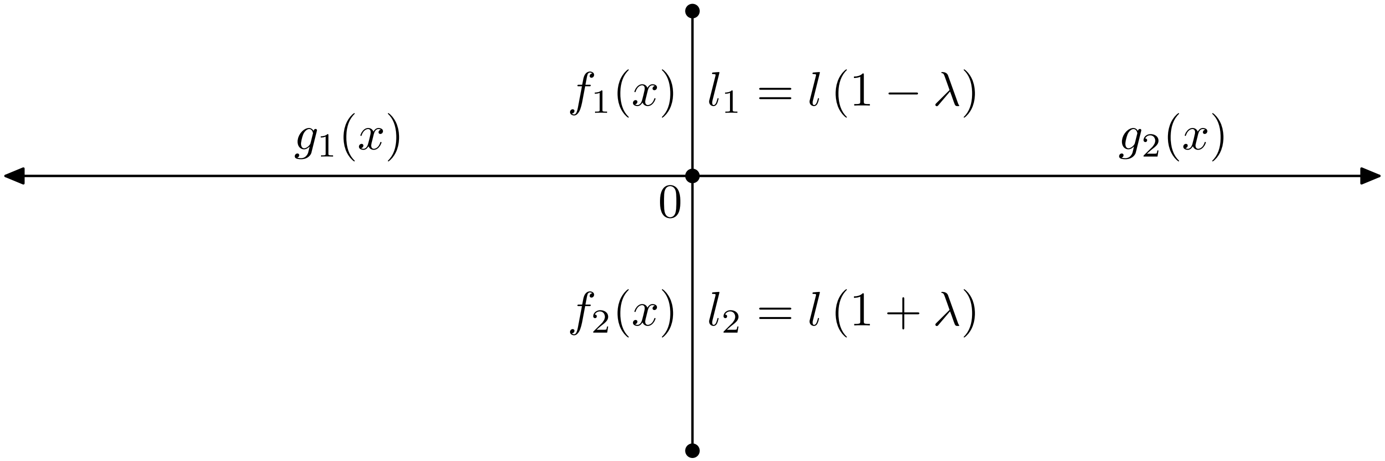


The trajectory of the resonance pole starting at $k_0 = 2\pi$ for the coefficients values $\alpha_1^{-1} = 1$, $\alpha_2^{-1} = 1$, $\tilde{\alpha}_1^{-1} = 1$, $\tilde{\alpha}_2^{-1} = 1$, $|\gamma_1|^2 = 1$, $|\gamma_2|^2 = 1$, $n = 2$. The colour coding is the same as above.

Example: a cross-shaped graph



Example: a cross-shaped graph



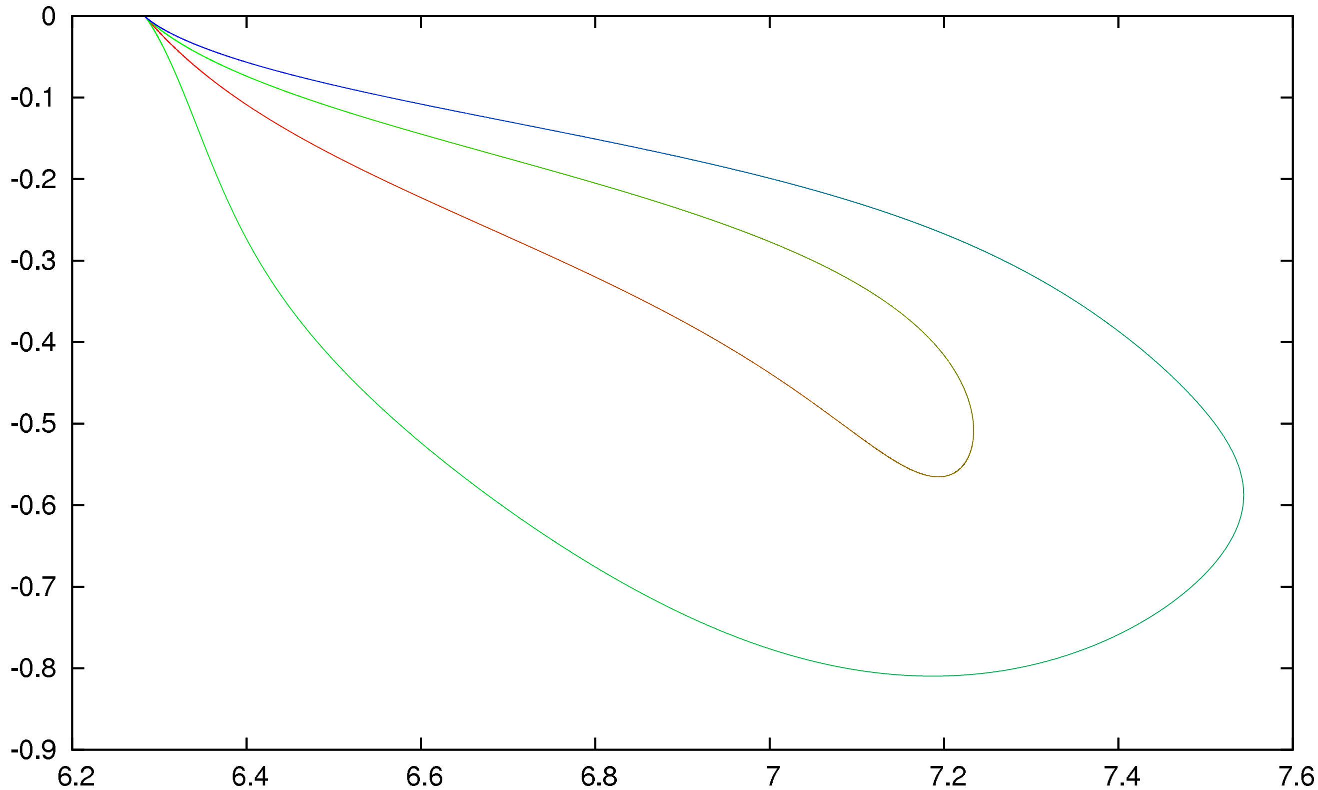
This time we restrict ourselves to the δ coupling as the boundary condition at the vertex and we consider Dirichlet conditions at the loose ends, i.e.

$$\begin{aligned} f_1(0) &= f_2(0) = g_1(0) = g_2(0)\,, \\ f_1(l_1) &= f_2(l_2) = 0\,, \\ \alpha f_1(0) &= f_1'(0) + f_2'(0) + g_1'(0) + g_2'(0)\,. \end{aligned}$$

leading to the resonance condition

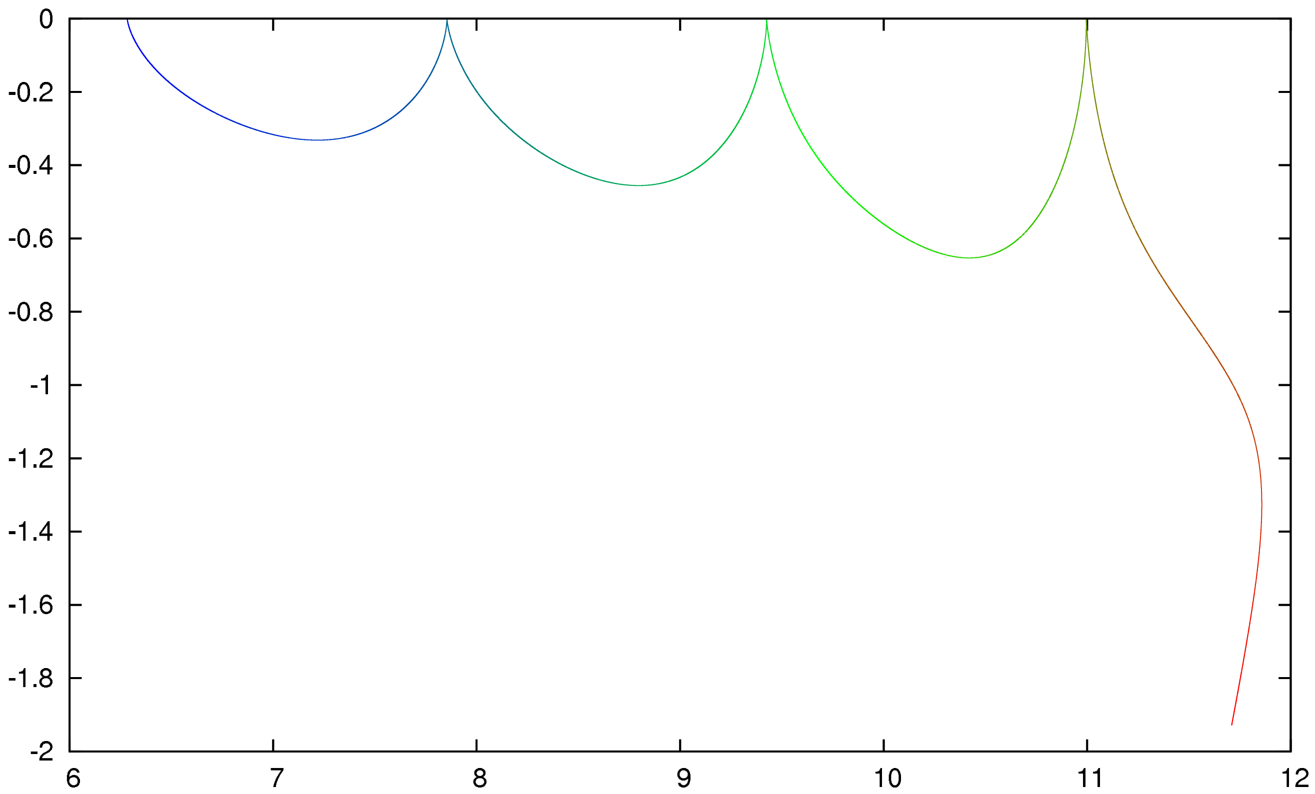
$$2k \sin 2kl + (\alpha - 2ik)(\cos 2kl\lambda - \cos 2kl) = 0$$

Pole trajectory



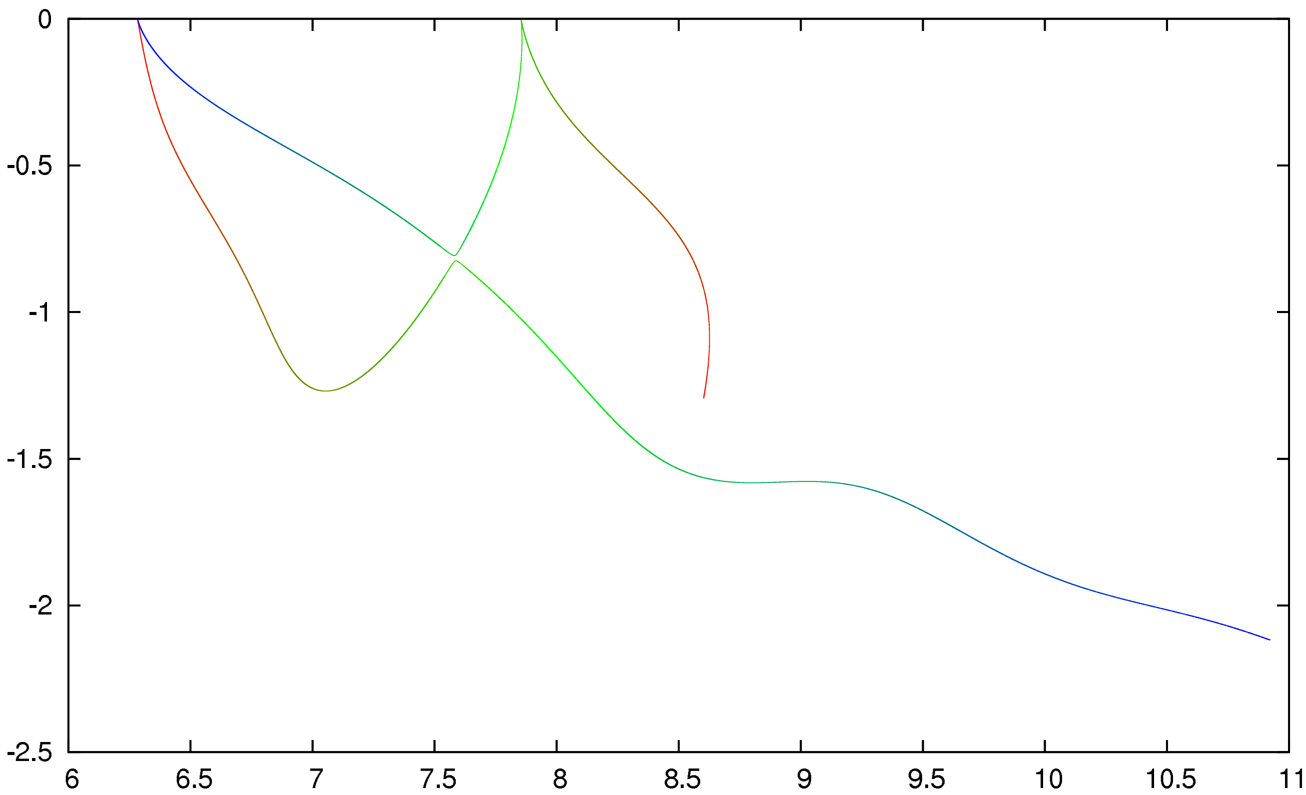
The trajectory of the resonance pole starting at $k_0 = 2\pi$ for the coefficients values $\alpha = 10$, $n = 2$. The colour coding is the same as in the previous figures.

Pole trajectory



The trajectory of the resonance pole for the coefficients values $\alpha = 1$, $n = 2$. The colour coding is the same as above.

Pole trajectory



The trajectories of two resonance poles for the coefficients values $\alpha = 2.596$, $n = 2$. We can see an avoided resonance crossing – the former eigenvalue "travelling from the left to the right" interchanges with the former resonance "travelling the other way" and ending up as an embedded eigenvalue. The colour coding is the same as above.

Multiplicity preservation

In a similar way resonances can be generated in the general case. What is important, nothing is “lost”:

Theorem [E-Lipovský’10]: Let Γ have N finite edges of lengths l_i, M infinite edges, and the coupling given by $U = \begin{pmatrix} U_1 & U_2 \\ U_3 & U_4 \end{pmatrix}$, where U_4 refers to infinite edge coupling. Let k_0 satisfying $\det[(1-k_0)U_4 - (1+k_0)I] \neq 0$ be a pole of the resolvent $(H - \lambda\,\mathrm{id})^{-1}$ of a multiplicity d. Let Γ_ε be a geometrically perturbed quantum graph with edge lengths $l_i(1+\varepsilon)$ and the same coupling. Then there is $\varepsilon_0 > 0$ s.t. for all $\vec{\varepsilon} \in \mathcal{U}_{\varepsilon_0}(0)$ the sum of multiplicities of the resolvent poles in a sufficiently small neighbourhood of k_0 is d.

Multiplicity preservation

In a similar way resonances can be generated in the general case. What is important, nothing is “lost”:

Theorem [E-Lipovský’10]: Let Γ have N finite edges of lengths l_i, M infinite edges, and the coupling given by $U = \begin{pmatrix} U_1 & U_2 \\ U_3 & U_4 \end{pmatrix}$, where U_4 refers to infinite edge coupling. Let k_0 satisfying $\det[(1-k_0)U_4 - (1+k_0)I] \neq 0$ be a pole of the resolvent $(H - \lambda\,\mathrm{id})^{-1}$ of a multiplicity d. Let Γ_ε be a geometrically perturbed quantum graph with edge lengths $l_i(1+\varepsilon)$ and the same coupling. Then there is $\varepsilon_0 > 0$ s.t. for all $\vec{\varepsilon} \in \mathcal{U}_{\varepsilon_0}(0)$ the sum of multiplicities of the resolvent poles in a sufficiently small neighbourhood of k_0 is d.

Remark: The result holds only perturbatively, for larger values of ε poles may, e.g., escape to infinity.

Second problem: (non-)Weyl asymptotics

Let us now look into *high-energy asymptotics* of graph resonances. Introduce *counting function* $N(R, F)$ as the number of zeros of $F(k)$ in the circle $\{k : \; |k| < R\}$ of given radius $R > 0$, algebraic multiplicities taken into account.

If F comes from resonance secular equation we count in this way *number of resonances* within a given (semi)circle

Second problem: (non-)Weyl asymptotics

Let us now look into *high-energy asymptotics* of graph resonances. Introduce *counting function* $N(R, F)$ as the number of zeros of $F(k)$ in the circle $\{k : |k| < R\}$ of given radius $R > 0$, algebraic multiplicities taken into account.

If F comes from resonance secular equation we count in this way *number of resonances* within a given (semi)circle

You have heard Brian Davies talk containing an intriguing observation [Davies-Pushnitski'10]:

if the coupling is *Kirchhoff* and some external vertices are *balanced*, i.e. connecting the same number of internal and external edges, then the leading term in the asymptotics may be *less than Weyl formula prediction*

Second problem: (non-)Weyl asymptotics

Let us now look into *high-energy asymptotics* of graph resonances. Introduce *counting function* $N(R,F)$ as the number of zeros of $F(k)$ in the circle $\{k:\,|k|<R\}$ of given radius $R>0$, algebraic multiplicities taken into account.

If F comes from resonance secular equation we count in this way *number of resonances* within a given (semi)circle

You have heard Brian Davies talk containing an intriguing observation [Davies-Pushnitski'10]:

if the coupling is *Kirchhoff* and some external vertices are *balanced*, i.e. connecting the same number of internal and external edges, then the leading term in the asymptotics may be *less than Weyl formula prediction*

Let us look how the situation looks like for graphs with more general vertex couplings

Recall the resonance condition

Denote $e_j^{\pm} := \mathrm{e}^{\pm ikl_j}$ and $e^{\pm} := \Pi_{j=1}^{N} e_j^{\pm}$, then secular eq-n is

$$0 = \det\left\{\frac{1}{2}[(U-I)+k(U+I)]E_1(k) + \frac{1}{2}[(U-I)+k(U+I)]E_2 + k(U+I)E_3 \right.$$
$$\left. + (U-I)E_4 + [(U-I)-k(U+I)]\,\mathrm{diag}\,(0,\dots,0,I_{M\times M})\right\},$$

where $E_i(k) = \mathrm{diag}\left(E_i^{(1)}, E_i^{(2)}, \dots, E_i^{(N)}, 0, \dots, 0\right)$, $i = 1,2,3,4$, consists of N nontrivial 2×2 blocks

$$E_1^{(j)} = \begin{pmatrix} 0 & 0 \\ -ie_j^{+} & e_j^{+} \end{pmatrix},\; E_2^{(j)} = \begin{pmatrix} 0 & 0 \\ ie_j^{-} & e_j^{-} \end{pmatrix},\; E_3^{(j)} = \begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix},\; E_4^{(j)} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

and the trivial $M \times M$ part.

Recall the resonance condition

Denote $e_j^{\pm} := \mathrm{e}^{\pm ikl_j}$ and $e^{\pm} := \Pi_{j=1}^{N} e_j^{\pm}$, then secular eq-n is

$$0 = \det \left\{ \frac{1}{2}[(U-I) + k(U+I)]E_1(k) + \frac{1}{2}[(U-I) + k(U+I)]E_2 + k(U+I)E_3 \right.$$

$$\left. + (U-I)E_4 + [(U-I) - k(U+I)] \,\mathrm{diag}\,(0,\dots,0,I_{M\times M}) \right\},$$

where $E_i(k) = \mathrm{diag}\left(E_i^{(1)}, E_i^{(2)}, \dots, E_i^{(N)}, 0, \dots, 0\right)$,

$i = 1,2,3,4$, consists of N nontrivial 2×2 blocks

$$E_1^{(j)} = \begin{pmatrix} 0 & 0 \\ -ie_j^{+} & e_j^{+} \end{pmatrix},\; E_2^{(j)} = \begin{pmatrix} 0 & 0 \\ ie_j^{-} & e_j^{-} \end{pmatrix},\; E_3^{(j)} = \begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix},\; E_4^{(j)} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

and the trivial $M\times M$ part.

Looking for zeros of the *rhs* we can employ a modification of a classical result on zeros of exponential sums [Langer'31]

Exponential sum zeros

Theorem: Let $F(k) = \sum_{r=0}^{n} a_r(k)\, \mathrm{e}^{ik\sigma_r}$, where $a_r(k)$ are rational functions of the complex variable k with complex coefficients, and $\sigma_r \in \mathbb{R}$, $\sigma_0 < \sigma_1 < \ldots < \sigma_n$. Suppose that $\lim_{k\to\infty} a_0(k) \neq 0$ and $\lim_{k\to\infty} a_n(k) \neq 0$. There exist a compact $\Omega \subset \mathbb{C}$, real numbers m_r and positive K_r, $r = 1, \ldots, n$, such that the zeros of $F(k)$ outside Ω lie in the logarithmic strips bounded by the curves $-\mathrm{Im}\, k + m_r \log |k| = \pm K_r$ and the counting function behaves in the limit $R \to \infty$ as

$$N(R, F) = \frac{\sigma_n - \sigma_0}{\pi} R + \mathcal{O}(1)$$

Application of the theorem

We need the coefficients at $e^{\pm}$ in the resonance condition. Let us pass to the effective b.c. formulation,

$$\begin{aligned} 0 &= \det\Big\{ \frac{1}{2}[(\tilde{U}(k)-I)+k(\tilde{U}(k)+I)]\tilde{E}_1(k) \\ &+ \frac{1}{2}[(\tilde{U}(k)-I)-k(\tilde{U}(k)+I)]\tilde{E}_2(k)+k(\tilde{U}(k)+I)\tilde{E}_3+(\tilde{U}(k)-I)\tilde{E}_4 \Big\}, \end{aligned}$$

where $\tilde{E}_j$ are the nontrivial $2N \times 2N$ parts of the matrices E_j and I denotes the $2N \times 2N$ unit matrix

Application of the theorem

We need the coefficients at $e^{\pm}$ in the resonance condition. Let us pass to the effective b.c. formulation,

$$\begin{aligned} 0 &= \det\Big\{ \frac{1}{2}[(\tilde U(k)-I)+k(\tilde U(k)+I)]\tilde E_1(k) \\ &\quad + \frac{1}{2}[(\tilde U(k)-I)-k(\tilde U(k)+I)]\tilde E_2(k) + k(\tilde U(k)+I)\tilde E_3 + (\tilde U(k)-I)\tilde E_4 \Big\}, \end{aligned}$$

where $\tilde E_j$ are the nontrivial $2N\times 2N$ parts of the matrices E_j and I denotes the $2N\times 2N$ unit matrix

By a direct computation we get

Lemma: The coefficient of $e^{\pm}$ in the above equation is $\left(\frac{i}{2}\right)^N \det\left[(\tilde U(k)-I)\pm k(\tilde U(k)+I)\right]$

The resonance asymptotics

Theorem [Davies-E-Lipovský'10]: Consider a quantum graph (Γ, H_U) corresponding to Γ with finitely many edges and the coupling at vertices $\mathcal{X}_j$ given by unitary matrices U_j. The asymptotics of the resonance counting function as $R \to \infty$ is of the form

$$N(R,F) = \frac{2W}{\pi} R + \mathcal{O}(1)\,,$$

where W is the effective size of the graph. One always has

$$0 \leq W \leq V := \sum_{j=1}^{N} l_j\,.$$

Moreover $W < V$ (graph is non-Weyl in the terminology of [Davies-Pushnitski'10] if and only if there exists a vertex where the corresponding energy dependent coupling matrix $\tilde{U}_j(k)$ has an eigenvalue $(1-k)/(1+k)$ or $(1+k)/(1-k)$.

Permutation invariant couplings

Now we apply the result to graphs with coupling invariant w.r.t. edge permutations. These are described by matrices $U_j = a_j J + b_j I$, where $a_j,\, b_j \in \mathbb{C}$ such that $|b_j| = 1$ and $|b_j + a_j \deg \mathcal{X}_j| = 1$; matrix J has all entries equal to one.

Note that δ and δ'_{s} are particular cases of such a coupling

Permutation invariant couplings

Now we apply the result to graphs with coupling invariant w.r.t. edge permutations. These are described by matrices $U_j = a_j J + b_j I$, where $a_j,\, b_j \in \mathbb{C}$ such that $|b_j| = 1$ and $|b_j + a_j \deg \mathcal{X}_j| = 1$; matrix J has all entries equal to one.

Note that δ and δ'_s are particular cases of such a coupling

We need two simple auxiliary statements:

Lemma: The matrix $U = aJ_{n\times n} + bI_{n\times n}$ has $n-1$ eigenvalues b and one eigenvalue $na+b$. Its inverse is $U^{-1} = -\frac{a}{b(an+b)} J_{n\times n} + \frac{1}{b} I_{n\times n}$.

Permutation invariant couplings

Now we apply the result to graphs with coupling invariant w.r.t. edge permutations. These are described by matrices $U_j = a_j J + b_j I$, where $a_j,\, b_j \in \mathbb{C}$ such that $|b_j| = 1$ and $|b_j + a_j \deg \mathcal{X}_j| = 1$; matrix J has all entries equal to one.

Note that δ and $\delta'_{\rm s}$ are particular cases of such a coupling

We need two simple auxiliary statements:

Lemma: The matrix $U = aJ_{n\times n} + bI_{n\times n}$ has $n-1$ eigenvalues b and one eigenvalue $na+b$. Its inverse is $U^{-1} = -\frac{a}{b(an+b)} J_{n\times n} + \frac{1}{b} I_{n\times n}$.

Lemma: Let p internal and q external edges be coupled with b.c. given by $U = aJ_{(p+q)\times(p+q)} + bI_{(p+q)\times(p+q)}$. Then the energy-dependent effective matrix is

$$\tilde{U}(k) = \frac{ab(1-k) - a(1+k)}{(aq+b)(1-k) - (k+1)} J_{p\times p} + bI_{p\times p}\,.$$

Asymptotics in the symmetric case

Combining them with the above theorem we find easily that there are only two cases which exhibit non-Weyl asymptotics here

Asymptotics in the symmetric case

Combining them with the above theorem we find easily that there are only two cases which exhibit non-Weyl asymptotics here

Theorem [Davies-E-Lipovský'10]: Let (Γ, H_U) be a quantum graph with permutation-symmetric coupling conditions at the vertices, $U_j = a_j J + b_j I$. Then it has *non-Weyl asymptotics* if and only if at least one of its vertices is balanced, $p = q$, and the coupling at this vertex is either

(a) $f_j = f_n, \quad \forall j, n \le 2p, \quad \sum_{j=1}^{2p} f'_j = 0,$
i.e. $U = \frac{1}{p} J_{2p\times 2p} - I_{2p\times 2p}$, or

(b) $f'_j = f'_n, \quad \forall j, n \le 2p, \quad \sum_{j=1}^{2p} f_j = 0,$
i.e. $U = -\frac{1}{p} J_{2p\times 2p} + I_{2p\times 2p}$.

Unbalanced non-Weyl graphs

On the other hand, in graphs with *unbalanced* vertices there are many cases of non-Weyl behaviour. To this end we employ a trick based on the unitary transformation $W^{-1}UW$, where W is block diagonal with a nontrivial unitary $q \times q$ part W_4,

$$W = \begin{pmatrix} e^{i\varphi} I_{p\times p} & 0 \\ 0 & W_4 \end{pmatrix}$$

Unbalanced non-Weyl graphs

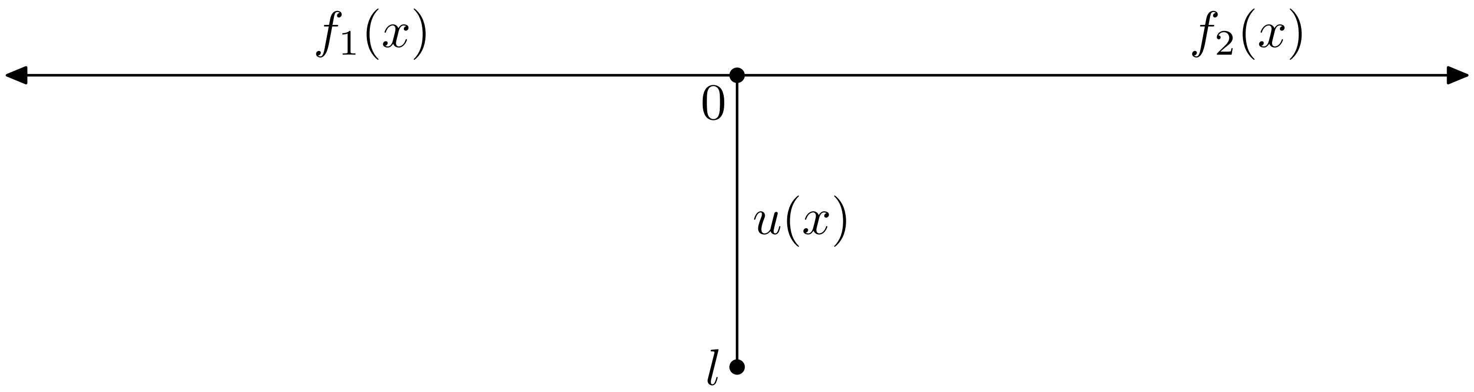
On the other hand, in graphs with *unbalanced* vertices there are many cases of non-Weyl behaviour. To this end we employ a trick based on the unitary transformation $W^{-1}UW$, where W is block diagonal with a nontrivial unitary $q \times q$ part W_4,

$$W = \begin{pmatrix} e^{i\varphi} I_{p\times p} & 0 \\ 0 & W_4 \end{pmatrix}$$

One can check easily the following claim

Lemma: The family of resonances of H_U does not change if the original coupling matrix U is replaced by $W^{-1}UW$.

Example: line with a stub



The Hamiltonian acts as $-\mathrm{d}^2/\mathrm{d}x^2$ on graph Γ consisting of two half-lines and one internal edge of length l. Its domain contains functions from $W^{2,2}(\Gamma)$ which satisfy

$$\begin{aligned} 0 &= (U-I)\,(u(0), f_1(0), f_2(0))^{\mathrm{T}} + i(U+I)\,(u'(0), f_1'(0), f_2'(0))^{\mathrm{T}}\,, \\ 0 &= u(l) + cu'(l)\,, \end{aligned}$$

$f_i(x)$ referring to half-lines and $u(x)$ to the internal edge.

Example, continued

We start from the matrix $U_0 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \mathrm{e}^{i\psi} \end{pmatrix}$, describing one half-line separated from the rest of the graph. As mentioned above such a graph has non-Weyl asymptotics (obviously, it cannot have more than two resonances)

Example, continued

We start from the matrix $U_0 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \mathrm{e}^{i\psi} \end{pmatrix}$, describing one half-line separated from the rest of the graph. As mentioned above such a graph has non-Weyl asymptotics (obviously, it cannot have more than two resonances)

Using $U_W = W^{-1}UW$ with $W = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r\mathrm{e}^{i\varphi_1} & \sqrt{1-r^2}\,\mathrm{e}^{i\varphi_2} \\ 0 & \sqrt{1-r^2}\,\mathrm{e}^{i\varphi_3} & -r\mathrm{e}^{i(\varphi_2+\varphi_3-\varphi_1)} \end{pmatrix}$

we arrive at a three-parameter family with the same resonances — *thus non-Weyl* — described by

$$U = \begin{pmatrix} 0 & r\mathrm{e}^{i\varphi_1} & \sqrt{1-r^2}\mathrm{e}^{i\varphi_2} \\ r\mathrm{e}^{-i\varphi_1} & (1-r^2)\mathrm{e}^{i\psi} & -r\sqrt{1-r^2}\mathrm{e}^{-i(-\psi+\varphi_1-\varphi_2)} \\ \sqrt{1-r^2}\mathrm{e}^{-i\varphi_2} & -r\sqrt{1-r^2}\mathrm{e}^{i(\psi+\varphi_1-\varphi_2)} & r^2\mathrm{e}^{i\psi} \end{pmatrix}$$

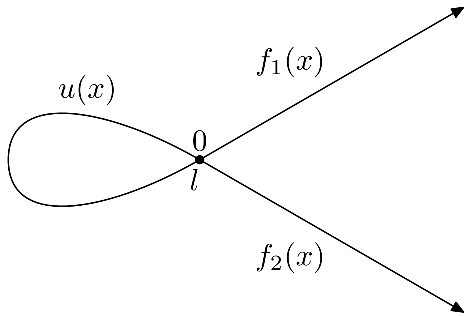
Remark

In particular, for Dirichlet condition both at the end of the separated half-line, $\psi = \pi$, and at the remote end of the internal edge, $c = 0$, one obtains a family of Hamiltonians which have no resonances at all. This includes $\varphi_1 = \varphi_2 = 0$ and $r = 1/\sqrt{2}$, or the conditions

$$f_1(0) = f_2(0), \quad u(0) = \sqrt{2} f_1(0), \quad f_1'(0) - f_2'(0) = -\sqrt{2} u'(0)\,,$$

where the fact of resonance absence was first noted in [E-Šerešová'94], and a similar behavior for $\varphi_1 = \varphi_2 = \pi$ and $r = 1/\sqrt{2}$. Notice that the absence of resonances is easily understood if one regards the graph in question as a tree and employs a unitary equivalence proposed first in [Naimark-Solomyak'00], see also [Sobolev-Solomyak'02], etc.

Example: a loop with two leads



To illustrate how the asymptotics can *change with the graph geometry*, consider the above graph. The Hamiltonian acts as above with coupling conditions

$$u(0) = f_1(0)\,, \quad u(l) = f_2(0)\,,$$
$$\alpha u(0) = u'(0) + f_1'(0) + \beta(-u'(l) + f_2'(0))\,,$$
$$\alpha u(l) = \beta(u'(0) + f_1'(0)) - u'(l) + f_2'(0)$$

with real parameters α, $\beta \in \mathbb{R}$. The choice $\beta = 1$ gives the "overall" δ-condition of strength α, while $\beta = 0$ corresponds to a line with two δ-interactions at the distance l.

Example, continued

Using $\mathrm{e}_\pm = \mathrm{e}^{\pm ikx}$ we write the resonance condition as

$$8\frac{i\alpha^2\mathrm{e}_+ + 4k\alpha\beta - i[\alpha(\alpha-4ik)+4k^2(\beta^2-1)]\,\mathrm{e}_-}{4(\beta^2-1)+\alpha(\alpha-4i)} = 0\,.$$

The coefficient of e^+ vanishes *iff* $\alpha = 0$, the second term vanishes for $\beta = 0$ or if $|\beta| \neq 1$ and $\alpha = 0$, while the polynomial multiplying e^- does not vanish for any combination of α and β.

Example, continued

Using $\mathrm{e}_\pm = \mathrm{e}^{\pm ikx}$ we write the resonance condition as

$$8\,\frac{i\alpha^2 \mathrm{e}_+ + 4k\alpha\beta - i[\alpha(\alpha-4ik)+4k^2(\beta^2-1)]\,\mathrm{e}_-}{4(\beta^2-1)+\alpha(\alpha-4i)} = 0\,.$$

The coefficient of e^+ vanishes *iff* $\alpha = 0$, the second term vanishes for $\beta = 0$ or if $|\beta| \neq 1$ and $\alpha = 0$, while the polynomial multiplying e^- does not vanish for any combination of α and β.

In other words, the graph has a *non-Weyl asymptotics* *iff* $\alpha = 0$. If, in addition, $|\beta| \neq 1$, than all resonances are confined to some circle, i.e. the graph "size" is zero. The exceptions are Kirchhoff condition, $\beta = 1$ and $\alpha = 0$, and its counterpart, $\beta = -1$ and $\alpha = 0$, for which "one half" of the resonances is preserved, the "size" being $l/2$.

Example, continued

Let us look at the δ-condition, $\beta = 1$, to illustrate the disappearance of half of the resonances when the coupling strength vanishes. The resonance equation becomes

$$\frac{-\alpha \sin kl + 2k(1 + i \sin kl - \cos kl)}{\alpha - 4i} = 0$$

Example, continued

Let us look at the δ-condition, $\beta = 1$, to illustrate the disappearance of half of the resonances when the coupling strength vanishes. The resonance equation becomes

$$\frac{-\alpha \sin kl + 2k(1 + i \sin kl - \cos kl)}{\alpha - 4i} = 0$$

A simple calculation shows that there is a sequence of embedded ev's, $k = 2n\pi/l$ with $n \in \mathbb{Z}$, and a family of resonances given by solutions to $\mathrm{e}^{ikl} = -1 + \frac{4ik}{\alpha}$. The former do not depend on α, while the latter behave for small α as

$$\mathrm{Im}\, k = -\frac{1}{l} \log \frac{1}{\alpha} + \mathcal{O}(1)\,, \quad \mathrm{Re}\, k = n\pi + \mathcal{O}(\alpha)\,,$$

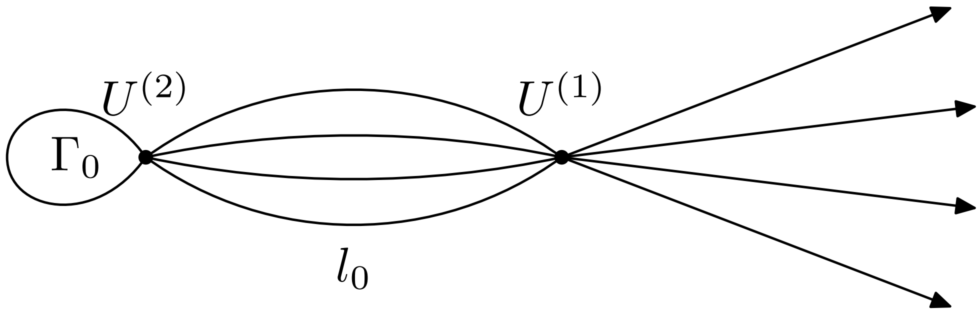
thus all the (true) resonances escape to infinity as $\alpha \to 0$.

What can cause a non-Weyl asymptotics?

We will argue that (anti)Kirchhoff conditions at balanced vertices are too easy to decouple diminishing in this way effectively the graph size

What can cause a non-Weyl asymptotics?

We will argue that (anti)Kirchhoff conditions at balanced vertices are too easy to decouple diminishing in this way effectively the graph size



Consider the above graph with a balanced vertex $\mathcal{X}_1$ which connects p internal edges of the same length l_0 and p external edges with the coupling given by a unitary $U^{(1)} = aJ_{2p\times 2p} + bI_{2p\times 2p}$. The coupling to the rest of the graph, denoted as Γ_0, is described by a $q \times q$ matrix $U^{(2)}$, where $q \geq p$; needless to say such a matrix can hide different topologies of this part of the graph

Unitary equivalence again

Proposition: Consider Γ be the the coupling given by arbitrary $U^{(1)}$ and $U^{(2)}$. Let V be an arbitrary unitary $p \times p$ matrix, $V^{(1)} := \operatorname{diag}(V, V)$ and $V^{(2)} := \operatorname{diag}(I_{(q-p)\times(q-p)}, V)$ be $2p \times 2p$ and $q \times q$ block diagonal matrices, respectively. Then H on Γ is *unitarily equivalent* to the Hamiltonian H_V on topologically the same graph with the coupling given by the matrices $[V^{(1)}]^{-1} U^{(1)} V^{(1)}$ and $[V^{(2)}]^{-1} U^{(2)} V^{(2)}$.

Unitary equivalence again

Proposition: Consider Γ be the the coupling given by arbitrary $U^{(1)}$ and $U^{(2)}$. Let V be an arbitrary unitary $p \times p$ matrix, $V^{(1)} := \mathrm{diag}\,(V, V)$ and $V^{(2)} := \mathrm{diag}\,(I_{(q-p)\times(q-p)}, V)$ be $2p \times 2p$ and $q \times q$ block diagonal matrices, respectively. Then H on Γ is *unitarily equivalent* to the Hamiltonian H_V on topologically the same graph with the coupling given by the matrices $[V^{(1)}]^{-1}U^{(1)}V^{(1)}$ and $[V^{(2)}]^{-1}U^{(2)}V^{(2)}$.

Remark: The assumption that the same edge length is made for convenience only; we can always get it fulfilled by adding Kirchhhoff vertices

Application to symmetric coupling

Let now $U^{(1)} = aJ_{2p\times 2p} + bI_{2p\times 2p}$ at $\mathcal{X}_1$. We choose columns of W as an orthonormal set of eigenvectors of the $p\times p$ block $aJ_{p\times p} + bI_{p\times p}$, the first one being $\frac{1}{\sqrt{p}}(1,1,\dots,1)^{\mathrm{T}}$. The transformed matrix $[V^{(1)}]^{-1}U^{(1)}V^{(1)}$ decouples into blocks connecting only pairs (v_j, g_j).

Application to symmetric coupling

Let now $U^{(1)} = aJ_{2p\times 2p} + bI_{2p\times 2p}$ at $\mathcal{X}_1$. We choose columns of W as an orthonormal set of eigenvectors of the $p \times p$ block $aJ_{p\times p} + bI_{p\times p}$, the first one being $\frac{1}{\sqrt{p}}(1, 1, \ldots, 1)^{\mathrm{T}}$. The transformed matrix $[V^{(1)}]^{-1}U^{(1)}V^{(1)}$ decouples into blocks connecting only pairs (v_j, g_j).

The first one corresponding to a *symmetrization* of all the u_j's and f_j's, leads to the 2×2 matrix $U_{2\times 2} = apJ_{2\times 2} + bI_{2\times 2}$, while the other lead to *separation of the corresponding internal and external edges* described by the Robin conditions, $(b-1)v_j(0) + i(b+1)v_j'(0) = 0$ and $(b-1)g_j(0) + i(b+1)g_j'(0) = 0$ for $j = 2, \ldots, p$.

Application to symmetric coupling

Let now $U^{(1)} = aJ_{2p\times 2p} + bI_{2p\times 2p}$ at $\mathcal{X}_1$. We choose columns of W as an orthonormal set of eigenvectors of the $p \times p$ block $aJ_{p\times p} + bI_{p\times p}$, the first one being $\frac{1}{\sqrt{p}}(1, 1, \dots, 1)^{\mathrm{T}}$. The transformed matrix $[V^{(1)}]^{-1}U^{(1)}V^{(1)}$ decouples into blocks connecting only pairs (v_j, g_j).

The first one corresponding to a *symmetrization* of all the u_j's and f_j's, leads to the 2×2 matrix $U_{2\times 2} = apJ_{2\times 2} + bI_{2\times 2}$, while the other lead to *separation of the corresponding internal and external edges* described by the Robin conditions, $(b-1)v_j(0) + i(b+1)v_j'(0) = 0$ and $(b-1)g_j(0) + i(b+1)g_j'(0) = 0$ for $j = 2, \dots, p$.

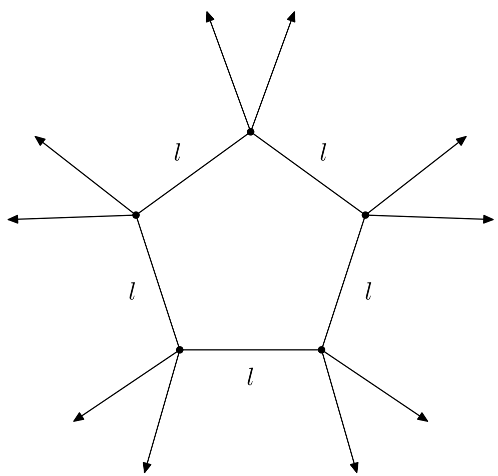
The "overall" Kirchhoff/anti-Kirchhoff condition at $\mathcal{X}_1$ is transformed to the "line" Kirchhoff/anti-Kirchhoff condition in the subspace of permutation-symmetric functions, *reducing the graph size by l_0*. In all the other cases the point interaction corresponding to the matrix $apJ_{2\times 2} + bI_{2\times 2}$ is nontrivial, and consequently, *the graph size is preserved*.

Effective size is a global property

One may ask whether there are geometrical rules that would quantify the effect of each balanced vertex on the asymptotics. The following *example* shows that this is not likely:

Effective size is a global property

One may ask whether there are geometrical rules that would quantify the effect of each balanced vertex on the asymptotics. The following *example* shows that this is not likely:



For a fixed integer $n \geq 3$ we start with a regular n-gon, each edge having length ℓ, and attach two semi-infinite leads to each vertex, so that each vertex is balanced; thus the *effective size* W_n *is strictly less than* $V_n = n\ell$.

Example, continued

Proposition: The effective size of the graph Γ_n is given by

$$W_n = \begin{cases} n\ell/2 & \text{if } n \neq 0 \mod 4, \\ (n-2)\ell/2 & \text{if } n = 0 \mod 4. \end{cases}$$

Example, continued

Proposition: The effective size of the graph Γ_n is given by

$$W_n = \begin{cases} n\ell/2 & \text{if } n \neq 0 \mod 4, \\ (n-2)\ell/2 & \text{if } n = 0 \mod 4. \end{cases}$$

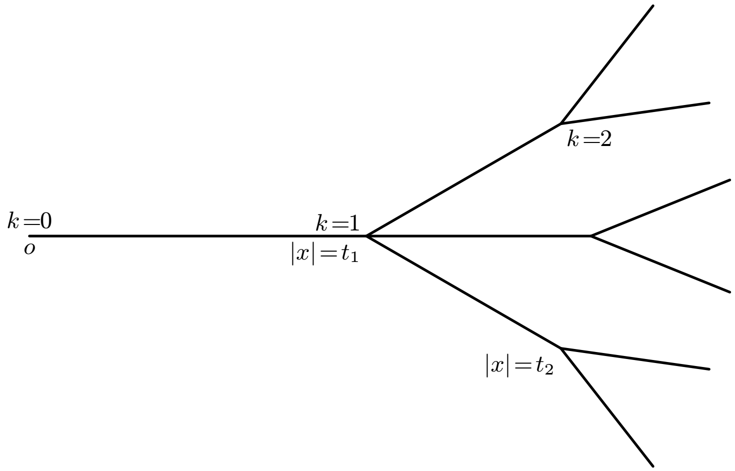
Sketch of the proof: We employ Bloch/Floquet decomposition of H w.r.t. the cyclic rotation group $\mathbb{Z}_n$. It leads to analysis of one segment with "quasimomentum" ω satisfying $\omega^n = 1$; after a short computation we find that H_ω has a resonance *iff*

$$-2(\omega^2 + 1) + 4\omega e^{-ik\ell} = 0.$$

Hence the effective size W_ω of the system of resonances of H_ω is $\ell/2$ if $\omega^2 + 1 \neq 0$ but it is zero if $\omega^2 + 1 = 0$. Now $\omega^2 + 1 = 0$ is not soluble if $\omega^n = 1$ and $n \neq 0 \mod 4$, but it has two solutions if $n = 0 \mod 4$. $\square$

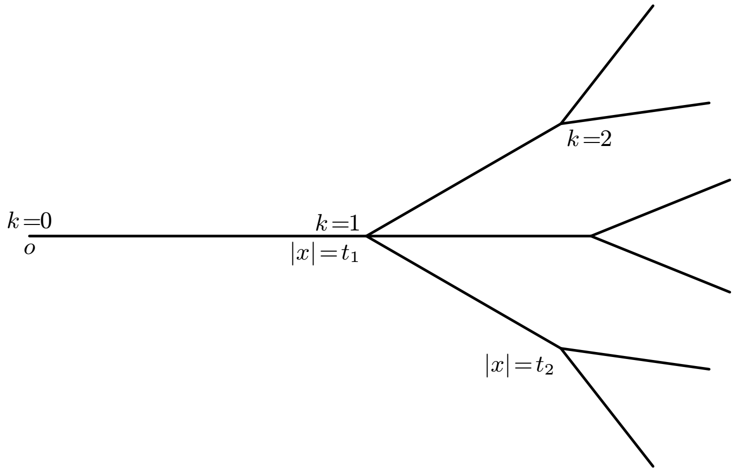
Finally, a spectral problem for trees

As the third problem of this talk, we consider Laplacian on a *radial rooted trees*.



Finally, a spectral problem for trees

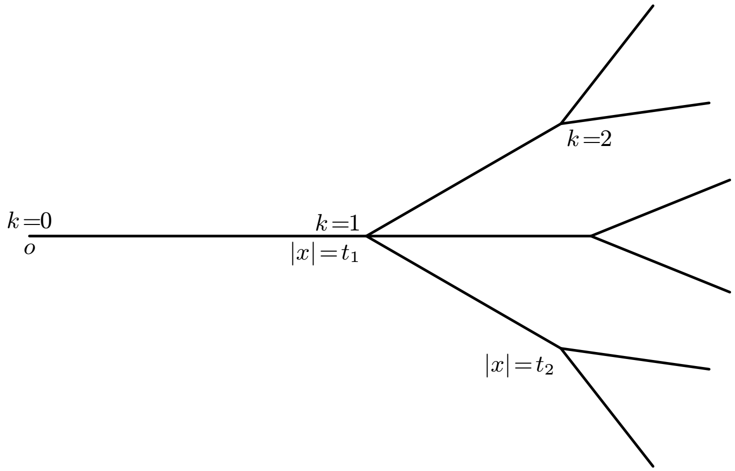
As the third problem of this talk, we consider Laplacian on a *radial rooted trees*.



Recently [Breuer-Frank'09] demonstrated that $\sigma_{\mathrm{ac}}(H_{\mathrm{tree}}) = \emptyset$ is the coupling is *Kirchoff* and the tree is *"sparse"*

Finally, a spectral problem for trees

As the third problem of this talk, we consider Laplacian on a *radial rooted trees*.



Recently [Breuer-Frank'09] demonstrated that $\sigma_{\rm ac}(H_{\rm tree}) = \emptyset$ is the coupling is *Kirchoff* and the tree is *"sparse"*

Let us look what will happen for a *more general coupling*

Preliminaries

- The method: we are going to employ the *Solomyak trick* mentioned above by which we can rephrase the question as a family of halfline problems

Preliminaries

- The method: we are going to employ the *Solomyak trick* mentioned above by which we can rephrase the question as a family of halfline problems
- it is possible if the tree is *radial*, i.e. the branching number $b(v)$ is the same in each generation, and the same for the edge lenghts

Preliminaries

- The method: we are going to employ the *Solomyak trick* mentioned above by which we can rephrase the question as a family of halfline problems
- it is possible if the tree is *radial*, i.e. the branching number $b(v)$ is the same in each generation, and the same for the edge lenghts
- For radial graphs t_k is the distance between the root and the vertices in the k-th generation, and b_k as the branching number of the k-th generation vertices; for the root we put $b_0 = 1$ and $t_0 = 0$. We defines the *branching function* $g_0(t) : \mathbb{R}_+ \to \mathbb{N}$ by

$$g_0(t) := b_0 b_1 \dots b_k \quad \text{for} \quad t \in (t_k, t_{k+1}) \,.$$

Preliminaries

- Tree graph vertices are naturally ordered, we write $w \succeq v$ and defines the *vertex subtree* $\Gamma_{\succeq v}$ as the set of vertices and edges succeeding v, and the *edge subtree* $\Gamma_{\succeq e}$ as the union of the edge e and the vertex subtree corresponding to its vertex remoter from the origin.

Preliminaries

- Tree graph vertices are naturally ordered, we write $w \succeq v$ and defines the *vertex subtree* $\Gamma_{\succeq v}$ as the set of vertices and edges succeeding v, and the *edge subtree* $\Gamma_{\succeq e}$ as the union of the edge e and the vertex subtree corresponding to its vertex remoter from the origin.
- Permutation properties of graph edges: fix a radial tree v of degree $b \equiv b_k$. Denote edges emanating from v by $e_j, j \in \{1, \dots, b\}$. We define Q_v on $L^2(\Gamma_{\succeq v})$ acting as

$$Q_v : f_j \mapsto f_{j+1} \,;$$

each f_j refers to functions on all the edges $\succeq e_j$.

Preliminaries

- Tree graph vertices are naturally ordered, we write $w \succeq v$ and defines the *vertex subtree* $\Gamma_{\succeq v}$ as the set of vertices and edges succeeding v, and the *edge subtree* $\Gamma_{\succeq e}$ as the union of the edge e and the vertex subtree corresponding to its vertex remoter from the origin.
- Permutation properties of graph edges: fix a radial tree v of degree $b \equiv b_k$. Denote edges emanating from v by $e_j, j \in \{1, \dots, b\}$. We define Q_v on $L^2(\Gamma_{\succeq v})$ acting as

$$Q_v : f_j \mapsto f_{j+1} \,;$$

each f_j refers to functions on all the edges $\succeq e_j$.
- Since $Q_v^b = \mathrm{id}$, the operator has eigenvalues $\mathrm{e}^{2\pi i s/b}$, $s \in \{0, \dots, b-1\}$ with the eigenspaces denoted by $L^2_s(\Gamma_{\succeq v}) := \mathrm{Ker}(Q_v - \mathrm{e}^{2\pi i s/b}\,\mathrm{id})$. The function $f \in L^2(\Gamma_{\succeq v})$ is *s-radial at the vertex* v if $f \in L^2_s(\Gamma_{\succeq v})$ and $f \in L^2_0(\Gamma_{\succeq v'})$ holds for all $v' \succeq v$; we write $f \in L^2_{s,\mathrm{rad}}(\Gamma_{\succeq v})$. In particular, the 0-radial functions will be simply called *radial*.

Vertex coupling

By assumption, they are *the same for vertices of the same generation*. Out of possible $(b_k+1)^2$ real parameters we choose a $[(b_k-1)^2+4]$-parameter subfamily

Vertex coupling

By assumption, they are *the same for vertices of the same generation*. Out of possible $(b_k+1)^2$ real parameters we choose a $[(b_k-1)^2+4]$-parameter subfamily

$$\sum_{j=1}^{b_k} f'_{vj+} - f'_{v-} = \frac{\alpha_{\mathrm{t}k}}{2}\left(\frac{1}{b_k}\sum_{j=1}^{b_k} f_{vj+} + f_{v-}\right) + \frac{\gamma_{\mathrm{t}k}}{2}\left(\sum_{j=1}^{b_k} f'_{vj+} + f'_{v-}\right),$$

$$\frac{1}{b_k}\sum_{j=1}^{b_k} f_{vj+} - f_{v-} = -\frac{\bar\gamma_{\mathrm{t}k}}{2}\left(\frac{1}{b_k}\sum_{j=1}^{b_k} f_{vj+} + f_{v-}\right) + \frac{\beta_{\mathrm{t}k}}{2}\left(\sum_{j=1}^{b_k} f'_{vj+} + f'_{v-}\right),$$

$$(U_k - I)V_k\Psi_v + i(U_k + I)V_k\Psi'_v = 0\,,$$

where j distinguishes the edges emanating from v, the subscript $-$ refers to the ingoing (or preceding) edge, and

$$\Psi_v := (f_{v1+}, f_{v2+}, \dots, f_{vb_k+})^{\mathrm{T}}, \quad \Psi'_v := (f'_{v1+}, f'_{v2+}, \dots, f'_{vb_k+})^{\mathrm{T}}$$

Vertex coupling, continued

Coupling of Ψ_v and Ψ'_v is described by a $(b_k-1)\times(b_k-1)$ unitary matrix U_k, while V_k is the (b_k-1)-*dimensional projection* to the orthogonal complement of $(1,1,\dots,1)$, and the vectors $V_k(f_1(\cdot),\dots,f_{b_k}(\cdot))^{\mathrm{T}}$ form an orthonormal basis in the “non-radial” subspace $L^2(\Gamma_{\succeq v})\ominus L^2_{0,\mathrm{rad}}(\Gamma_{\succeq v})$

Vertex coupling, continued

Coupling of Ψ_v and Ψ'_v is described by a $(b_k-1)\times(b_k-1)$ unitary matrix U_k, while V_k is the (b_k-1)*-dimensional projection* to the orthogonal complement of $(1,1,\dots,1)$, and the vectors $V_k(f_1(\cdot),\dots,f_{b_k}(\cdot))^{\mathrm{T}}$ form an orthonormal basis in the "non-radial" subspace $L^2(\Gamma_{\succeq v})\ominus L^2_{0,\mathrm{rad}}(\Gamma_{\succeq v})$

At the tree root, we assume Robin boundary conditions

$$f'_o + f_o \tan\frac{\theta_0}{2} = 0\,, \quad \theta_0 \in (-\pi/2, \pi/2]\,.$$

Vertex coupling, continued

Coupling of Ψ_v and Ψ'_v is described by a $(b_k-1)\times(b_k-1)$ unitary matrix U_k, while V_k is the (b_k-1)-*dimensional projection* to the orthogonal complement of $(1,1,\dots,1)$, and the vectors $V_k(f_1(\cdot),\dots,f_{b_k}(\cdot))^{\mathrm{T}}$ form an orthonormal basis in the “non-radial” subspace $L^2(\Gamma_{\succeq v})\ominus L^2_{0,\mathrm{rad}}(\Gamma_{\succeq v})$

At the tree root, we assume Robin boundary conditions

$$f'_o + f_o \tan\frac{\theta_0}{2} = 0\,, \quad \theta_0 \in (-\pi/2,\pi/2]\,.$$

By a direct computation we get

Lemma: Laplacian on the radial tree with the above boundary conditions is a self-adjoint operator

Generalized point interactions

For Laplacian on a halfline we consider the conditions

$$\begin{aligned} y'_+ - y'_- &= \frac{\alpha}{2}(y_+ + y_-) + \frac{\gamma}{2}(y'_+ + y'_-)\,, \\ y_+ - y_- &= -\frac{\bar\gamma}{2}(y_+ + y_-) + \frac{\beta}{2}(y'_+ + y'_-) \end{aligned}$$

characterized by a $\mathcal{A} = \begin{pmatrix} \alpha & \gamma \\ -\bar\gamma & \beta \end{pmatrix}$ with $\alpha, \beta \in \mathbb{R}$ and $\gamma \in \mathbb{C}$

– not universal, but describe almost all s-a extensions

Generalized point interactions

For Laplacian on a halfline we consider the conditions

$$
\begin{aligned}
y'_+ - y'_- &= \frac{\alpha}{2}(y_+ + y_-) + \frac{\gamma}{2}(y'_+ + y'_-)\,, \\
y_+ - y_- &= -\frac{\bar\gamma}{2}(y_+ + y_-) + \frac{\beta}{2}(y'_+ + y'_-)
\end{aligned}
$$

characterized by a $\mathcal{A} = \begin{pmatrix} \alpha & \gamma \\ -\bar\gamma & \beta \end{pmatrix}$ with $\alpha, \beta \in \mathbb{R}$ and $\gamma \in \mathbb{C}$

– not universal, but describe almost all s-a extensions

There are other possible forms, e.g.

$$
\begin{pmatrix} y'_+ \\ -y'_- \end{pmatrix} = \begin{pmatrix} a & c \\ \bar c & d \end{pmatrix} \begin{pmatrix} y_+ \\ y_- \end{pmatrix} \quad \text{or} \quad (U-I)\begin{pmatrix} y_+ \\ y_- \end{pmatrix} + i(U+I)\begin{pmatrix} y'_+ \\ -y'_- \end{pmatrix} = 0
$$

with $a, d \in \mathbb{R}$ and $c \in \mathbb{C}$ and U unitary; we will not need them

Tree-to-halfline map

The main idea of *Solomyak trick* is to identify “symmetric” functions, $f \in L^2_{0,\mathrm{rad}}(\Gamma)$, with the function on the halfline through the isometry $\Pi : f \to \varphi$, $\varphi(t) = f(x)$ for $t = |x|$ of $L^2_{0,\mathrm{rad}}(\Gamma)$ into the weighted space $L^2(\mathbb{R}_+, g_0)$ with the norm

$$\|\varphi\|^2_{L^2(\mathbb{R}_+,g_0)} = \int_{\mathbb{R}_+} |\varphi(t)|^2 g_0(t)\,\mathrm{d}t\,,$$

and then isometrically to $L^2(\mathbb{R})$ by $y(t) := g_0^{1/2}(t)\varphi(t)$ and the relations $y_{k+} = (b_0 \cdot \ldots \cdot b_k)^{1/2}\varphi_{k+}$, $y_{k-} = (b_0 \cdot \ldots \cdot b_{k-1})^{1/2}\varphi_{k-}$ for the boundary values at the vertices.

Tree-to-halfline map

The main idea of *Solomyak trick* is to identify "symmetric" functions, $f \in L^2_{0,\mathrm{rad}}(\Gamma)$, with the function on the halfline through the isometry $\Pi : f \to \varphi$, $\varphi(t) = f(x)$ for $t = |x|$ of $L^2_{0,\mathrm{rad}}(\Gamma)$ into the weighted space $L^2(\mathbb{R}_+, g_0)$ with the norm

$$\|\varphi\|^2_{L^2(\mathbb{R}_+,g_0)} = \int_{\mathbb{R}_+} |\varphi(t)|^2 g_0(t)\, \mathrm{d}t\,,$$

and then isometrically to $L^2(\mathbb{R})$ by $y(t) := g_0^{1/2}(t)\varphi(t)$ and the relations $y_{k+} = (b_0 \cdot \ldots \cdot b_k)^{1/2}\varphi_{k+}$, $y_{k-} = (b_0 \cdot \ldots \cdot b_{k-1})^{1/2}\varphi_{k-}$ for the boundary values at the vertices. In our case

$$f_{v-} \longrightarrow y_{k-}\,, \qquad f'_{v-} \longrightarrow y'_{k-}\,,$$

$$\frac{1}{b_k}\sum_{j=1}^{b_k} f_{vj+} \longrightarrow b_k^{-1/2} y_{k+}\,, \qquad \sum_{j=1}^{b_k} f'_{jv+} \longrightarrow b_k^{1/2} y'_{k+}\,.$$

Tree-to-halfline map, continued

By a direct computation we get

Proposition: The tree vertex coupling is mapped into

$$\begin{aligned} y'_{k+} - y'_{k-} &= \frac{\alpha_{\mathrm{h}k}}{2}(y_{k+} + y_{k-}) + \frac{\gamma_{\mathrm{h}k}}{2}(y'_{k+} + y'_{k-})\,, \\ y_{k+} - y_{k-} &= -\frac{\bar\gamma_{\mathrm{h}k}}{2}(y_{k+} + y_{k-}) + \frac{\beta_{\mathrm{h}k}}{2}(y'_{k+} + y'_{k-})\,, \\ y(0+)' &+ y(0+)\,\tan\frac{\theta_0}{2} = 0\,, \end{aligned}$$

where

$$\begin{aligned} \alpha_{\mathrm{h}k} &:= \frac{16\alpha_{\mathrm{t}k}}{4(b_k^{1/2}+1)^2 + \det\mathcal{A}_{\mathrm{t}k}(b_k^{1/2}-1)^2 + 4(1-b_k)\,\mathrm{Re}\,\gamma_{\mathrm{t}k}}\,, \\ \beta_{\mathrm{h}k} &:= \frac{16\,b_k\,\beta_{\mathrm{t}k}}{4(b_k^{1/2}+1)^2 + \det\mathcal{A}_{\mathrm{t}k}(b_k^{1/2}-1)^2 + 4(1-b_k)\,\mathrm{Re}\,\gamma_{\mathrm{t}k}}\,, \\ \gamma_{\mathrm{h}k} &:= 2\cdot\frac{(1-b_k)(4+\det\mathcal{A}_{\mathrm{t}k}) + 8ib_k^{1/2}\,\mathrm{Im}\,\gamma_{\mathrm{t}k} + 4(b_k+1)\,\mathrm{Re}\,\gamma_{\mathrm{t}k}}{4(b_k^{1/2}+1)^2 + \det\mathcal{A}_{\mathrm{t}k}(b_k^{1/2}-1)^2 + 4(1-b_k)\,\mathrm{Re}\,\gamma_{\mathrm{t}k}}\,. \end{aligned}$$

Tree-to-halfline map, continued

Proposition, continued: The conditions $f_{v+} = f_{v-} = 0$ or $\sum_{j=1}^{b_k} f'_{vj+} = f'_{v-} = 0$ transform similarly to $y_{k+} = y_{k-} = 0$ or $y'_{k+} = y'_{k-} = 0$, respectively. Finally, the tree coupling with $\alpha_{tk} = 0$, $\beta_{tk} \neq 0$, $\gamma_{tk} = 2\frac{b_k^{1/2}+1}{b_k^{1/2}-1}$ changes to

$$y'_{k+} = -y'_{k-}, \quad y_{k+} + y_{k-} = \frac{\beta_{tk}}{2}(b_k^{1/2} - 1)^2(-y'_{k-}),$$

while conditions $\alpha_{tk} \neq 0$, $\beta_{tk} = 0$, $\gamma_{tk} = 2\frac{b_k^{1/2}+1}{b_k^{1/2}-1}$ change to

$$y_{k+} = -y_{k-}, \quad y'_{k+} + y'_{k-} = -\frac{\alpha_{tk}}{2}(b_k^{-1/2} - 1)^2 y_{k-}.$$

The unitary equivalence

Since U_k is unitary, there are $\theta_{k,j},\ j=1,\dots,b_k-1$, such that $U_k = W_k^{-1} D_k W_k$, where $D_k := \mathrm{diag}\,(\mathrm{e}^{i\theta_{k,1}},\dots,\mathrm{e}^{i\theta_{k,b_k-1}})$. For a given vertex v of the k-th generation we can then define the operator R_v on $H^2(\Gamma_{\succeq v}) \ominus L^2_{0,\,\mathrm{rad}}(\Gamma_{\succeq v})$ by

$$R_v : \begin{pmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_{b_k}(x) \end{pmatrix} \mapsto \begin{pmatrix} \sum_{j=1}^{b_k} (W_k \cdot V_k)_{1j}\, f_j(x) \\ \sum_{j=1}^{b_k} (W_k \cdot V_k)_{2j}\, f_j(x) \\ \vdots \\ \sum_{j=1}^{b_k} (W_k \cdot V_k)_{(b_k-1)j}\, f_j(x) \end{pmatrix},$$

where $f_j(x)$ is the function component on the j-th subtree.

The unitary equivalence

Since U_k is unitary, there are $\theta_{k,j},\ j = 1, \dots, b_k - 1$, such that $U_k = W_k^{-1} D_k W_k$, where $D_k := \mathrm{diag}\,(\mathrm{e}^{i\theta_{k,1}}, \dots, \mathrm{e}^{i\theta_{k,b_k-1}})$. For a given vertex v of the k-th generation we can then define the operator R_v on $H^2(\Gamma_{\succeq v}) \ominus L^2_{0,\,\mathrm{rad}}(\Gamma_{\succeq v})$ by

$$R_v : \begin{pmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_{b_k}(x) \end{pmatrix} \mapsto \begin{pmatrix} \sum_{j=1}^{b_k} (W_k \cdot V_k)_{1j}\, f_j(x) \\ \sum_{j=1}^{b_k} (W_k \cdot V_k)_{2j}\, f_j(x) \\ \vdots \\ \sum_{j=1}^{b_k} (W_k \cdot V_k)_{(b_k-1)j}\, f_j(x) \end{pmatrix},$$

where $f_j(x)$ is the function component on the j-th subtree.

Lemma: f satisfies tree coupling conditions *iff* $R_v f$ satisfies $(R_v f)'_{vs+} + (R_v f)_{vs+} \tan\frac{\theta_{ks}}{2} = 0$ for all $s \in \{1, \dots, b(v) - 1\}$.

The unitary equivalence, continued

Let v be a vertex belonging to the n-th generation. Denoting $n := \operatorname{gen} v$ we introduce the operator acting as $H_{L_{ns}} := -\frac{\mathrm{d}^2}{\mathrm{d}t^2}$ with the domain consisting of all the $f \in \bigoplus_{k=n}^{\infty} H^2(t_k, t_{k+1})$ satisfying conditions of the Proposition at the points t_k, $k > n$, and $y' + \tan\frac{\theta_{ns}}{2} y = 0$ at t_n.

The unitary equivalence, continued

Let v be a vertex belonging to the n-th generation. Denoting $n := \text{gen}\, v$ we introduce the operator acting as $H_{L_{ns}} := -\frac{\mathrm{d}^2}{\mathrm{d}t^2}$ with the domain consisting of all the $f \in \bigoplus_{k=n}^{\infty} H^2(t_k, t_{k+1})$ satisfying conditions of the Proposition at the points t_k, $k > n$, and $y' + \tan\frac{\theta_{ns}}{2} y = 0$ at t_n.

Theorem [E-Lipovský'10]: The radial-tree Hamiltonian $\mathbf{H}$ is unitarily equivalent to

$$\mathbf{H} \cong H_{L_0} \oplus \bigoplus_{n=1}^{\infty} \bigoplus_{s=1}^{b_n - 1} (\oplus\, b_0 \dots b_{n-1}) H_{L_{ns}}\,,$$

where $(\oplus\, m) H_{L_{ns}}$ is the m-tuple copy of the operator $H_{L_{ns}}$.

The unitary equivalence, continued

Let v be a vertex belonging to the n-th generation. Denoting $n := \mathrm{gen}\, v$ we introduce the operator acting as $H_{L_{ns}} := -\frac{\mathrm{d}^2}{\mathrm{d}t^2}$ with the domain consisting of all the $f \in \bigoplus_{k=n}^{\infty} H^2(t_k, t_{k+1})$ satisfying conditions of the Proposition at the points t_k, $k > n$, and $y' + \tan\frac{\theta_{ns}}{2} y = 0$ at t_n.

Theorem [E-Lipovský'10]: The radial-tree Hamiltonian $\mathbf{H}$ is unitarily equivalent to

$$\mathbf{H} \cong H_{L_0} \oplus \bigoplus_{n=1}^{\infty} \bigoplus_{s=1}^{b_n - 1} (\oplus\, b_0 \dots b_{n-1}) H_{L_{ns}}\,,$$

where $(\oplus\, m) H_{L_{ns}}$ is the m-tuple copy of the operator $H_{L_{ns}}$.

Remarks: The claim remain valid when a regular *radial* potential is added, as well as for *tree-like graphs*, i.e. when edges are replaced by identical compact graphs

Sparse point interactions

Modifying result of [Remling'07], [Breuer-Frank'09] we get

Theorem [E-Lipovský'10]: Let H be the halfline GPI Hamiltonian with Dirichlet b.c. at $t=0$ and described coupling at $t=t_n$. Suppose that there are $N \in \mathbb{N}$, $K \in (0,\infty)$ and $\delta > 0$ such that for all $n > N$ one of the following hypotheses holds: either

(a) $|\beta_n| > \delta > 0$ and $|c_n| > \delta > 0$, or

(b) $\beta_n = 0$, $|\gamma_n| < K$, and at least one of the following conditions is valid for all $n > N$: $\mathrm{Re}\,\gamma_n > \delta$ or $\mathrm{Re}\,\gamma_n < -\delta$ or $\alpha_n > \delta$ or $\alpha_n < -\delta$.

Let further the number of GPI's described by separating conditions be at most finite. Let $\varepsilon = \inf_{n,m;n\neq m} |t_n - t_m| > 0$. If $\limsup_{n\to\infty}(t_{n+1}-t_n) = \infty$, the *ac* spectrum of H is empty.

Absence of transport on trees

Putting now the above results together we get

Theorem [E-Lipovský'10]: Let $\mathbf{H}$ act as $-\mathrm{d}^2/\mathrm{d}x^2$ on a radial tree graph with branching numbers b_n and the domain consisting of all functions $f \in \bigoplus_{e\in\Gamma} H^2(e)$ satisfying the above coupling conditions at t_n, $n \in \mathbb{N}$, among which the number of separating ones is at most finite. Suppose that there are $K \in (0,\infty)$ and $N \in \mathbb{N}$ such that for all $n > N$ the following conditions hold:

(i) $\limsup_{n\to\infty}(t_{n+1} - t_n) = \infty$,

(ii) $\inf_{m,n}(t_m - t_n) > 0$,

(iii) either $\mathrm{Im}\,\gamma_{\mathrm{t}n} \neq 0$, or both $\det\mathcal{A}_{\mathrm{t}n} \neq 4$ and condition $\det\mathcal{A}_{\mathrm{t}n}(\sqrt{b_k} - 1) + 4(1 - b_n)\,\mathrm{Re}\,\gamma_{\mathrm{t}n} + 4(1 + \sqrt{b_n}) \neq 0$ are valid,

Absence of transport on trees, continued

Theorem, continued:

(iv) the following conditions hold

$$\frac{1}{K} < \left|4 - 2\sqrt{b_n}(\det\mathcal{A}_{\mathrm{t}n} - 4) + \det\mathcal{A}_{\mathrm{t}n} + b_n(4 + \det\mathcal{A}_{\mathrm{t}n} - 4\mathrm{Re}\,\gamma_{\mathrm{t}n}) + 4\mathrm{Re}\,\gamma_{\mathrm{t}n}\right| < K$$

$$\frac{1}{K} < 4b_n\det\mathcal{A}_{\mathrm{t}n} + (1-b_n)[(4+\det\mathcal{A}_{\mathrm{t}n}+4\mathrm{Re}\,\gamma_{\mathrm{t}n})^2 - b_n(4+\det\mathcal{A}_{\mathrm{t}n} - 4\mathrm{Re}\,\gamma_{\mathrm{t}n})^2] < K$$

(v) finally, one of the following conditions holds:

(a) $b_n|\beta_{\mathrm{t}n}| > \frac{1}{K}$ and $\frac{b_n^{1/2}}{|\beta_{\mathrm{t}n}|}\sqrt{(-4+\det\mathcal{A}_{\mathrm{t}n})^2 + (4\,\mathrm{Im}\,\gamma_{\mathrm{t}n})^2} > 1/K$ is valid for all $n > N$,

(b) $\beta_{\mathrm{t}n} = 0$, and either the right-hand side of $\alpha_{\mathrm{t}k}$ is larger than $1/K$ for all $n > N$ or smaller than $-1/K$ for all $n > N$, or the *rhs* of $\beta_{\mathrm{t}k}$ is larger than $1/K$ for all $n > N$ or smaller than $-1/K$ for all $n > N$.

Then the absolutely continuous spectrum of $\mathbf{H}$ is empty.

Existence of transport on sparse trees

While generically the *ac* spectrum of a sparse tree graph is thus empty, there are exceptions:

Examples: Suppose that there is an N that for all $n \in \mathbb{N}$, $n \geq N$ one has $\alpha_{\mathrm{t}n} = \beta_{\mathrm{t}n} = 0$, while $\gamma_{\mathrm{t}n} = 2\frac{b_n^{1/2}-1}{b_n^{1/2}+1}$. Then the spectrum of H contains an *ac* part, in particular, if $N = 1$, then the spectrum is *purely absolutely continuous*.

Existence of transport on sparse trees

While generically the *ac* spectrum of a sparse tree graph is thus empty, there are exceptions:

Examples: Suppose that there is an N that for all $n \in \mathbb{N}$, $n \geq N$ one has $\alpha_{\mathrm{t}n} = \beta_{\mathrm{t}n} = 0$, while $\gamma_{\mathrm{t}n} = 2\frac{b_n^{1/2}-1}{b_n^{1/2}+1}$. Then the spectrum of H contains an *ac* part, in particular, if $N = 1$, then the spectrum is *purely absolutely continuous*.

It is obvious since the "transformed" coupling constants at t_n are those of the free Hamiltonian, $\alpha_{\mathrm{h}n} = \beta_{\mathrm{h}n} = \gamma_{\mathrm{h}n} = 0$ (note that the above conclusions are not sensitive to the distribution of the points $\{t_n\}$).

Existence of transport on sparse trees

While generically the *ac* spectrum of a sparse tree graph is thus empty, there are exceptions:

Examples: Suppose that there is an N that for all $n \in \mathbb{N}$, $n \geq N$ one has $\alpha_{\mathrm{t}n} = \beta_{\mathrm{t}n} = 0$, while $\gamma_{\mathrm{t}n} = 2\frac{b_n^{1/2}-1}{b_n^{1/2}+1}$. Then the spectrum of H contains an *ac* part, in particular, if $N = 1$, then the spectrum is *purely absolutely continuous*.

It is obvious since the "transformed" coupling constants at t_n are those of the free Hamiltonian, $\alpha_{\mathrm{h}n} = \beta_{\mathrm{h}n} = \gamma_{\mathrm{h}n} = 0$ (note that the above conclusions are not sensitive to the distribution of the points $\{t_n\}$).

The same conclusion can be made, with the classical Deift-Killip result in mind, if a *radial L^2 potential is added*.

Concluding remarks

The present results inspire various questions, e.g.

- Can the inequality for resonances be valid *not only asymptotically*, in the spirit of Pólya conjecture for Dirichlet Laplacians?

Concluding remarks

The present results inspire various questions, e.g.

- Can the inequality for resonances be valid *not only asymptotically*, in the spirit of Pólya conjecture for Dirichlet Laplacians?
- How these effects will look like for *more general operators*? in [DEL'10] criteria for occurrence of non-Weyl asymptotics were derived for weighted Laplacians, but there are others

Concluding remarks

The present results inspire various questions, e.g.

- Can the inequality for resonances be valid *not only asymptotically*, in the spirit of Pólya conjecture for Dirichlet Laplacians?
- How these effects will look like for *more general operators*? in [DEL'10] criteria for occurrence of non-Weyl asymptotics were derived for weighted Laplacians, but there are others
- Can the above tree result extended to some classes of *non-radial* tree graphs?

Concluding remarks

The present results inspire various questions, e.g.

- Can the inequality for resonances be valid *not only asymptotically*, in the spirit of Pólya conjecture for Dirichlet Laplacians?
- How these effects will look like for *more general operators*? in [DEL'10] criteria for occurrence of non-Weyl asymptotics were derived for weighted Laplacians, but there are others
- Can the above tree result extended to some classes of *non-radial* tree graphs?
- etc.

The results discussed here come from

[AGA08] P.E., J.P. Keating, P. Kuchment, T. Sunada, A. Teplyaev, eds.: Analysis on Graphs and Applications, *Proceedings of a Isaac Newton Institute programme*, January 8–June 29, 2007; 670 p.; AMS "Proceedings of Symposia in Pure Mathematics" Series, vol. 77, Providence, R.I., 2008

[BF09] J. Breuer, R. Frank: Singular spectrum for radial trees, *Rev. Math. Phys.* **21** (2009), 1–17.

[DP10] E.B. Davies, A. Pushnitski: Non-Weyl resonance asymptotics for quantum graphs, `arXiv: 1003.0051 [math.SP]`

[DEL10] E.B. Davies, P.E., J. Lipovský: Non-Weyl asymptotics for quantum graphs with general coupling conditions, *J. Phys. A: Math. Theor.* **A43** (2010), to appear; `arXiv: 1004.08560 [math-phys]`

[EL10] P.E., J. Lipovský: Resonances from perturbations of quantum graphs with rationally related edges, *J. Phys. A: Math. Theor.* **A43** (2010), 105301

Thank you for your attention!